Asymmetric Treatment of Identical Agents in Teams

by

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Abstract

We investigate when identical agents will be treated asymmetrically in a simple team setting. Asymmetric treatment is optimal when the agents' individual contributions to team performance are complements. Symmetric treatment of identical agents is optimal when the agents' contributions are substitutes or when they are independent.

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1 Introduction.

Studies have found that individuals often are averse to inequitable treatment (e.g., Prasnikar and Roth, 1992; Charness and Rabin, 2002) and that the properties of optimal reward structures can change substantially when agents are averse to inequity (e.g., Fehr and Schmidt, 1999; Demougin et al., 2006; Desiraju and Sappington, 2007). In light of these findings, it is important to understand when identical economic agents are likely to be treated asymmetrically in relevant economic circumstances, absent explicit attempts to limit inequity.

The present research examines this issue in a simple team setting where the owner of a project ("the principal") hires two identical workers ("agents") to operate the project. To illustrate, the principal might be the owner of a research and development (R&D) enterprise (e.g., a pharmaceutical company) and the agents might be scientists hired to undertake a specific R&D project (e.g., develop a new drug). The contributions that the agents make to project (e.g., the scientists' creative research efforts) may be independent, or they may be complements or substitutes.[1] In addition, these contributions can be delivered either simultaneously or sequentially.[2] In all cases, the agents' contributions are not verifiable. Consequently, a team moral hazard problem arises. The principal induces the agents to contribute to the project by linking their compensation to the observed performance of the project (e.g., whether the scientists successfully develop an effective new drug). We investigate when, if ever, the principal will implement asymmetric reward structures for the two identical agents.

We find that the principal will implement identical reward structures for the two risk-neutral agents when their contributions to the project are independent or when their contri-

[1] The contributions may be complements, for example, when a discovery by one scientist provides useful information that increases the productivity of the other scientist's research efforts. The contributions may be substitutes when initial progress by one scientist reduces the chances that the other scientist will be able to provide useful incremental progress.

[2] For example, the scientists might be instructed to conduct their individual experiments simultaneously or sequentially.

butions are substitutes. In contrast, the principal will present the two identical agents with different reward structures when their contributions are complements.

The finding that the principal optimally implements identical reward structures when the contributions of the identical agents are independent is not surprising. The actions of one agent do not influence the relevant incentive problem that the principal faces in motivating the other agent when the agents' contributions are independent. Therefore, the principal optimally presents the two identical agents with the same reward structure. This is the case whether the agents deliver their contributions simultaneously or sequentially.

When the agents' contributions are complements, the principal optimally directs one agent (the first mover) to deliver his contribution before the other agent (the second mover) makes his contribution. This sequential arrangement motivates the first mover to increase his contribution in order to increase the productivity of the second mover's contribution and thereby induce the second mover to increase his contribution to the project. When the first mover faces this natural incentive to increase his contribution to the project, the principal can reduce the compensation that she delivers to the first mover without unduly reducing his contribution to the project. Therefore, the principal optimally implements a less generous reward structure for the (identical) first mover.

When the agents' contributions are substitutes, a first mover would be reluctant to deliver a large contribution because it would reduce the productivity of the second mover's contribution and thereby induce the second mover to reduce his contribution. To avoid this disincentive for the first mover, the principal directs the two agents to deliver their contributions simultaneously. When the agents act simultaneously, the principal effectively faces the same incentive problem in motivating each of the identical agents and so implements the same reward structure for the two agents.

Our finding that identical agents may optimally be treated asymmetrically is not unique. Hermalin (1994), for example, observes that identical Cournot competitors may implement distinct intra-firm reward structures in equilibrium.[3] Similarly, our finding that a principal may gain by directing the activities of agents to proceed sequentially rather than simultaneously (or vice versa) is not novel. This is an important theme, for example, in the literature on the financing of public goods (e.g., Varian, 1994; Andreoni, 2006; Yildirim, 2006).[4] Our primary contribution to the literature is the simple, systematic link that we identify between the production technology in a team and the optimal structuring of rewards and timing of actions within the team. Our findings also suggest when aversion to inequity is particularly likely to pose problems for the optimal design of reward structures within teams.

The analysis proceeds as follows. Section 2 reviews the key elements of our model. Section 3 presents our main findings. Section 4 concludes and suggests directions for future research. The proofs of all formal conclusions are presented in the Appendix.

2 The Model.

A principal hires two identical agents (A and B) to operate a project. The project ultimately succeeds or fails. Success generates value $V_S > 0$. Failure generates the smaller value V_F. The probability that the project succeeds is $p = p^A + p^B + \gamma\, p^A p^B \in [0, 1]$, where $p^i \geq 0$ denotes the contribution (e.g., the effort) delivered by agent $i \in \{A, B\}$. γ is a parameter that reflects the interaction between the contributions of the two agents. When $\gamma = 0$, these contributions are independent, and the probability of project success is simply the sum of the contributions of the two agents. When $\gamma > 0$, the agents' contributions are complements because the marginal impact of one agent's contribution on the probability of project success increases as the contribution of the other agent increases (i.e., $\frac{\partial^2 p}{\partial p^A \partial p^B} = \gamma > 0$). In contrast, the agents' contributions are substitutes when $\gamma < 0$. In this case, the

[3] The distinct reward structures in Hermalin's (1994) model reflect the different output levels that the firms produce in equilibrium.

[4] The analyses in this literature differ from our analysis in many respects. For example, the analyses typically do not consider the optimal design of reward structures, which is the focus of our analysis. Furthermore, moral hazard problems generally are not central in these analyses. The analyses typically assume that the individual (monetary) contributions of "agents" are readily observed or can be revealed costlessly when such revelation is advantageous.

marginal productivity of an agent's contribution declines as the contribution of the other agent increases.

The ultimate success or failure of the project is observable and verifiable. However, the agents' contributions are not verifiable. Therefore, payments from the principal to the agents vary only according to whether the project succeeds or fails.[5] T_S^i will denote the payment from the principal to agent $i \in \{A, B\}$ when the project succeeds. T_F^i will denote the corresponding payment when the project fails. All payments must be non-negative because the agents have no wealth.

The personal cost that each agent incurs is an increasing, convex function of his contribution to project success. To facilitate closed-form solutions to the principal's problem and explicit comparisons of optimal outcomes in distinct institutional settings,[6] we assume that agent i incurs personal cost $\frac{k}{\theta}(p^i)^\theta$ when he delivers contribution p^i, where $k > 0$ and $\theta \geq 2$.[7] Each agent chooses his contribution to maximize his expected profit, which is the difference between his expected payment from the principal and the personal cost he incurs. An agent will work on the project if and only if he anticipates non-negative profit from doing so.[8]

The risk-neutral principal acts to maximize her expected profit, which is the difference between her expected return from the project and her expected payments to the agents. In addition to specifying the payments that she will deliver to the agents, the principal can determine whether the agents will deliver their contributions simultaneously or sequentially. Without loss of generality, we assume that when the agents act sequentially, agent A delivers

[5] Verifiable communication between the principal and the agents is assumed to be prohibitively costly.

[6] To determine the optimal sequencing of the agents' contributions to project success, it is necessary to compare the principal's maximum expected profit when the agents deliver their contributions simultaneously with the corresponding profit when the agents deliver their contributions sequentially. This exercise requires both a complete characterization of the optimal reward structures and an analytic ranking of the principal's maximum expected profit in the two distinct settings. As the proof of Proposition 2 reveals, this exercise is challenging even when the identified functional forms for the agents' cost structure and the probability of project success are adopted.

[7] The assumption $\theta \geq 2$ ensures that each agent's cost function is sufficiently convex that the principal optimally induces a strictly positive contribution from both agents even when their contributions are substitutes. We restrict attention to values of k that are sufficiently large that p^A, p^B, and p are all less than 1 in equilibrium.

[8] The agents' preferences and capabilities are common knowledge.

his contribution before agent B delivers his contribution.

