A note on concomitants of records

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Abstract

We study the asymptotic distribution properties of concomitants of records and Pfiefer records and provide some interesting examples on the different possible behaviour of the limits. We also show that under suitable conditions, the unnormalised partial sum of concomitants of lower and upper records converges in distribution and the limit is infinitely divisible.

1 Introduction

Study of records was initiated by Chandler [10] and after that contributions of numerous researchers have enriched the field of theory of records and its ramifications. See Arnold et. al. [1] for a comprehensive bibliography. An associated field of interest is the study of concomitants of records defined as follows in the i.i.d. set up. Let $\{(X_n, Y_n)\}_{n\geq 1}$ be a sequence of i.i.d. bivariate random variables, with a common joint cdf $F(x, y)$. We assume that F is continuous. Let us consider $\{X_n\}$, the i.i.d. sequence of random variables with common cdf F_X, which is the marginal of F. We define the sequence of records for $\{X_n\}$ in the usual way as follows:

Let $L_0 = 1$. Suppose L_i are defined for all $i < n$. Then define $L_n = \inf\{k > L_{n-1} : X_k > X_{L_{n-1}}\}$. Define $R_n = X_{L_n}$. Then $\{R_n\}$ is the sequence of upper records for $\{X_n\}$. We denote the corresponding Y-coordinates, Y_{L_n} by $R_{[n]}$ and they are defined to be the concomitants of upper records. When there is no chance of confusion we simply refer to

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them as concomitants of records. We define lower records for X_n and concomitants for lower records in an analogous way.

Concomitants of records can arise in several applications. Suppose individuals are to be selected on the basis of measurement of an attribute **A** whose high value is desirable. Suppose **B** is an associated attribute which is known to be positively (or negatively) correlated with **A**. While **B** is easy to measure, **A** is not. So the individuals are first measured on the basis of their **B** values and only those having **B** value bigger (or smaller) than all previous observations on **B** qualify to be measured for their **A** values and sequence of **A** values thus measured are concomitants of records. One can think of many such examples where concomitants of records are useful.

Concomitants of order statistics attracted a considerable amount of attention in the literature (see for example, Balasubramanian et. al. [3], Beg et. al. [4], David et. al. [12] etc.) but concomitants of records received comparatively little attention. Some important references are Chacko et. al. [9], Houchens [14]. In this note we will discuss some interesting results on asymptotic behaviour of concomitants of records. In Section 2.1, we derive a relation between the limiting distribution of normalized records and that of concomitants. In Section 2.2 we consider the concomitants of Pfeifer records (see Pfeifer [16]). In Section 3, we present the asymptotic behaviour of sums of concomitants under certain conditions. Except for Section 2.2, everywhere else we assume $\{(X_i, Y_i)\}$ to be a sequence of i.i.d. bivariate random variables.

2 Main Results

Our basic is that the records sequence itself has a nondegenerate limit distribution. For conditions under which this happens, see Resnick [17],, [18] or Arnold et. al. [1].

Assumption I. There exist sequences of reals $\{a_n\}$ and $\{b_n\}$ and a nondegenerate random variable R such that

$$\frac{R_n - b_n}{a_n} \xrightarrow{\mathcal{D}} R \quad \text{as} \quad n \to \infty. \tag{2.1}$$

We also assume the following:

Assumption II. $F(x, y)$ is absolutely continuous with density $f(x, y)$.

The following result is an analogue of a result for concomitants of order statistics (see David [11]), and does not seem to be available in the literature.

Theorem 1. *Suppose $\{(X_i, Y_i)\}$ are i.i.d. and $\{R_n\}$ is the record sequence of the X variables. Suppose **Assumption I** and **II** hold. Suppose $G : \mathcal{R}^2 \to [0, 1]$ is continuous in r, such that for each fixed y,*

$$F_{Y|X=a_n r+b_n}(c_n y + d_n) \to G(r, y) \quad \textit{as} \quad n \to \infty,$$

locally uniformly in r, where $F_{Y|X}$ is the conditional cdf of Y given X.Then

$$\mathbb{P}\left(\frac{R_{[n]} - d_n}{c_n} \le y\right) \to \int_{-\infty}^{\infty} G(r, y) dF_R(r),$$

where F_R is the cdf of R.

Proof: The proof is easy and similar to that available for concomitants of order statistics. But for completeness we give it here. Now,

$$\begin{aligned} \mathbb{P}(R_{[n]} \le d_n + c_n y) &= \int_{-\infty}^{\infty} \mathbb{P}(R_{[n]} \le d_n + c_n y | R_n = x) F_{R_n}(dx) \\ &= \int_{-\infty}^{\infty} \mathbb{P}(Y_{L_n} \le d_n + c_n y | X_{L_n} = x) F_{R_n}(dx). \end{aligned} \tag{2.2}$$

Since (X_i, Y_i) are i.i.d., by Lemma 1 in the Appendix,

$$\mathbb{P}(Y_{L_n} \le d_n + c_n y | X_{L_n} = x) = F_{Y|X=x}(d_n + c_n y).$$

Now putting $x = a_n r + b_n$, we have

$$\mathbb{P}(R_{[n]} \le d_n + c_n y) = \int_{-\infty}^{\infty} F_{Y|X=a_n r + b_n}(d_n + c_n y) F_{\frac{R_n - b_n}{a_n}}(dr).$$