In the case of sequential contributions, the principal's problem, [P-SQ], is the following:

$$\underset{\{T_S^i,\,T_F^i\}}{Maximize}\;\left[p^A+p^B+\gamma\,p^Ap^B\right]\left[V_S-T_S^A-T_S^B\right]+\left[1-p^A-p^B-\gamma\,p^Ap^B\right]\left[V_F-T_F^A-T_F^B\right] \tag{1}$$

subject to, for $i \in \{A, B\}$:

$$\left[p^A+p^B+\gamma\,p^Ap^B\right]T_S^i+\left[1-p^A-p^B-\gamma\,p^Ap^B\right]T_F^i-\frac{k}{\theta}(p^i)^\theta\;\geq\;0\,; \tag{2}$$

$$p^B\left(p^A\right)\;=\;\underset{0\leq\,\widetilde{p}^B\,\leq 1}{\arg\max}\left\{\left[p^A+\widetilde{p}^B+\gamma\,p^A\widetilde{p}^B\right]T_S^B+[1-p^A-\widetilde{p}^B-\gamma\,p^A\widetilde{p}^B]T_F^B-\frac{k}{\theta}(\widetilde{p}^B)^\theta\right\}; \tag{3}$$

$$\begin{aligned} p^A \;=\; &\underset{0\leq\,\widetilde{p}^A\,\leq 1}{\arg\max}\,\{\left[\widetilde{p}^A+p^B\left(\widetilde{p}^A\right)+\gamma\,\widetilde{p}^Ap^B\left(\widetilde{p}^A\right)\right]T_S^A \\ &+\;\left[1-\widetilde{p}^A-p^B\left(\widetilde{p}^A\right)-\gamma\,\widetilde{p}^Ap^B\left(\widetilde{p}^A\right)\right]T_F^A-\frac{k}{\theta}(\widetilde{p}^A)^\theta\}\,; \end{aligned} \tag{4}$$

$$T_S^i\;\geq\;0\;;\qquad T_F^i\;\geq\;0\;; \tag{5}$$

$$p^i\;\in\;[0,1]\;;\quad\text{and}\quad p^A+p^B+\gamma\,p^Ap^B\;\in\;[0,1]. \tag{6}$$

Expression (1) reflects the principal's objective to maximize her expected profit. Inequality (2) ensures that each agent anticipates non-negative profit. Expression (3) states that agent B will choose his contribution (p^B) to maximize his expected profit, taking as given the contribution delivered by agent A. Expression (4) indicates that agent A will choose his contribution to maximize his expected profit, anticipating the impact of his choice of p^A on agent B's choice of p^B. Inequality (5) restricts all payments to be non-negative. Inequality (6) ensures that all relevant probabilities are well defined.

The principal's problem when the agents choose their contributions simultaneously, [P-S], is analogous to [P-SQ]. The only difference is that agent A takes agent B's contribution (p^B) as given when he chooses p^A. Therefore, expression (4) becomes:

$$p^A = \underset{0 \leq \widetilde{p}^A \leq 1}{\arg\max} \left\{ \left[\widetilde{p}^A + p^B + \gamma\, \widetilde{p}^A p^B\right] T_S^A + \left[1 - \widetilde{p}^A - p^B - \gamma\, \widetilde{p}^A p^B\right] T_F^A - \frac{k}{\theta} (\widetilde{p}^A)^\theta \right\}. \quad (7)$$

In the ensuing discussion, we will denote by Π^S the value of the principal's objective function (expression (1)) at the solution to [P-S]. Π^{SQ} will denote the corresponding value at the solution to [P-SQ].

3 Findings.

Lemma 1 reports that the principal will always deliver the minimum possible payment to both agents whenever the project fails. By implementing the smallest possible payment for failure, the principal limits the agents' rents while providing them with the strongest feasible incentives to increase the probability of project success.

Lemma 1. $T_F^A = T_F^B = 0$ *at the solution to [P-S] and at the solution to [P-SQ].*

Lemma 1 implies that if the identical agents ever face asymmetric reward structures in the present setting, the asymmetry will pertain to the payments that the agents receive when the project succeeds.

Proposition 1 reports that the two agents will face identical reward structures when their contributions to project success are independent (so $\gamma = 0$). In this case, the problem that the principal faces in motivating each agent is the same regardless of the contribution delivered by the other agent. Consequently, the principal is effectively designing reward structures for two identical, independent agents. The optimal such reward structures are identical for the two agents. This is the case whether the agents deliver their contributions simultaneously or sequentially. Proposition 1 refers to π^i, which is the expected profit of agent $i \in \{A, B\}$.

Proposition 1. *Suppose* $\gamma = 0$. *Then* $\Pi^S = \Pi^{SQ}$, *so the principal's expected profit is the same whether the agents deliver their contributions simultaneously or sequentially. In either case, the principal implements the same reward structure for the two agents (*$T_S^A =$

T_S^B). *Furthermore, the agents deliver the same contribution to project success* ($p^A = p^B$) *and secure the same expected profit* ($\pi^A = \pi^B$).

Distinct considerations arise when the agents' contributions to project success are not independent. In this case, the contribution delivered by one agent affects the impact of the other agent's contribution on the probability of project success. Consequently, externalities between the agents' contributions arise. These externalities influence both the details of the optimal reward structures and the principal's preference regarding the timing of contributions.

When the agents' contributions are complements (so $\gamma > 0$), beneficial externalities between the agents' contributions arise. In particular, an increase in one agent's contribution increases the rate at which the project success probability rises as the contribution of the other agent increases. Therefore, when the agents deliver their contributions sequentially, agent A (the first mover) recognizes that an increase in p^A will increase the marginal productivity of p^B (i.e., $\frac{\partial}{\partial p^A}\left(\frac{\partial p}{\partial p^B}\right) = \gamma > 0$) and thereby motivate agent B to increase p^B. Agent A benefits from the resulting increase in p^B because the associated increase in the probability of project success increases agent A's expected payment from the principal.

Consequently, agent A anticipates both a direct and an indirect benefit from increasing p^A when $\gamma > 0$ and contributions are delivered sequentially. The direct benefit stems from the increase in the project success probability (p) that arises as p^A increases, holding p^B constant. The increase in p increases agent A's expected payment from the principal ($p\,T_S^A$). The indirect benefit arises from the induced increase in p^B, which increases p and thereby increases $p\,T_S^A$.

The principal is able to capture some of the indirect benefit that agent A derives from increasing p^A when $\gamma > 0$ and the agents deliver their contributions sequentially. She does so by reducing the payment that she makes to agent A when the project succeeds. The smaller payment to agent A increases the principal's expected profit as long as the induced reduction in the probability of project success is not too pronounced. The induced reduction in p is limited because, despite the reduced payment, agent A continues to deliver a relatively large contribution to project success. She does so in order to induce agent B to make a corresponding substantial contribution to project success. Therefore, the principal optimally directs the agents to deliver their contributions sequentially, as Proposition 2 reports.

Proposition 2. *Suppose $\gamma > 0$. Then $\Pi^{SQ} > \Pi^{S}$, so the principal directs the agents to deliver their contributions sequentially. The principal pays agent A (the first mover) less than she pays agent B (the second mover) when the project succeeds ($T_S^A < T_S^B$). Agent A secures less profit and delivers a higher contribution to project success than agent B ($\pi^A < \pi^B$ and $p^A > p^B$).*

Proposition 2 implies that agent A experiences a first-mover disadvantage in equilibrium when the agents' contributions are complements. The principal substitutes the indirect benefit that agent A anticipates from increasing p^A for some of the payment for success that the principal would otherwise deliver to agent A. The diminished payment reduces agent A's expected profit below the level achieved by agent B.[9]

While the sequential delivery of contributions allows the principal to capture some of the surplus derived from the beneficial externalities that arise when the agents' contributions are complements, the simultaneous delivery of contribution limits the incidence of and insulates the principal from the detrimental externalities that arise when the agents' contributions are substitutes (so $\gamma < 0$). When $\gamma < 0$, an increase in p^A reduces the rate at which the project success probability (p) increases as p^B increases (i.e., $\frac{\partial}{\partial p^A}\left(\frac{\partial p}{\partial p^B}\right) = \gamma < 0$). The reduced impact of p^B on p reduces the contribution that agent B will deliver. Consequently, when the agents deliver their contributions sequentially, agent A anticipates an indirect loss from increasing p^A. The indirect loss is the induced reduction in p^B, which reduces the probability of project success and thus agent A's expected payment from the principal. This indirect

[9] Although agent A receives a lower expected profit than agent B when $\gamma > 0$, numerical solutions indicate that agent A's expected profit typically is higher at the solution to [P-SQ] than at the solution to [P-S] when $\gamma > 0$. Thus, the beneficial externalities that arise when $\gamma > 0$ often provide Pareto gains.

loss discourages agent A from increasing p^A.

The principal can eliminate this indirect loss by directing the agents to deliver their contributions simultaneously when $\gamma < 0$. Under simultaneous delivery, agent A takes p^B as given when he determines his contribution to project success. In particular, agent A does not expect agent B to reduce p^B in response to an increase in p^A, and so agent A anticipates no indirect loss from increasing p^A. In the absence of this indirect loss, the principal can motivate agent A to deliver any desired contribution with a smaller payment for success (T_S^A). The associated reduction in the cost of motivating agent A to contribute to the success of the project underlies the principal's preference for the simultaneous delivery of contributions when $\gamma < 0$. Under such simultaneous delivery, the identical agent optimally operate in symmetric environments in equilibrium, and so face identical reward structures. These conclusions are recorded formally in Proposition 3.

Proposition 3. *Suppose $\gamma < 0$. Then $\Pi^S > \Pi^{SQ}$, so the principal directs the agents to deliver their contributions simultaneously. The principal presents the same reward structure ($T_S^A = T_S^B$) to the two agents. The agents deliver the same contribution to project success ($p^A = p^B$) and secure the same expected profit ($\pi^A = \pi^B$).*

For emphasis, Corollary 1 summarizes the primary implications of Propositions 1 - 3.

Corollary 1. *The principal will implement asymmetric reward structures for the identical agents if and only if their contributions to project success are complementary (i.e., $\gamma > 0$).*

4 Conclusions.

We have analyzed a simple model of team production to determine when identical team members ("agents") will be treated asymmetrically in equilibrium. We found that such asymmetric treatment will arise if and only if the agents' contributions to the team project are complementary. In this case, the agents are optimally directed to deliver their contributions sequentially. The first mover is then naturally motivated to deliver a substantial contribution to the project in order to enhance the productivity of the second mover's contribution. This natural motivation reduces the need for generous compensation to induce the first mover to work diligently. Consequently, the first mover optimally is afforded a less generous reward structure than his teammate.

In contrast, when the agents' contributions are substitutes, a first mover would be reluctant to deliver a substantial contribution to the project, for fear of unduly reducing the productivity of the second mover's contribution. To eliminate this disincentive to labor diligently, the principal directs the agents to deliver their contributions simultaneously. The resulting symmetric positioning of the identical agents renders identical reward structures for the two agents optimal.

The structured model that we analyzed allowed us to characterize optimal reward structures completely and to compare analytically the principal's expected profit under distinct timings of the agents' contributions. The key forces that underlie our main conclusions seem likely to arise in many plausible alternative settings, and so we suspect that our central qualitative conclusions will hold more generally.[10] It would be useful to test this conjecture in models that consider more general production technologies, alternative cost functions, and more than two agents. Future research also might investigate the effects of risk aversion,[11] limited flexibility in the timing of agents' contributions, and positive but limited agent wealth.[12] In addition, it would be interesting to determine how the properties of optimal reward structures and the optimal timing of contributions to a team project vary in response to explicit agent aversion to inequitable treatment.

We analyzed a setting with identical agents in order to identify most clearly the properties

[10] As the proofs of Propositions 2 and 3 reveal, the principal continues to prefer that the agents deliver their contributions sequentially (respectively, simultaneously) when $\gamma > 0$ (respectively, $\gamma < 0$) when the agents have different production costs. Also, it is apparent that Proposition 1 holds in many alternative settings, including settings with more than two agents.

[11] Itoh (1991) analyzes the design of reward structures for risk averse agents in settings where each agent assumes primary responsibility for a distinct project and where the success or failure of each distinct project is observable and verifiable.

[12] McAfee and McMillan (1991) demonstrate how team moral hazard problems can be resolved when all risk neutral agents have ample wealth.

of the team production technology that promote asymmetric treatment of team members. However, as the proofs of Propositions 1 – 3 reveal, our analysis admits a complete characterization of optimal reward structures when the agents have different innate capabilities. Additional issues of interest arise in this case. For instance, when the agents' contributions are complementary, the principal may prefer that the least capable agent move first. This is indeed the case when the agents' capabilities are sufficiently similar.[13] Whether this preference persists more generally is an open question of interest.

[13] Formally, suppose agent i's personal cost of delivering contribution p^i is $\frac{k^i}{\theta}(p^i)^\theta$ for $i \in \{A, B\}$. Then when $\gamma > 0$, the principal's maximum expected profit is higher when agent i delivers his contribution before agent j delivers his contribution if and only if $k^i > k^j$, where $j \neq i$, $i, j \in \{A, B\}$, $|k^i - k^j| = \varepsilon > 0$, and ε is arbitrarily small. Numerical solutions indicate that the principal also typically prefers to have agent i deliver his contribution first when k^i is substantially larger than k^j.

Appendix

The proof of Lemma 1 is a straightforward proof by contradiction, and so is omitted.

The proofs of Propositions 1 – 3 follow, beginning with the proof of Proposition 2, which lays the groundwork for the proofs of Propositions 1 and 3. To limit the length of the proof of Proposition 2, the details of some algebraic manipulation are omitted. These details are contained in a Technical Appendix that is available from the authors.

Proof of Proposition 2.

The proof proceeds as follows. Observation 1 establishes that the solution to [P-SQ] can be identified by solving two nonlinear equations. Observation 2 provides an alternative specification of these two nonlinear equations. Observation 3 demonstrates that a unique solution to these alternative nonlinear equations exists. Conclusions 1 and 2 characterize this solution. Conclusion 3 ranks the expected profit of the two agents at the identified solution. Conclusion 4 reports that the principal secures greater expected profit at the solution to [P-SQ] than at the solution to [P-S].

Let [P-SQ]′ denote the counterpart to problem [P-SQ] in the more general setting where agent i's cost of delivering contribution p^i is $\frac{1}{\theta}k^B(\widetilde{p}^B)^\theta$ for $i \in \{A, B\}$. Suppose without loss of generality that $V_F = 0$. Then the natural counterparts to (3) and (4) are readily shown to imply:

$$T_S^A = \frac{k^A \left(p^A\right)^{\theta-1}}{1 + \left[\frac{\theta}{\theta-1}\right]\gamma p^B} \quad \text{and} \quad T_S^B = \frac{k^B \left(p^B\right)^{\theta-1}}{1 + \gamma p^A}. \tag{8}$$

It can be shown that the natural counterpart to (2) will hold when it is not imposed. Therefore, [P-SQ]′ can be written as:

$$\underset{\{p^A,\, p^B\}}{Maximize} \;\; [p^A + p^B + \gamma p^A p^B] \left[V_S - \frac{k^A \left(p^A\right)^{\theta-1}}{1 + \left[\frac{\theta}{\theta-1}\right]\gamma p^B} - \frac{k^B \left(p^B\right)^{\theta-1}}{1 + \gamma p^A} \right] \tag{9}$$

subject to:

$$\frac{k^A \left(p^A\right)^{\theta-1}}{1 + \left[\frac{\theta}{\theta-1}\right]\gamma p^B} \geq 0; \qquad \frac{k^B \left(p^B\right)^{\theta-1}}{1 + \gamma p^A} \geq 0;$$

$$0 < p^i \leq 1 \quad \text{for } i = A, B; \quad \text{and} \quad 0 < p^A + p^B + \gamma p^A p^B \leq 1.$$