Fix $\epsilon > 0$. Choose $K > 0$ so large that $\int_{-\infty}^{-K} F_R(dr) + \int_K^{\infty} F_R(dr) < \epsilon$. Then using (2.1),

$$\int_{-\infty}^{-K} F_{Y|X=a_n r + b_n}(d_n + c_n y) F_{\frac{R_n - b_n}{a_n}}(dr) + \int_{-K}^{\infty} F_{Y|X=a_n r + b_n}(d_n + c_n y) F_{\frac{R_n - b_n}{a_n}}(dr) < \epsilon$$

for all n large enough. Now since by our assumption, for each y, $F_{Y|X=a_n r + b_n}(d_n + c_n y) \longrightarrow G(r, y)$ locally uniformly in r,

$$\begin{aligned} &\int_{-K}^{K} F_{Y|X=a_n r + b_n}(d_n + c_n y) F_{\frac{R_n - b_n}{a_n}}(dr) \\ \le &\int_{-K}^{K} |F_{Y|X=a_n r + b_n}(d_n + c_n y) - G(r, y)| F_{\frac{R_n - b_n}{a_n}}(dr) \quad + \quad \int_{-K}^{K} G(r, y) F_{\frac{R_n - b_n}{a_n}}(dr) \\ &\longrightarrow \int_{-K}^{K} G(r, y) F_R(dr), \end{aligned} \tag{2.3}$$

since $G(\cdot, y)$ is a bounded and continuous (by assumption) function. □

2.1 Examples

Example 1: Consider Gumbel's bivariate exponential distribution,

$$F(x, y) = 1 - \exp(-x) - \exp(-y) + \exp\left(-(x + y + \theta x y)\right), \ x, y > 0, \ 0 \le \theta \le 1.$$

Its p.d.f. is given by

$$f(x, y) = \exp(-x - y - \theta x y) \left((1 + \theta x)(1 + \theta y) - \theta\right), \quad x > 0, y > 0.$$

It is easy to see that the marginal cdf F_X of X is $\mathbb{E}(1)$, the standard exponential. So (see Resnick [18])

$$\frac{R_n - n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1), \quad \text{the standard normal variate.}$$

Let $r_n := \sqrt{n}r + n, \quad -\infty < r < \infty$. Then

$$\begin{aligned} F_{Y|X=r_n}(c_n y + d_n) &= \int_0^{c_n y + d_n} f_{Y|X=r_n}(u) du \\ &= \int_0^{c_n y + d_n} \exp(-u(1+\theta r_n))((1+\theta r_n) + \theta u(1+\theta r_n) - \theta) du \quad (2.4) \end{aligned}$$

Now putting $c_n y + d_n = y_n$ we have

$$\begin{aligned} F_{Y|X=r_n}(y_n) &= \int_0^{y_n} \exp(u(1+\theta r_n))((1+\theta r_n) + \theta u(1+\theta r_n) - \theta) du \\ &= 1 - \exp(-y_n(1+\theta r_n))(1+\theta y_n). \quad (2.5) \end{aligned}$$

If we take $c_n = 1/n$ and $d_n = 0$, i.e. $y_n = y/n$, then for any fixed $y > 0$,

$$1 - \exp\left[-(y/n)(1+\theta(\sqrt{n}r + n))\right](1+\theta y/n) \longrightarrow 1 - \exp(-y\theta) \quad \text{as} \quad n \to \infty,$$

uniformly in $r \in [-K, K]$, for any $K > 0$. So in this example $G(r, y) = 1 - \exp(-y\theta)$, $y > 0$. Therefore by the above theorem

$$\mathbb{P}(nR_{[n]} \le y) \to \int_{-\infty}^{\infty} (1 - \exp(-y\theta))\Phi(dr),$$

where Φ is the standard normal cdf. Hence $\mathbb{P}(nR_{[n]} \le y) \to 1 - \exp(-y\theta)$, that is, $nR_{[n]} \xrightarrow{\mathcal{D}} \mathbb{E}(\theta)$.

Incidentally, it also follows that $R_{[n]} \xrightarrow{\mathcal{P}} 0$, though $R_n \xrightarrow{a.s.} \infty$. But this is not surprising since one can show that correlation of (X, Y) for $F(X, Y)$ in the above example, is

$$-1 + \int_0^{\infty} \frac{\exp(-y)}{1+\theta y} dy < 0.$$

Example 2: Consider the bivariate Normal distribution $N(0, 0, 1, 1, \rho)$. Here conditional cdf of Y given $X = x$ is $\Phi(\rho x, 1 - \rho^2)$, where $\Phi(\cdot)$ is the cdf of the standard normal variate. In this case, the asymptotic distribution of concomitants can be easily calculated using linearity of the regression of the normal distribution (see Arnold et al. [1] p.273).