The Lagrangian corresponding to [P-SQ]′ is:

$$\mathcal{L} = [p^A + p^B + \gamma p^A p^B] \left[V_S - \frac{k^A \left(p^A\right)^{\theta-1}}{1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B} - \frac{k^B \left(p^B\right)^{\theta-1}}{1 + \gamma p^A} \right]$$
$$+\lambda \left[1 - p^A - p^B - \gamma p^A p^B\right]. \tag{10}$$

Differentiating $\mathcal{L}$ provides:

$$
\begin{aligned}
\mathcal{L}_{p^A} &= \left[1+\gamma p^B\right]\left[V_S - \frac{k^A\left(p^A\right)^{\theta-1}}{1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B} - \frac{k^B\left(p^B\right)^{\theta-1}}{1+\gamma p^A}\right] - \lambda\left[1+\gamma p^B\right] \\
&\quad +[p^A+p^B+\gamma p^A p^B]\left[-\frac{(\theta-1)\,k^A\left(p^A\right)^{\theta-2}}{1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B} + \frac{\gamma k^B\left(p^B\right)^{\theta-1}}{\left(1+\gamma p^A\right)^2}\right]; \qquad (11)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{p^B} &= \left[1+\gamma p^A\right]\left[V_S - \frac{k^A\left(p^A\right)^{\theta-1}}{1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B} - \frac{k^B\left(p^B\right)^{\theta-1}}{1+\gamma p^A}\right] - \lambda\left[1+\gamma p^A\right] \\
&\quad +[p^A+p^B+\gamma p^A p^B]\left[-\frac{(\theta-1)\,k^B\left(p^B\right)^{\theta-2}}{1+\gamma p^A} + \frac{\left(\frac{\theta}{\theta-1}\right)\gamma k^A\left(p^A\right)^{\theta-1}}{\left(1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right)^2}\right];\ \text{and} \qquad (12)
\end{aligned}
$$

$$
\mathcal{L}_\lambda = 1 - \left[p^A+p^B+\gamma p^A p^B\right]. \qquad (13)
$$

Given the maintained focus on values of k^A and k^B such that $p^A \in (0,1)$, $p^B \in (0,1)$, and $p^A+p^B+\gamma p^A p^B \in (0,1)$, (11) and (12) imply:

$$
\begin{aligned}
V_S &- \frac{k^A\left(p^A\right)^{\theta-1}}{1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B} - \frac{k^B\left(p^B\right)^{\theta-1}}{1+\gamma p^A} \\
&= -\left[\frac{p^A+p^B+\gamma p^A p^B}{1+\gamma p^B}\right]\left[-\frac{(\theta-1)\,k^A\left(p^A\right)^{\theta-2}}{1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B} + \frac{\gamma k^B\left(p^B\right)^{\theta-1}}{\left(1+\gamma p^A\right)^2}\right];\ \text{and} \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
V_S &- \frac{k^A\left(p^A\right)^{\theta-1}}{1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B} - \frac{k^B\left(p^B\right)^{\theta-1}}{1+\gamma p^A} \\
&= -\left[\frac{p^A+p^B+\gamma p^A p^B}{1+\gamma p^A}\right]\left[-\frac{[\theta-1]\,k^B\left(p^B\right)^{\theta-2}}{1+\gamma p^A} + \frac{\left[\frac{\theta}{\theta-1}\right]\gamma k^A\left(p^A\right)^{\theta-1}}{\left(1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B\right)^2}\right]. \qquad (15)
\end{aligned}
$$

(14) and (15) imply:

$$
\begin{aligned}
&\left[1+\gamma p^A\right]\left[-\frac{[\theta-1]\,k^A\left(p^A\right)^{\theta-2}}{1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B} + \frac{\gamma k^B\left(p^B\right)^{\theta-1}}{\left(1+\gamma p^A\right)^2}\right] \\
&\qquad = \left[1+\gamma p^B\right]\left[-\frac{[\theta-1]\,k^B\left(p^B\right)^{\theta-2}}{1+\gamma p^A} + \frac{\left[\frac{\theta}{\theta-1}\right]\gamma k^A\left(p^A\right)^{\theta-1}}{\left(1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B\right)^2}\right]. \qquad (16)
\end{aligned}
$$

Consider:

$$
p^A+p^B+\gamma p^A p^B = \delta. \qquad (17)
$$

Then we have:

<u>Observation 1</u>. If $p^{A*} > 0$ and $p^{B*} > 0$ solve [P-SQ]$'$, then p^{A*} and p^{B*} satisfy (16). Furthermore, there exists a $\delta \in (0, 1]$ such that p^{A*} and p^{B*} solve (17).

The remainder of the proof considers the case where $\gamma > 0$. (The proofs of Propositions 1 and 3 consider the cases of $\gamma = 0$ and $\gamma < 0$, respectively.)

Define:

$$f\left(p^A\right) = f_1\left(p^A\right) - f_2\left(p^A\right), \quad \text{where} \tag{18}$$

$$f_1\left(p^A\right) = k^B \left(p^B\right)^{\theta-2} [\theta - 1] \left[1 + \left(\frac{\theta}{\theta - 1}\right) \gamma p^B\right]^3, \quad \text{and} \tag{19}$$

$$\begin{aligned} f_2\left(p^A\right) = k^A \left(p^A\right)^{\theta-2} \Bigg[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta - 1} - \frac{\theta\gamma\delta}{\theta - 1} - \gamma^2 \left(p^A\right)^2 \\ + \gamma p^A \left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta - 1} + \frac{\theta}{\theta - 1}\right\}\Bigg]. \end{aligned} \tag{20}$$

<u>Observation 2</u>. p^A and p^B solve (16) and (17) if and only if $f\left(p^A\right) = 0$ and $p^B = \frac{\gamma - p^A}{1 + \gamma p^A}$.

Proof. (16) holds if and only if:

$$\begin{aligned} &-\frac{[\theta - 1] k^A \left(p^A\right)^{\theta-2}}{1 + \left[\frac{\theta}{\theta-1}\right] \gamma p^B} + \frac{\gamma k^B \left(p^B\right)^{\theta-1}}{\left[1 + \gamma p^A\right]^2} \\ &= -\frac{\left[1 + \gamma p^B\right] [\theta - 1] k^B \left(p^B\right)^{\theta-2}}{\left[1 + \gamma p^A\right]^2} + \left[\frac{1 + \gamma p^B}{1 + \gamma p^A}\right] \left[\frac{\left(\frac{\theta}{\theta-1}\right) \gamma k^A \left(p^A\right)^{\theta-1}}{\left(1 + \left[\frac{\theta}{\theta-1}\right] \gamma p^B\right)^2}\right]. \end{aligned} \tag{21}$$

Note that:

$$\begin{aligned} &\frac{\gamma k^B \left(p^B\right)^{\theta-1}}{\left[1 + \gamma p^A\right]^2} + \frac{\left[1 + \gamma p^B\right] [\theta - 1] k^B \left(p^B\right)^{\theta-2}}{\left[1 + \gamma p^A\right]^2} \\ &= \frac{k^B \left(p^B\right)^{\theta-2} \left[\gamma p^B + \theta - 1 - \gamma p^B + \theta\gamma p^B\right]}{\left[1 + \gamma p^A\right]^2} = \frac{k^B \left(p^B\right)^{\theta-2} \left[\theta - 1 + \theta\gamma p^B\right]}{\left[1 + \gamma p^A\right]^2}. \end{aligned} \tag{22}$$

Also:

$$\begin{aligned} &\frac{[\theta - 1] k^A \left(p^A\right)^{\theta-2}}{1 + \left[\frac{\theta}{\theta-1}\right] \gamma p^B} + \left[\frac{1 + \gamma p^B}{1 + \gamma p^A}\right] \left[\frac{\left(\frac{\theta}{\theta-1}\right) \gamma k^A \left(p^A\right)^{\theta-1}}{\left(1 + \left[\frac{\theta}{\theta-1}\right] \gamma p^B\right)^2}\right] \\ &= \frac{k^A \left(p^A\right)^{\theta-2} \left\{[\theta - 1] \left[1 + \gamma p^A\right] \left[1 + \left(\frac{\theta}{\theta-1}\right) \gamma p^B\right] + \left[1 + \gamma p^B\right] \left[\frac{\theta}{\theta-1}\right] \gamma p^A\right\}}{\left[1 + \gamma p^A\right] \left[1 + \left(\frac{\theta}{\theta-1}\right) \gamma p^B\right]^2}. \end{aligned} \tag{23}$$

(21), (22), and (23) imply that (16) holds if and only if:

$$\frac{k^B\left(p^B\right)^{\theta-2}\left[\theta-1+\theta\gamma p^B\right]}{\left[1+\gamma p^A\right]^2} = \frac{\gamma k^B\left(p^B\right)^{\theta-1}}{\left[1+\gamma p^A\right]^2}+\frac{\left[1+\gamma p^B\right]\left[\theta-1\right]k^B\left(p^B\right)^{\theta-2}}{\left[1+\gamma p^A\right]^2}$$

$$= \frac{k^A\left(p^A\right)^{\theta-2}\left\{\left[\theta-1\right]\left[1+\gamma p^A\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]+\left[1+\gamma p^B\right]\left[\frac{\theta}{\theta-1}\right]\gamma p^A\right\}}{\left[1+\gamma p^A\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^2}$$

$$\Leftrightarrow \quad k^B\left(p^B\right)^{\theta-2}\left[\theta-1+\theta\gamma p^B\right]$$

$$= \frac{\left[1+\gamma p^A\right]k^A\left(p^A\right)^{\theta-2}\left\{\left[\theta-1\right]\left[1+\gamma p^A\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]+\left[1+\gamma p^B\right]\left[\frac{\theta}{\theta-1}\right]\gamma p^A\right\}}{\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^2}. \quad (24)$$

Observe that:

$$\left[\theta-1\right]\left[1+\gamma p^A\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]+\left[1+\gamma p^B\right]\left[\frac{\theta}{\theta-1}\right]\gamma p^A$$

$$= \left[1+\gamma p^B\right]\left[\frac{\theta}{\theta-1}\right]\gamma p^A+\left[\theta-1+(\theta-1)\gamma p^A\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]$$

$$= \theta-1+\theta\gamma p^B+\left[\frac{\gamma^2\theta^2}{\theta-1}\right]p^Ap^B+\left[\theta-1+\frac{\theta}{\theta-1}\right]\gamma p^A. \quad (25)$$

(24) and (25) imply that (16) holds if and only if:

$$k^B\left(p^B\right)^{\theta-2}\left[\theta-1+\theta\gamma p^B\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^2$$

$$= \left[1+\gamma p^A\right]k^A\left(p^A\right)^{\theta-2}\left\{\theta-1+\theta\gamma p^B+\left[\frac{\gamma^2\theta^2}{\theta-1}\right]p^Ap^B+\left[\theta-1+\frac{\theta}{\theta-1}\right]\gamma p^A\right\}. \quad (26)$$

Since $p^A+p^B+\gamma p^Ap^B=\delta$:

$$\theta-1+\theta\gamma p^B+\left[\frac{\gamma^2\theta^2}{\theta-1}\right]p^Ap^B+\left[\theta-1+\frac{\theta}{\theta-1}\right]\gamma p^A$$

$$= \theta-1+\theta\gamma p^B+\left[\theta-1+\frac{\theta}{\theta-1}\right]\gamma p^A+\left[\frac{\gamma\theta^2}{\theta-1}\right]\left[\delta-p^A-p^B\right]$$

$$= \theta-1+\frac{\delta\gamma\theta^2}{\theta-1}-\gamma p^A-\frac{\gamma\theta p^B}{\theta-1} = \theta-1+\frac{\delta\gamma\theta^2}{\theta-1}-\gamma\left[p^A+\frac{\theta p^B}{\theta-1}\right]. \quad (27)$$

(26) and (27) imply that (16) holds if and only if:

$$k^B\left(p^B\right)^{\theta-2}\left[\theta-1\right]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^3$$

$$= \left[1+\gamma p^A\right] k^A \left(p^A\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \gamma\left(p^A + \frac{\theta p^B}{\theta-1}\right)\right] \tag{28}$$

$$\begin{aligned} &= k^A \left(p^A\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \gamma p^A - \frac{\theta\gamma p^B}{\theta-1}\right] \\ &+k^A \left(p^A\right)^{\theta-2} \left\{\left[\theta-1\right]\gamma p^A + \left[\frac{\delta\gamma^2\theta^2}{\theta-1}\right] p^A - \gamma^2 \left(p^A\right)^2 - \left[\frac{\theta\gamma^2}{\theta-1}\right] p^A p^B\right\}. \end{aligned} \tag{29}$$

Since $p^A + p^B + \gamma p^A p^B = \delta$, (29) implies that (16) holds if and only if:

$$\begin{aligned} k^B \left(p^B\right)^{\theta-2} \left[\theta-1\right] \left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^3 &= k^A \left(p^A\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \gamma p^A - \frac{\theta\gamma p^B}{\theta-1}\right] \\ &+k^A \left(p^A\right)^{\theta-2} \left\{\left[\theta-1\right]\gamma p^A + \left[\frac{\delta\gamma\theta^2}{\theta-1}\right]\gamma p^A - \gamma^2\left(p^A\right)^2 - \left[\frac{\theta\gamma}{\theta-1}\right]\left[\delta - p^A - p^B\right]\right\} \\ &= k^A \left(p^A\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \frac{\theta\gamma\delta}{\theta-1} - \gamma^2\left(p^A\right)^2 + \gamma p^A\left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right]. \end{aligned} \tag{30}$$

(18) - (20) and (30) imply that (16) holds if and only if $f(\cdot) = 0$. □

<u>Observation 3</u>. For any $\delta \in (0,1)$, there exists a unique $p^A \in (0,\delta]$ and $p^B \in (0,\delta]$ that solve (16) and (17).

Proof. The conclusion in the Observation follows from Observation 2 if we can show that $f\left(p^A = 0\right) > 0$, $f\left(p^A = \delta\right) < 0$, and $f'\left(p^A\right) < 0$ for all $p^A \in (0,\delta)$. We now demonstrate that these three inequalities hold.

Since $p^A + p^B + \gamma p^A p^B = \delta$, $p^B = \delta$ when $p^A = 0$. Therefore, (18) implies:

$$f\left(p^A = 0\right) = k^B \left(\delta\right)^{\theta-2} \left[\theta-1\right] \left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma\delta\right]^3 > 0. \tag{31}$$

Also, $p^B = 0$ when $p^A = \delta$. Therefore, (18) implies:

$$\begin{aligned} f\left(p^A = \delta\right) &= -k^A \left(\delta\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \frac{\theta\gamma\delta}{\theta-1} - \gamma^2\left(\delta\right)^2 + \gamma\delta\left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right] \\ &= -\, k^A \left(\delta\right)^{\theta-2} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \gamma^2\left(\delta\right)^2 + \gamma\delta\left[\theta-2\right] + \left(\frac{\delta^2\gamma^2\theta^2}{\theta-1}\right)\right]. \end{aligned} \tag{32}$$

Since $\theta > 1$, $\frac{\theta^2}{\theta-1} > 1 \Leftrightarrow z(\theta) \equiv \theta^2 - \theta + 1 > 0$. $z(1) = 1$ and $z'(\theta) = 2\theta - 1 > 0$. Hence, $z(\theta) > 0$ for all $\theta > 1$, and so $\frac{\theta^2}{\theta-1} > 1$. Therefore, $\delta^2\gamma^2\left[\frac{\theta^2}{\theta-1}\right] > \delta^2\gamma^2$. Consequently, (32) implies:

$$f\left(p^A = \delta\right) < 0. \tag{33}$$

It remains to prove that $f'\left(p^A\right) < 0$ for all $p^A \in (0, \delta)$. From (18), it will suffice to show that $f_1'\left(p^A\right) < 0$ and $f_2'\left(p^A\right) > 0$ for all $p^A \in (0, \delta)$. Differentiating (19) provides:

$$\begin{aligned} f_1'\left(p^A\right) &= [\theta - 2]\, k^B \left(p^B\right)^{\theta-3} [\theta - 1] \left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^3 \frac{\partial p^B}{\partial p^A} \\ &\qquad + 3k^B \left(p^B\right)^{\theta-2} [\theta - 1] \left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^2 \left[\frac{\gamma\theta}{\theta-1}\right] \frac{\partial p^B}{\partial p^A} \qquad (34) \\ &= k^B \left(p^B\right)^{\theta-3} [\theta - 1] \left[(\theta - 2)\left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^3\right. \\ &\qquad \left. + 3p^B \left[1 + \left(\frac{\theta}{\theta-1}\right)\gamma p^B\right]^2 \left(\frac{\gamma\theta}{\theta-1}\right)\right] \frac{\partial p^B}{\partial p^A}. \qquad (35) \end{aligned}$$