However, we can also apply Theorem 1 to compute the proper centering and scaling and obtain the asymptotic distribution of concomitants as follows: Let $(X, Y) \sim N(0, 0, 1, 1, \rho)$. Then (see Arnold et al. [1], p.19),

$$\frac{R_n - \psi(n)}{\psi(n + \sqrt{n}) - \psi(n)} \xrightarrow{\mathcal{D}} N(0,1),$$

where $\psi(n) = \Phi^{-1}(1-e^{-n})$ and conditional distribution of Y given $X=x$ is $N(\rho x, 1-\rho^2)$. A little calculation shows $b_n := \psi(n) \sim \sqrt{2n}$ and $a_n := \psi(n+\sqrt{n}) - \psi(n) \sim 1/\sqrt{2}$. Put $r_n = \frac{r}{\sqrt{2}} + \sqrt{2n}$. Then

$$F_{Y|X=r_n}(c_n y + d_n) = \Phi\left(\frac{c_n y + d_n - \rho r_n}{\sqrt{1-\rho^2}}\right).$$

If we choose $c_n = 1$ and $d_n = \sqrt{2n}\rho$, then it is easy to see that

$$G(r,y) := \lim_{n\to\infty} F_{Y|X=r_n}(c_n y + d_n) = \Phi(\frac{y - \frac{\rho r}{\sqrt{2}}}{\sqrt{1-\rho^2}}).$$

Therefore, as $n \to \infty$,

$$\mathbb{P}(R_{[n]} - \sqrt{2n}\rho \le y) \to \int_{-\infty}^{\infty} \Phi\left(\frac{y - \frac{\rho r}{\sqrt{2}}}{\sqrt{1-\rho^2}}\right) \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr.$$

Computing the integral by first differentiating the integrand with respect to y under sign of integration and then integrating with respect to y, we get the known result (see Arnold et. al. [1], p. 272)

$$\mathbb{P}(R_{[n]} - \sqrt{2n}\rho \le y) \to \Phi\left(\frac{y}{\sqrt{1-\frac{\rho^2}{2}}}\right).$$

Example 3: Theorem 1 is not applicable if (2.1) does not hold. For example, consider Mardia's bivariate Pareto distribution, with joint pdf (see Arnold et. al. [1], p.273),

$$f_{X,Y}(x,y) = \frac{\alpha(\alpha+1)}{\sigma_X \sigma_Y}\left(1 + \frac{x}{\sigma_X} + \frac{y}{\sigma_Y}\right)^{-(\alpha+2)}.$$

Then

$$R_{[n]} = c(R_n)(V-1), \quad \text{where} \quad c(x) = \sigma_Y(1+\frac{x}{\sigma_X}) \quad \text{and} \quad V \sim \ \text{Pareto}(\alpha+1, 1).$$

In this case

$$c(R_n) \stackrel{\mathcal{D}}{=} \sigma_X(\Pi_{j=0}^n U_j)^{-\frac{1}{\alpha}},$$

where $\alpha > 0$ and $\{U_j\}$ are i.i.d. with common distribution $U[0,1]$, and are independent of V. Therefore,

$$R_{[n]} \stackrel{\mathcal{D}}{=} \sigma_Y(\Pi_{j=0}^n U_j)^{-\frac{1}{\alpha}}(V-1).$$

Then

$$\log R_{[n]} \stackrel{\mathcal{D}}{=} K - \frac{1}{\alpha}\sum_{j=0}^{n} \log U_j + \log(V-1).$$

The above equation yields

$$\alpha\left[\frac{\log R_{[n]} - K + \frac{n}{\alpha}}{\sqrt{n}}\right] \stackrel{\mathcal{D}}{=} \frac{-\sum_{j=0}^n \log U_j - n}{\sqrt{n}} + \frac{\log(V-1)}{\sqrt{n}} \stackrel{\mathcal{D}}{\to} Z \sim N(0,1).$$

2.2 Concomitants of Pfeifer Records

There is one important case where it is possible to drop the assumption of identical distribution on $\{(X_i, Y_i)\}_{i\geq 1}$ in Theorem 1. This is the case of Pfeifer records, and is defined as follows (see Pfeifer [16]):

Consider a rectangular array of independent bivariate random variables $\{(X_{ij}, Y_{ij})\}$, $i = 1, 2, ...,$ $j = 0, 1, ...$ and $(X_{in}, Y_{in})_{i\geq 1}$ are i.i.d. random variables with common distribution $F_n(x, y)$. We also assume that the marginals $\{(F_n)_X\}_{n\geq 0}$ satisfy $1 - (F_n)_X(x) = (1 - (F_0)_X(x))^{\alpha_n}$, where $\{\alpha_n\}$ is a sequence of positive numbers. We define Pfeifer records of X as follows:
We define X_{01} as the 0th Pfeifer record R_0. After defining the nth record R_n, from the sequence $\{(X_{in}, Y_{in})\}_{i\geq 1}$, we move on to the $(n+1)$st row and consider the first $i \geq 1$ for which $X_{i(n+1)} > R_n$ and for that i, $X_{i(n+1)}$ is defined to be R_{n+1}, the $(n+1)$st Pfeifer record. In this set-up, we call the Y co-ordinate, corresponding to R_n, as concomitant of nth Pfeifer record and denote it by $R_{[n]}$.

It is easy to see that in this model,

$$\mathbb{P}(R_{[n]} \leq y | R_n = x) = (F_n)_{Y|X=x}(y).$$

The proof is similar to Lemma 1 in the Appendix. The following modification of Theorem 1 holds. The proof is similar to that of Theorem 1 and is omitted.