Note that:

$$p^A + p^B + \gamma p^A p^B = \delta \quad \Rightarrow \quad \frac{\partial p^B}{\partial p^A} = -\left[\frac{1 + \gamma p^B}{1 + \gamma p^A}\right] < 0. \qquad (36)$$

Therefore, since $\theta \geq 2$, (35) and (36) imply:

$$f_1'\left(p^A\right) < 0. \qquad (37)$$

Differentiating (20) provides:

$$\begin{aligned} f_2'\left(p^A\right) &= [\theta - 2]\, k^A \left(p^A\right)^{\theta-3} \left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \frac{\theta\gamma\delta}{\theta-1} - \gamma^2 \left(p^A\right)^2 + \gamma p^A \left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right] \\ &\qquad + k^A \left(p^A\right)^{\theta-2} \left[-2\gamma^2 p^A + \gamma\left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right] \qquad (38) \\ &= k^A \left(p^A\right)^{\theta-3} \left\{[\theta - 2]\left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \frac{\theta\gamma\delta}{\theta-1} - \gamma^2 \left(p^A\right)^2 + \gamma p^A \left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right]\right. \\ &\qquad \left. + p^A \left[-2\gamma^2 p^A + \gamma\left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right]\right\}. \qquad (39) \end{aligned}$$

(39) implies:

$$\begin{aligned} \frac{f_2'\left(p^A\right)}{k^A \left(p^A\right)^{\theta-3}} &= [\theta - 2]\left[\theta - 1 + \frac{\delta\gamma\theta^2}{\theta-1} - \frac{\theta\gamma\delta}{\theta-1} - \gamma^2 \left(p^A\right)^2\right. \\ &\qquad \left. + \gamma p^A \left\{\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right\}\right] - 2\gamma^2 \left(p^A\right)^2 + \gamma p^A \left[\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right] \qquad (40) \\ &= [\theta - 2][\theta - 1 + \delta\gamma\theta] + [\theta - 2]\gamma p^A \left[\theta - 2 + \frac{\delta\gamma\theta^2}{\theta-1} + \frac{\theta}{\theta-1}\right] \\ &\qquad + \gamma p^A \left[\theta - 2 + \frac{\theta}{\theta-1}\right] + \theta\gamma^2 p^A \left[\frac{\delta\theta}{\theta-1} - p^A\right]. \qquad (41) \end{aligned}$$

The equality in (41) holds because $[\theta-2]\left[\frac{\delta\gamma\theta^2-\theta\delta\gamma}{\theta-1}\right]=[\theta-2]\left[\frac{\delta\gamma\theta(\theta-1)}{\theta-1}\right]=[\theta-2]\delta\gamma\theta$.

$\frac{\delta\theta}{\theta-1}>p^A$ since $p^A<\delta$ and $\frac{\theta}{\theta-1}>1$. Therefore, (41) implies that if $\theta\geq 2$, then:

$$f_2'\left(p^A\right)>0. \tag{42}$$

(18), (37), and (42) imply:

$$f'\left(p^A\right)<0. \tag{43}$$

(31), (33), (43), and Observation 2 imply that there exists a unique $p^A\in(0,\delta]$ and $p^B\in(0,\delta]$ that solve (16) and (17). □

Conclusion 1. Suppose $k^A=k^B$. Then $p^A>p^B$ at the solution to [P-SQ]′.

Proof. From Observation 3, there exists a $\delta\in(0,1)$ such that if $p^{A*}>0$ and $p^{B*}>0$ solve [P-SQ]′, then p^{A*} and p^{B*} satisfy (16) and (17). Thus, it suffices to show that for any $\delta\in(0,1)$, if $p^{A*}>0$ and $p^{B*}>0$ satisfy (16) and (17), then $p^{A*}>p^{B*}$.

Let $\overline{p}^A$ be the value of p^A at which $p^A=p^B$. We will first show that $f(\overline{p}^A)>0$. Therefore, since $f\left(p^{A*}\right)=0$ (from Observation 2) and since $f'\left(p^A\right)<0$ (from (43)), it will follow that $\overline{p}^A>p^{A*}$.

Since $k^A=k^B$, (18) implies:

$$\begin{aligned}
f\left(\overline{p}^A\right) &= k^A\left(\overline{p}^A\right)^{\theta-2}[\theta-1]\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\overline{p}^A\right]^3 \\
&\quad -k^A\left(\overline{p}^A\right)^{\theta-2}\left[\theta-1+\frac{\delta\gamma\theta^2}{\theta-1}-\frac{\theta\gamma\delta}{\theta-1}-\gamma^2\left(\overline{p}^A\right)^2+\gamma\overline{p}^A\left\{\theta-2+\frac{\delta\gamma\theta^2}{\theta-1}+\frac{\theta}{\theta-1}\right\}\right] \\
&= k^A\left(\overline{p}^A\right)^{\theta-2}\left[(\theta-1)\left\{1+3\gamma^2\left[\frac{\theta}{\theta-1}\right]^2\left(\overline{p}^A\right)^2+3\left[\frac{\gamma\theta}{\theta-1}\right]\overline{p}^A+\left[\frac{\gamma\theta}{\theta-1}\right]^3\left(\overline{p}^A\right)^3\right\}\right. \\
&\quad \left.-[\theta-1]-\frac{\delta\gamma\theta^2}{\theta-1}+\frac{\theta\gamma\delta}{\theta-1}+\gamma^2\left(\overline{p}^A\right)^2-\gamma\overline{p}^A\left\{\theta-2+\frac{\delta\gamma\theta^2}{\theta-1}+\frac{\theta}{\theta-1}\right\}\right] \qquad (44) \\
&= k^A\left(\overline{p}^A\right)^{\theta-2}\left[(\theta-1)\left\{3\gamma^2\left[\frac{\theta}{\theta-1}\right]^2\left(\overline{p}^A\right)^2+3\left[\frac{\gamma\theta}{\theta-1}\right]\overline{p}^A+\left[\frac{\gamma\theta}{\theta-1}\right]^3\left(\overline{p}^A\right)^3\right\}\right. \\
&\quad \left.-\theta\gamma\delta+\gamma^2\left(\overline{p}^A\right)^2-\gamma\overline{p}^A\left\{\theta-2+\frac{\delta\gamma\theta^2}{\theta-1}+\frac{\theta}{\theta-1}\right\}\right] \qquad (45) \\
&= k^A\left(\overline{p}^A\right)^{\theta-2}\left[(\theta-1)\left[\frac{\gamma\theta}{\theta-1}\right]^3\left(\overline{p}^A\right)^3+\gamma^2\left(\overline{p}^A\right)^2\left\{3[\theta-1]\left[\frac{\theta}{\theta-1}\right]^2+1\right\}\right. \\
&\quad \left.+\gamma\overline{p}^A\left\{3\theta-\theta+2-\frac{\delta\gamma\theta^2}{\theta-1}-\frac{\theta}{\theta-1}\right\}-\theta\gamma\delta\right] \qquad (46)
\end{aligned}$$

$$= k^A \left(\overline{p}^A\right)^{\theta-2} \left[(\theta-1) \left[\frac{\gamma\theta}{\theta-1}\right]^3 \left(\overline{p}^A\right)^3 + \gamma^2 \left(\overline{p}^A\right)^2 \left[\frac{3\theta^2}{\theta-1}+1\right] \right.$$

$$\left. + \gamma\overline{p}^A \left\{ 2[\theta+1] - \frac{\theta}{\theta-1} \right\} - \left[\frac{\delta\gamma^2\theta^2}{\theta-1}\right] \overline{p}^A - \theta\gamma\delta \right]. \tag{47}$$

The equality in (45) holds because $-\frac{\delta\gamma\theta^2-\theta\gamma\delta}{\theta-1} = -\frac{[\theta-1]\theta\gamma\delta}{\theta-1} = -\theta\gamma\delta$.

Since $p^A = p^B = \overline{p}^A$ and $p^A + p^B + \gamma p^A p^B = \delta$:

$$2\overline{p}^A + \gamma\left(\overline{p}^A\right)^2 = \delta. \tag{48}$$

Using (48) in (47) provides:

$$f\left(\overline{p}^A\right) = k^A \left(\overline{p}^A\right)^{\theta-2} \left[[\theta-1] \left[\frac{\gamma\theta}{\theta-1}\right]^3 \left(\overline{p}^A\right)^3 + \gamma^2 \left(\overline{p}^A\right)^2 \left[\frac{3\theta^2}{\theta-1}+1\right] \right.$$

$$\left. + \gamma\overline{p}^A \left[2(\theta+1) - \frac{\theta}{\theta-1} \right] - \frac{\delta\gamma^2\theta^2\overline{p}^A}{\theta-1} - \theta\gamma\left(2\overline{p}^A + \gamma\left(\overline{p}^A\right)^2\right) \right] \tag{49}$$

$$= k^A \left(\overline{p}^A\right)^{\theta-2} \left[(\theta-1) \left[\frac{\gamma\theta}{\theta-1}\right]^3 \left(\overline{p}^A\right)^3 + \gamma^2 \left(\overline{p}^A\right)^2 \left[\frac{1}{\theta-1}\right] \left[2\theta^2+2\theta-1\right] \right.$$

$$\left. + \gamma\overline{p}^A \left[\frac{1}{\theta-1}\right] [\theta-2] - \frac{\delta\gamma^2\theta^2\overline{p}^A}{\theta-1} \right]. \tag{50}$$

Using (48) in (50) provides:

$$f\left(\overline{p}^A\right) = k^A \left(\overline{p}^A\right)^{\theta-2} \left[(\theta-1) \left[\frac{\gamma\theta}{\theta-1}\right]^3 \left(\overline{p}^A\right)^3 + \gamma^2 \left(\overline{p}^A\right)^2 \left[\frac{1}{\theta-1}\right] \left[2\theta^2+2\theta-1\right] \right.$$

$$\left. + \gamma\overline{p}^A \left[\frac{1}{\theta-1}\right] [\theta-2] - \left[\frac{\gamma^2\theta^2\overline{p}^A}{\theta-1}\right] \left(2\overline{p}^A + \gamma\left(\overline{p}^A\right)^2\right) \right] \tag{51}$$

$$= k^A \left(\overline{p}^A\right)^{\theta-2} \left\{ \left[\frac{\gamma^3\theta^2}{(\theta-1)^2}\right] \left(\overline{p}^A\right)^3 + \gamma^2 \left(\overline{p}^A\right)^2 \left[\frac{2\theta-1}{\theta-1}\right] + \left[\frac{\theta-2}{\theta-1}\right] \gamma\overline{p}^A \right\} > 0. \tag{52}$$

Since $f\left(\overline{p}^A\right) > 0$, $f'\left(p^A\right) < 0$, and $f\left(p^{A*}\right) = 0$, it follows that $p^{A*} > \overline{p}^A$.

(36) implies $\frac{\partial p^B}{\partial p^A} < 0$. Also $\overline{p}^B = p^B\left(\overline{p}^A\right) = \overline{p}^A$. Therefore, since $p^{A*} > \overline{p}^A$, it follows that $p^{B*} < \overline{p}^B$.

In summary, $p^{A*} > \overline{p}^A = \overline{p}^B > p^{B*}$, which completes the proof. □

Conclusion 2. Suppose $k^A = k^B$. Then $T_S^A < T_S^B$ at the solution to [P-SQ]′.

Proof. From (8):

$$T_S^A = \frac{k^A\left(p^A\right)^{\theta-1}}{1+\left[\frac{\theta}{\theta-1}\right]\gamma p^B} \quad \text{and} \quad T_S^B = \frac{k^B\left(p^B\right)^{\theta-1}}{1+\gamma p^A}. \tag{53}$$

(53) implies that we need to show:

$$\frac{\left(p^{A*}\right)^{\theta-1}}{1+\left[\frac{\theta}{\theta-1}\right]\gamma p^{B*}} < \frac{\left(p^{B*}\right)^{\theta-1}}{1+\gamma p^{A*}}. \tag{54}$$

First consider p^A and $p^B = \frac{\delta - p^A}{1+\gamma p^A}$ such that $p^A > p^B$ and $T_S^A = T_S^B$. Denote this value of p^A by $\widetilde{p}^A$. We will demonstrate that $f\left(\widetilde{p}^A\right) < 0$. Then, since $f'\left(p^A\right) < 0$ (43), $f\left(\widetilde{p}^A\right) < 0$ implies that $\widetilde{p}^A > p^{A*}$.

Since $\frac{\partial p^B}{\partial p^A} < 0$ from (36), T_S^A is (strictly) increasing in p^A and T_S^B is (strictly) decreasing in p^A. Therefore, $T_S^{A*} < T_S^{B*}$ if $T_S^A\left(\widetilde{p}^A\right) = T_S^B\left(\widetilde{p}^A\right)$ and $\widetilde{p}^A > p^{A*}$. Hence, the conclusion is proved if $f\left(\widetilde{p}^A\right) < 0$.

With $k^A = k^B$, (24) and (18) imply:

$$\begin{aligned} f\left(\widetilde{p}^A\right) &= k^A\left(\widetilde{p}^B\right)^{\theta-2}(\theta-1)\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]^3 \\ &- \left[1+\gamma\widetilde{p}^A\right]k^A\left(\widetilde{p}^A\right)^{\theta-2}\left[(\theta-1)\left(1+\gamma\widetilde{p}^A\right)\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]+\left[1+\gamma\widetilde{p}^B\right]\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^A\right]. \end{aligned} \tag{55}$$

(55) implies:

$$\begin{aligned} f\left(\widetilde{p}^A\right)\widetilde{p}^A &= k^A\widetilde{p}^A\left(\widetilde{p}^B\right)^{\theta-2}[\theta-1]\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]^3 \\ &- \left[1+\gamma\widetilde{p}^A\right]k^A\left(\widetilde{p}^A\right)^{\theta-1}\left[(\theta-1)\left(1+\gamma\widetilde{p}^A\right)\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]+\left[1+\gamma\widetilde{p}^B\right]\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^A\right]. \end{aligned} \tag{56}$$

Also, since $T_S^A\left(\widetilde{p}^A\right) = T_S^B\left(\widetilde{p}^A\right)$, (53) implies:

$$\left(\widetilde{p}^A\right)^{\theta-1}\left[1+\gamma\widetilde{p}^A\right] = \left(\widetilde{p}^B\right)^{\theta-1}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]. \tag{57}$$