Theorem 2. *Suppose for some a_n and b_n, $\mathbb{P}\left(\frac{R_n - b_n}{a_n} \leq r\right) \to F_R(r)$ for some nondegenerate cdf F_R. If for some c_n and d_n and any fixed y, $(F_n)_{Y|X=a_n r+b_n}(c_n y+d_n) \to G(r, y)$ as $n \to \infty$, locally uniformly in r, where $G(r, y)$ continuous in r, then*

$$\mathbb{P}\left(\frac{R_{[n]} - d_n}{c_n} \leq y\right) \to \int_{-\infty}^{\infty} G(r, y) F_R(dr).$$

Example: Let $F_n(x, y)$ be the Ferlie-Gumbel-Morgenstern cdf

$$F_n(x, y) = (F_n)_X(x)(F_n)_Y(y)[1 + \theta(1 - (F_n)_X(x))(1 - (F_n)_Y(y))], \quad \text{where} \quad 0 < \theta < 1.$$

with exponential marginals $(F_n)_X(x)$ and $(F_n)_Y(y)$, given by

$$(F_n)_X = 1 - e^{\alpha_n x} \quad \text{and} \quad (F_n)_Y = 1 - e^{\beta_n x}$$

and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences of positive numbers, both increasing to ∞ in such a way that $\sum_{i=1}^{n}(1/\alpha_i^2)$ is divergent.

Then the nth conditional pdf

$$(f_n)_{Y|X=x}(y) = \beta_n e^{-\beta_n y}[1 + \theta(1 - 2e^{-\alpha_n x} - 2e^{-\beta_n y} + 4e^{-\alpha_n x - \beta_n y})], \; x, y > 0. \tag{2.6}$$

In this case, it is known that (see Arnold et. al. [1], (6.5.9), p. 201),

$$\frac{1}{a_n}(R_n - b_n) \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{where} \quad a_n = \left[\sum_{i=1}^{n}(1/\alpha_i^2)\right]^{1/2} \quad \text{and} \quad b_n = \sum_{i=1}^{n}(1/\alpha_i).$$

As before let $r_n := a_n r + b_n$, $-\infty < r < \infty$. Since

$$(F_n)_{Y|X=r_n}(c_n y + d_n) = \int_{-\infty}^{y_n} (f_n)_{Y|X=r_n}(u)du,$$

we have, using (2.6)

$$(F_n)_{Y|X=r_n}(y_n) = (1 - e^{-\beta_n y_n})[1 - \theta e^{-\beta_n y_n}(1 - 2e^{-\alpha_n r_n})].$$

If we let $c_n = \frac{1}{\beta_n}$ and $d_n = 0$, then for any $y > 0$, (as $\alpha_n r_n \to \infty$, uniformly over r, when r is in any compact interval)

$$(F_n)_{Y|X=r_n}(c_n y + d_n) \to (1 - e^{-y})(1 - \theta e^{-y}) \text{ locally uniformly in } r.$$

Therefore

$$G(r, y) = (1 - e^{-y})(1 - \theta e^{-y})$$

which is independent of r. Hence,

$$P(\beta_n R_{[n]} \leq y) \to (1 - e^{-y})(1 - \theta e^{-y}) =: G(y).$$

It is obvious that $G(\cdot)$ is a nondegenerate cdf.

It is interesting to note that the limiting distribution of normalized $R_{[n]}$ is independent of the limiting distribution of R_n. Indeed, when $\sum_{i=1}^{n}(1/\alpha_i^2) < \infty$, the limiting distribution of normalized R_n is discussed in Arnold et. al. [1], p. 201 and in that case too, $\beta_n R_{[n]}$ has the same limiting distribution as above.

3 Asymptotic behaviour of sums of concomitants

Limit distribution properties of sums of record values have been considered by several authors. See for example, Arnold et al. [2], Bose et al. [6], [7], [8], Iksanov [15]. We now investigate what happens to sums of comcomitants of record values. Let us first consider the very special case when $\{(X_i, Y_i)\}$ are i.i.d. bivariate normal with distribution $N(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho)$. Then

$$F_{R_{[i]}|R_i=r} = F_{Y_{L_i}|X_{L_i}=r} = F_{Y|X=r} \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(r - \mu_X), \sigma_Y^2(1 - \rho^2)\right).$$

Let

$$Z_i := R_{[i]} - (\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(R_i - \mu_X)).$$

If $\tilde{Z}_i := Y_i - (\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(R_i - \mu_X))$, then $\{\tilde{Z}_i\}_{i\geq 1}$ are i.i.d. normal $N(0, \sigma_Y^2(1 - \rho^2))$ and independent of $\{X_i\}_{i\geq 1}$. Therefore,

$$Z_i \equiv \tilde{Z}_{L_i} \sim N(0, \sigma_Y^2(1 - \rho^2))$$

and are mutually independent and are independent of $\{X_i\}_{i\geq 1}$.

In Bose et. al. [8], Theorem 2, it has been shown that

$$\frac{1}{\frac{n}{\sqrt{2}}}\sum_{i=1}^{n}\left(\frac{R_i-\mu_x}{\sigma_x}-\sqrt{2i}\right)\xrightarrow{\mathcal{D}} N(0,2/3).$$

Also by the CLT applied to the i.i.d. sequence $\{Z_i\}$, we have

$$\frac{1}{\sqrt{n}\sigma_Y\sqrt{1-\rho^2}}\sum_{i=1}^{n} Z_i \xrightarrow{\mathcal{D}} N(0,1).$$

Hence, $\frac{1}{\frac{n}{\sqrt{2}}\sigma_Y\rho}\sum_{i=1}^{n} Z_i \to 0$ in probability. Therefore,

$$\frac{1}{\frac{n}{\sqrt{2}}\rho\sigma_Y}\sum_{i=1}^{n}(R_{[i]}-\mu_Y-\sqrt{2i})\xrightarrow{\mathcal{D}} N(0,2/3).$$

3.1 Laplace transform of sums of concomitants

Bose et al. in [6] and [7] dealt with the asymptotic behaviour of sums of upper and lower records for a certain class of underlying distribution through studying their Laplace transforms. We now adapt that idea for sums of concomitants.

3.1.1 Sums of concomitants of lower records

In Bose et. al. [6] it is shown that under certain conditions on the underlying distribution F, the sums of lower records converge to an infinitely divisible random variable. We now provide a suitable extension to sums of concomitants.

Assumption III $F(x,y)$ is absolutely continuous and its support is a subset of $R_+ \times R_+$, i.e. $X>0$ and $Y>0$ and $F_X(0)=0$.

For convenience we denote the concomitants corresponding to the ith lower record by the same notation $R_{[i]}$.

Let

$$\xi_t(x) := \mathbb{E}\left[e^{-t\sum_{n=0}^{\infty} R_{[n]}}|R_0 \le x\right],\ \xi_t^n(x) := \mathbb{E}\left[e^{-t\sum_{i=0}^{n} R_{[i]}}|R_0 \le x\right]$$

and denote the conditional Laplace transform of Y given X by

$$L(x,t) := \int_0^{\infty} e^{-ty} f_{Y|X=x}(y)dy.$$

Theorem 3. *Suppose Assumption III holds. Further*

(i) for any fixed $t>0$, $\int_0^{\infty}\frac{1-L(x,t)}{F_X(x)}F_X(dx)<\infty$,

(ii) for any fixed $t > 0$, $L(x,t)$ is monotone in x in some neighbourhood of $x = 0$ and

(iii) $\frac{1-L(x,t)}{F_X(x)} = O(l(x))$ as $t \to 0+$, where $l(x)$ is integrable with respect to $F_X(dx)$.

Then $\sum_{i=0}^{n} R_{[i]}$ converges almost surely to a proper random variable with Laplace transform

$$L(t) := \mathbb{E}\left[e^{-t\sum_{i=0}^{\infty} R_{[i]}}\right] = \exp\left(-\int_0^\infty (1 - L(x,t))\frac{f_X(x)}{F_X(x)}dx\right).$$

Moreover, this limit is infinitely divisible.

Proof: Observe that

$$\begin{aligned}
\xi_t^n(x) :&= \mathbb{E}(e^{-t\sum_{i=0}^{n} R_{[i]}}|R_0 \leq x) \\
&= \int_0^\infty \int_0^x \mathbb{E}(e^{-t\sum_{i=0}^{n} R_{[i]}}|R_0 = u, R_{[0]} = v) f_{Y|X=u}(v)\frac{f_X(u)}{F_X(x)}dudv \\
&= \int_0^\infty \int_0^x e^{-tv}\mathbb{E}(e^{-t\sum_{i=1}^{n} R_{[i]}}|R_1 \leq u) f_{Y|X=u}(v)\frac{f_X(u)}{F_X(x)}dudv \\
&= \int_0^x \mathbb{E}(e^{-t\sum_{i=1}^{n} R_{[i]}}|R_1 \leq u)\frac{f_X(u)}{F_X(x)}\left(\int_0^\infty e^{-tv} f_{Y|X=u}(v)dv\right)du \\
&= \frac{1}{F_X(x)}\int_0^x \xi_t^{n-1}(u)L(u,t)f_X(u)du. \qquad (3.1)
\end{aligned}$$

(For the third equality we use Lemma 2 in Appendix). So (3.1) gives us an integral relation amongst $\{\xi_t^n\}_{n\geq 0}$. Since $R_{[i]}$ are all nonnegative, for any fixed $x > 0$, $\xi_t^n(x)$ is decreasing in n and for all n, $0 < \xi_t^n(x) < 1$. Therefore, $\lim_{n\to\infty}\xi_t^n(x)$ exists and lies between $[0,1]$ and by conditional DCT, $\lim_{n\to\infty}\xi_t^n(x) = \xi_t(x)$. Also from (3.1) and an application of DCT, it is evident that $\xi_t(\cdot)$ satisfies the integral equation

$$\xi_t(x) = \frac{1}{F_X(x)}\int_0^x \xi_t(u)L(u,t)F_X(du). \qquad (3.2)$$

We will now show that

(a) $\xi_t(\infty) := \lim_{x\to\infty}\xi_t(x)$ exists and

(b) $L(t) = \xi_t(\infty)$.

(a) follows easily by bounded convergence theorem. Indeed, in (3.2),

$$|\xi_t(u)L(u,t)I_{[0,x]}| \leq 1.$$

So by bounded convergence theorem,

$$\lim_{x\to\infty}\xi_t(x) = \int_0^\infty \xi_t(u)L(u,t)F_X(du).$$

The proof of (b) is similar to that given in Bose et. al. [6]. For the sake of completeness we give it here.

$$\begin{aligned}
&|\mathbb{E}(e^{-t\sum_{i=0}^{n}R_{[i]}}) - \xi_t(\infty)| \\
&\leq |\mathbb{E}(e^{-t\sum_{i=0}^{n}R_{[i]}}I_{(R_0\leq x)}) - \xi_t(x)| + |\mathbb{E}(e^{-t\sum_{i=0}^{n}R_{[i]}}I_{(R_0>x)})| + |\xi_t(x) - \xi_t(\infty)| \\
&\leq |F_X(x)\xi_t^n(x) - \xi_t(x)| + |\mathbb{P}(R_0 > x)| + |\xi_t(x) - \xi_t(\infty)| \\
&\leq F_X(x)|\xi_t^n(x) - \xi_t(x)| + \xi_t(x)|1 - F_X(x)| + \mathbb{P}(R_0 > x) + |\xi_t(x) - \xi_t(\infty)| \qquad (3.3)
\end{aligned}$$

(in the second inequality we use the fact that

$$F_X(x)\xi_t^n(x) = \mathbb{E}[\mathbb{E}(e^{-t\sum_{i=0}^{n}R_{[i]}}|R_0)I_{(R_0\leq x)})].$$

Clearly, (3.3) can be made arbitrarily small by taking n and x sufficiently large and hence (b) is proved. Therefore, $\xi_t(\infty)$ is $L(t)$, the Laplace transform of the sum of concomitants.

By condition (ii), using induction on n one can see that $\xi_t^{(n)}(x)$ is monotone in a neighbourhood of $(x = 0)$, for all n which implies $\xi_t(x)$ is also so. So $\lim_{x\to 0+}\xi_t(x)$ exists. We define $\xi_t(0) = \lim_{x\to 0+}\xi_t(x)$, so that $\xi_t(x)$ is continuous on $[0,\infty)$. Now consider the following integral equation for the function $h(\cdot)$:

$$h(x)F_X(x) = \int_0^x h(u)L(u,t)f_X(u)du. \qquad (3.4)$$

We have already seen that $\xi_t(\cdot)$ satisfies (3.4) and also it is a bounded continuous solution of (3.4). So we will consider only bounded continuous solutions which takes values in $[0,1]$. If $h(\cdot)$ is such a solution, we claim that $\xi_t(x) \geq h(x)$. The proof is similar to that given in Bose et. al. [6]:

$$\begin{aligned}
\xi_t^0(x) &= \mathbb{E}(e^{-tR_{[0]}}|R_0 \leq x) \\
&= \mathbb{E}(e^{-tY}|X \leq x) \\
&= \int_0^x L(u,t)\frac{f_X(u)}{F_X(u)}du \\
&\geq \int_0^x L(u,t)h(u)\frac{f_X(u)}{F_X(u)}du = h(x). \qquad (3.5)
\end{aligned}$$

Then using induction on n, it is easy to see that $\xi_t^n(x) \geq h(x)$ for all n and by taking limit as $n \to \infty$ we get $\xi_t(x) \geq h(x)$.

Define the function

$$g(x) := \exp\left(-\int_0^x \frac{(1 - L(u,t))F_X(du)}{F_X(u)}\right).$$

Now we will show that if $\eta(\cdot)$ is a solution of (3.4), then $\eta(\cdot)$ is a constant multiple of g. Since $\eta(\cdot)$ satisfies (3.4),

$$\eta(x)F_X(x) = \int_0^x \eta(u)L(u,t)f_X(u)du.$$

Since we consider only those solutions $\eta(\cdot)$, which are continuous on $[0, \infty)$, we can use fundamental theorem of calculus to have

$$\eta'(x)F_X(x) + \eta(x)f_X(x) = \eta(x)L(x,t)f_X(x).$$

Dividing both sides by $\eta(x)F_X(x)$ for $x > 0$, we have the following differential equation:

$$\frac{\eta'(x)}{\eta(x)} = (L(x,t) - 1)\frac{f_X(x)}{F_X(x)}. \tag{3.6}$$

Integrating both sides we get,

$$\eta(x) = C\exp\left(-\int_0^x (1 - L(u,t))\frac{f_X(u)}{F_X(u)}du\right) \tag{3.7}$$

where C is a constant lying in $[0,1]$. This shows that the solution $\eta(\cdot)$ is a constant multiple of $g(\cdot)$. Since $\xi_t(\cdot)$ is a solution greater than or equal to any other solution, $C = 1$ for $\xi_t(\cdot)$. Hence,

$$\xi_t(x) = \exp\left(-\int_0^x (1 - L(u,t))\frac{f_X(u)}{F_X(u)}du\right).$$

Taking limit as $x \to \infty$ we have

$$L(t) \equiv \xi_t(\infty) = \exp\left(-\int_0^\infty (1 - L(u,t))\frac{f_X(u)}{F_X(u)}du\right).$$

To see that it is a Laplace transform of a proper random variable it is enough to see that $L(t)$ is a completely monotone function of t, i.e. $(-1)^n \frac{d^n}{dt^n}L(t) \geq 0$ for al $n \geq 0$, (see Bondesson [5]). For any x in a neighbourhood of zero, the Taylor series expansion in t of $L(x,t)$ gives

$$1 - L(x,t) = -t\frac{d}{dt}L(x,t) + \frac{t^2}{2!}\frac{d^2}{dt^2}L(x,t) - \frac{t^3}{3!}\frac{d^3}{dt^3}L(x,t) + \cdots \tag{3.8}$$

Since $L(x,t)$ is a Laplace transform, it is a completely monotone function of t. Hence all the terms of the right hand side of (3.8) are nonnegative. Therefore, for all $n \geq 1$,

$$\left|\frac{\frac{t^n}{n!}\frac{d^n}{dt^n}L(x,t)}{F_X(x)}\right| \leq \frac{1 - L(x,t)}{F_X(x)},$$

which is integrable with respect to $F_X(dx)$ by condition (i). So differentiation with respect to t under the sign of integration is permissible. Performing this differentiation gives

$$(-1)\frac{d}{dt}L(t) = L(t)\int_0^\infty (-1)(\frac{d}{dt}L(u,t))\frac{f_X(u)}{F_X(u)}du \geq 0.$$

Using Leibnitz's chain rule and the fact that $L(x,t)$ is completely monotone, it is easy to see that $L(t)$ is completely monotone.

Now we invoke condition (iii). Let $h(t) := -\log L(t)$. Since the given hypothesis allows us to use DCT as $t \to 0+$, we have $\lim_{t\to 0+} h(t) = 0$. Therefore, $L(t) = e^{-h(t)}$ where $(d/dt)h(t)$

is a completely monotone function and $\lim_{t\to 0+} h(t) = 0$. So $L(t)$ is a Laplace transform of an infinitely divisible random variable, (see Feller [13], p.425) and that completes the proof of the Theorem. $\square$

Example 5: Let, for all $u > 0$,

$$f_{Y|X=u}(y) = \frac{1}{u} e^{-\frac{y}{u}}, \quad y > 0$$

and $F_X(u)$ be uniform cdf over $[0,1]$. Then $L(u,t) = \frac{1}{ut+1}$. Conditions (i), (ii) and (iii) of Theorem 3 are satisfied. So the sums of the lower concomitants converge to a random variable whose Laplace transform is $L(t) = (t+1)^{-1}$, i.e. the limiting distribution is Exp(1).

3.1.2 Concomitants of upper records

Next we consider the asymptotic behaviour of sums of concomitants of upper records. Here we assume $F(x,y)$ is absolutely continuous and the support of F_Y is a subset of R_+. For the sake of convenience, we assume that the support of F_X also is a subset of R_+, though it is not necessary. As before, define

$$L(x,t) = \mathbb{E}\left[e^{-tY} | X = x\right] \quad \text{and} \quad \zeta_t(x) := \mathbb{E}\left[e^{-t\sum_{i=0}^{\infty} R_{[i]}} | R_0 > x\right].$$

The proof of the following theorem is essentially similar to that in the lower records case and hence is omitted.

Theorem 4. *Suppose for any fixed $t > 0$,*

(a) $\int_0^\infty \frac{1-L(x,t)}{1-F_X(x)} F_X(dx) < \infty$ *and*

(b) $\frac{1-L(x,t)}{1-F_X(x)} = O(l(x))$ *as $t \to 0+$, where $l(x)$ is integrable with respect to $F_X(dx)$.*

Then $\sum_{i=0}^{\infty} R_{[i]}$ converges to a proper random variable whose Laplace transform is

$$L(t) = \exp\left(-\int_0^\infty \frac{1 - L(x,t)}{1 - F_X(x)} F_X(dx)\right), \quad t > 0.$$

In this case also, the limit is an infinitely divisible random variable.

Remark. Observe that condition (ii) in Theorem 3 is used only to prove that $\lim_{x\to 0+} \xi_t(x)$ exists, so that $\xi_t(x)$ can be made continuous on $[0,\infty)$ by removing the removable discontinuity at $x = 0$. Hence its analogue is not needed in upper records case, as $\zeta_t(x)$ has no discontinuity on $[0,\infty)$. Therefore, we do not need any extra condition near $x = 0$.

Observe that condition (b) in Theorem 4 is required only to ensure that $\lim_{t\to 0+}(-\log L(t)) = 0$. However, in the following example we can verify it directly instead of verifying condition (b).

Example 6: Let, for all $y > 0$ and $x > 0$,

$$f_{Y|X=x}(y) = xe^{-xy} \quad \text{and} \quad F_X(x) = 1 - \frac{\alpha}{x+\alpha}, \ x > 0,$$

for some constant $\alpha > 0$. Then condition (a) of the Theorem is satisfied. Computing directly $\int_0^\infty \frac{1-L(x,t)}{1-F_X(x)} F_X(dx)$ we have, for $t > 0$,

$$\begin{aligned} L(t) &= \left(\frac{\alpha}{t}\right)^{-\frac{t}{\alpha-t}}, && \text{for } t \neq \alpha \\ &= e^{-1}, && \text{for } t = \alpha. \end{aligned} \tag{3.9}$$

It is easy to see by direct computation that $\lim_{t\to 0+} \log L(t) = 0$ and $\frac{d}{dt}(-\log L(t))$ is completely monotone. Therefore, $\sum_{i=1}^n R_{[i]}$ converges to an infinitely divisible random variable whose Laplace transform is given by (3.9).

Remark. Iksanov [15], extended the results of Bose et al. [6], [7] to arbitrary distributions with simplified proofs using the idea of shot noise distribution. It would be interesting to see if his method can be applied to derive the limiting distribution of sums of concomitants.

4 Appendix

Lemma 1. *Let $\{(X_i, Y_i)\}_{i\geq 1}$ be a sequence of i.i.d. bivariate random variables. Let R_k and $R_{[k]}$ be kth record and its concomitant respectively. Then for any measurable function g, $\mathbb{E}(g(Y)|X) = \mathbb{E}(g(R_{[k]})|R_k)$, for all $k \geq 1$.*

Proof.

$$\begin{aligned} \mathbb{E}(g(R_{[k]})|R_k) &= \sum_{l=k}^{\infty} \mathbb{E}(g(R_{[k]}) I_{L_k=l}|R_k) \\ &= \sum_{l=k}^{\infty} \mathbb{E}(g(R_{[k]})|I_{L_k=l}, R_k)\mathbb{P}(L_k = l|R_k) = \sum_{l=k}^{\infty} \mathbb{E}(g(Y_l)|X_l, \mathcal{A})\mathbb{P}(L_k = l|R_k) \end{aligned} \tag{4.1}$$

where $\mathcal{A} \subset \sigma\{X_1, X_2, \dots X_{l-1}\}$. Since both X_l, Y_l are independent of $\mathcal{A}$, the last quantity in (4.1) is equal to

$$\sum_{l=k}^{\infty} \mathbb{E}(g(Y_l)|X_l)\mathbb{P}(L_k = l|R_k) = \mathbb{E}(g(Y)|X).$$

□

Lemma 2. *The set up is as in Lemma 1. Let $j \leq k_1 < k_2$. Then for any measurable functions g and h,*

$$\mathbb{E}\left(g(R_{[k_1]})h(R_{[k_2]})|R_i, R_{[i]}, i < j, R_{k_1}\right) = \mathbb{E}\left(g(R_{[k_1]})h(R_{[k_2]})|R_{k_1}\right).$$

Proof: Using the properties of conditional expectation we have,

$$\begin{aligned}
&\mathbb{E}\left(g(R_{[k_1]})h(R_{[k_2]})|R_i, R_{[i]}, i<j, R_{k_1}\right)\\
&= \sum_{j\le l_1<l_2<\infty} \mathbb{E}\left(g(R_{[k_1]})h(R_{[k_2]})I_{L_{k_1}=l_1}, I_{L_{k_2}=l_2}|R_i, R_{[i]}, i<j, R_{k_1}\right)\\
&= \sum_{j\le l_1<l_2<\infty} [\mathbb{E}\left(g(R_{[k_1]})h(R_{[k_2]})|\mathcal{A}\right)\mathbb{P}(L_{k_1}=l_1, L_{k_2}=l_2|R_i, R_{[i]}, i<j, R_{k_1})]\\
&= \sum_{j\le l_1<l_2<\infty} \mathbb{E}\left(g(Y_{l_1})h(Y_{l_2})|\mathcal{A}\right)\mathbb{P}(L_{k_1}=l_1, L_{k_2}=l_2|R_i, R_{[i]}, i<j, R_{k_1})\\
&= \sum_{j\le l_1<l_2<\infty} [\int_{x_2>R_{[k_1]}} \mathbb{E}[\mathbb{E}\left(g(Y_{l_1})h(Y_{l_2})|X_{l_1}, X_{l_2}, \mathcal{A}\right)|X_{l_2}, \mathcal{A}]f_{X_{l_2}|\mathcal{A}}(x_2)dx_2\\
&\quad\times \mathbb{P}(L_{k_1}=l_1, L_{k_2}=l_2|R_i, R_{[i]}, i<j, R_{k_1})]\\
&= \sum_{j\le l_1<l_2<\infty} [\int_{x_2>R_{[k_1]}} \mathbb{E}[\mathbb{E}(g(Y_{l_1})|X_{l_1})\mathbb{E}(h(Y_{l_2})|X_{l_2})|X_{L_2}, \mathcal{A}](x_2)f_{R_{k_2}|\mathcal{A}}(x_2)dx_2\\
&\quad\times \mathbb{P}(L_{k_1}=l_1, L_{k_2}=l_2|R_i, R_{[i]}, i<j, R_{k_1})] \qquad (4.2)
\end{aligned}$$

where $\mathcal{A}=\sigma\{I_{L_{k_1}=l_1}, I_{L_{k_2}=l_2}, R_i, R_{[i]}, i<j, R_{k_1}\}$. The last equality holds because Y_{l_i} depends only on X_{l_i}, $i=1,2$.

As $\{(X_n, Y_n)\}_{n\ge 1}$ are i.i.d. $\mathbb{E}(g(Y_{l_1})|X_{l_1})$ is independent of l_1 and is equal to $\mathbb{E}(R_{[k_1]}|R_{k_1})$ by Lemma 1 (the same is true for the conditional expectation of $h(Y)$). Also since the records of X form a Markov sequence and are independent of record times $\{I_{\{L_j=l_j\}}\}$, the last sum of (4.2) is equal to

$$\begin{aligned}
\mathbb{E}(g(R_{[k_1]})|R_{k_1}) &\int_{x_2>R_{k_1}} \mathbb{E}(h(R_{[k_2]})|R_{k_2})(x_2)f_{R_{k_2}|\mathcal{A}}(x_2)dx_2\\
&\times \sum_{j\le l_1<l_2<\infty} \mathbb{P}(L_{k_1}=l_1, L_{k_2}=l_2|R_i, R_{[i]}, i<j, R_{k_1})\\
&= \mathbb{E}(g(R_{[k_1]})|R_{k_1})\mathbb{E}(h(R_{[k_2]})|R_{k_1}). \qquad (4.3)
\end{aligned}$$

By the same argument as above we can show that

$$\mathbb{E}(g(R_{[k_1]})|R_{k_1})\mathbb{E}(h(R_{[k_2]})|R_{k_1}) = \mathbb{E}[g(R_{[k_1]})h(R_{[k_2]})|R_{k_1}],$$

and the result will follow. □

Remark. Instead of two indices k_1 and k_2 Lemma 2, can be extended to any finite number of indices.

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