Using (57) in (56) provides:

$$\begin{aligned} \frac{f\left(\widetilde{p}^A\right)\widetilde{p}^A}{k^A} &= \widetilde{p}^A\left(\widetilde{p}^B\right)^{\theta-2}[\theta-1]\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]^3 \\ &- \left(\widetilde{p}^B\right)^{\theta-1}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]\left[(\theta-1)\left[1+\gamma\widetilde{p}^A\right]\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]+\left[1+\gamma\widetilde{p}^B\right]\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^A\right] \end{aligned} \tag{58}$$

$$\Rightarrow \quad \frac{f\left(\widetilde{p}^A\right)\widetilde{p}^A}{k^A\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]} \;=\; \widetilde{p}^A\left[\theta-1\right]\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]^2$$

$$-\widetilde{p}^B\left\{\left[\theta-1\right]\left[1+\gamma\widetilde{p}^A\right]\left[1+\left(\frac{\theta\gamma}{\theta-1}\right)\widetilde{p}^B\right]+\left[1+\gamma\widetilde{p}^B\right]\left[\frac{\theta\gamma}{\theta-1}\right]\widetilde{p}^A\right\} \tag{59}$$

$$= \;\left[\theta-1\right]\left\{\gamma\widetilde{p}^A\widetilde{p}^B\left[\frac{\theta^2-\theta-1}{\left(\theta-1\right)^2}\right]+\widetilde{p}^A-\left[1+\frac{\theta\gamma\widetilde{p}^B}{\theta-1}\right]\widetilde{p}^B\right\}. \tag{60}$$

(60) implies:

$$\frac{f\left(\widetilde{p}^A\right)\widetilde{p}^A}{k^A\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]\left[\theta-1\right]} \;=\; \gamma\widetilde{p}^A\widetilde{p}^B\left[\frac{\theta^2-\theta-1}{\left(\theta-1\right)^2}\right]+\widetilde{p}^A-\left[1+\left(\frac{\theta\gamma\widetilde{p}^B}{\theta-1}\right)\right]\widetilde{p}^B. \tag{61}$$

(57) implies:

$$1+\left[\frac{\theta}{\theta-1}\right]\gamma\widetilde{p}^B=\frac{\left(\widetilde{p}^A\right)^{\theta-1}\left[1+\gamma\widetilde{p}^A\right]}{\left(\widetilde{p}^B\right)^{\theta-1}} \quad\Rightarrow\quad \left[1+\frac{\theta\gamma\widetilde{p}^B}{\theta-1}\right]\widetilde{p}^B=\frac{\left(\widetilde{p}^A\right)^{\theta-1}\left[1+\gamma\widetilde{p}^A\right]}{\left(\widetilde{p}^B\right)^{\theta-2}}. \tag{62}$$

(61) and (62) imply:

$$\frac{f\left(\widetilde{p}^A\right)\widetilde{p}^A}{k^A\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]\left[\theta-1\right]} \;=\; \gamma\widetilde{p}^A\widetilde{p}^B\left[\frac{\theta^2-\theta-1}{\left(\theta-1\right)^2}\right]+\widetilde{p}^A-\left[\frac{\left(\widetilde{p}^A\right)^{\theta-1}\left[1+\gamma\widetilde{p}^A\right]}{\left(\widetilde{p}^B\right)^{\theta-2}}\right]$$

$$= \;\left[\frac{\widetilde{p}^A}{\left(\widetilde{p}^B\right)^{\theta-2}}\right]\left\{\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\gamma\widetilde{p}^B\left(\frac{\theta^2-\theta-1}{\left[\theta-1\right]^2}\right)\right]-\left(\widetilde{p}^A\right)^{\theta-2}\left[1+\gamma\widetilde{p}^A\right]\right\}$$

$$\Rightarrow \quad \frac{f\left(\widetilde{p}^A\right)\widetilde{p}^A\left(\widetilde{p}^B\right)^{\theta-2}}{\widetilde{p}^A\left\{k^A\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\left(\frac{\theta}{\theta-1}\right)\gamma\widetilde{p}^B\right]\left[\theta-1\right]\right\}}$$

$$= \;\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\gamma\widetilde{p}^B\left(\frac{\theta^2-\theta-1}{\left[\theta-1\right]^2}\right)\right]-\left(\widetilde{p}^A\right)^{\theta-2}\left[1+\gamma\widetilde{p}^A\right]\;\equiv\; g\left(\widetilde{p}^A\right). \tag{63}$$

Let $\xi=\frac{\left[\theta^2-\theta-1\right]\gamma}{\left[\theta-1\right]^2}$. Then (63) implies:

$$g\left(\widetilde{p}^A\right) \;=\; \left(\widetilde{p}^B\right)^{\theta-2}\left[1+\xi\widetilde{p}^B\right]-\left(\widetilde{p}^A\right)^{\theta-2}\left[1+\gamma\widetilde{p}^A\right]. \tag{64}$$

(64) implies that the proof is complete if the following Claim can be shown to hold.

<u>Claim</u>. $g\left(\widetilde{p}^A\right)<0$.

The proof of the Claim is by contradiction. Suppose $g\left(\widetilde{p}^A\right)\geq 0$. Then (64) implies:

$$\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\xi\widetilde{p}^B\right] \;\geq\; \left(\widetilde{p}^A\right)^{\theta-2}\left[1+\gamma\widetilde{p}^A\right]. \tag{65}$$

Define $\alpha = \frac{\gamma\theta}{\theta-1}$. Then, from (57):

$$\left[1+\gamma\widetilde{p}^A\right]\left(\widetilde{p}^A\right)^{\theta-1} \;=\; \left[1+\alpha\widetilde{p}^B\right]\left(\widetilde{p}^B\right)^{\theta-1}. \tag{66}$$

(65) and (66) imply:

$$\frac{\left[1+\gamma\widetilde{p}^A\right]\left(\widetilde{p}^A\right)^{\theta-1}}{\left(\widetilde{p}^A\right)^{\theta-2}\left[1+\gamma\widetilde{p}^A\right]} \;\geq\; \frac{\left[1+\alpha\widetilde{p}^B\right]\left(\widetilde{p}^B\right)^{\theta-1}}{\left(\widetilde{p}^B\right)^{\theta-2}\left[1+\xi\widetilde{p}^B\right]} \quad\Rightarrow\quad \widetilde{p}^A \;\geq\; \left[\frac{1+\alpha\widetilde{p}^B}{1+\xi\widetilde{p}^B}\right]\widetilde{p}^B. \tag{67}$$

Using (67) in (66) provides:

$$\left[1+\alpha\widetilde{p}^B\right]\left(\widetilde{p}^B\right)^{\theta-1} \;\geq\; \left[1+\gamma\widetilde{p}^A\right]\left(\left[\frac{1+\alpha\widetilde{p}^B}{1+\xi\widetilde{p}^B}\right]\widetilde{p}^B\right)^{\theta-1}. \tag{68}$$

Using (67) in (68) provides:

$$\left[1+\alpha\widetilde{p}^B\right]\left(\widetilde{p}^B\right)^{\theta-1} \;\geq\; \left(1+\gamma\left[\frac{1+\alpha\widetilde{p}^B}{1+\xi\widetilde{p}^B}\right]\widetilde{p}^B\right)\left(\left[\frac{1+\alpha\widetilde{p}^B}{1+\xi\widetilde{p}^B}\right]\widetilde{p}^B\right)^{\theta-1}. \tag{69}$$

Denoting $\widetilde{p}^B$ by x, re-write equation (69) as:

$$\begin{aligned}
\left[1+\alpha x\right]x^{\theta-1} &\geq \left(1+\gamma\left[\frac{1+\alpha x}{1+\xi x}\right]x\right)\left(\left[\frac{1+\alpha x}{1+\xi x}\right]x\right)^{\theta-1} \\
\Rightarrow\quad 1+\alpha x &\geq \left(1+\gamma\left[\frac{1+\alpha x}{1+\xi x}\right]x\right)\left[\frac{1+\alpha x}{1+\xi x}\right]^{\theta-1} && (70)\\
\Rightarrow\quad 1+\alpha x &\geq \left(\frac{1+\xi x+\gamma x\left[1+\alpha x\right]}{1+\xi x}\right)\left[\frac{1+\alpha x}{1+\xi x}\right]^{\theta-1} \\
\Rightarrow\quad 1+\alpha x &\geq \left(\frac{1+\xi x+\gamma x\left[1+\alpha x\right]}{\left(1+\xi x\right)^{\theta}}\right)\left[1+\alpha x\right]^{\theta-1} \\
\Rightarrow\quad 1+\xi x^{\theta} &\geq \left[1+\alpha x\right]^{\theta-2}\left[1+\xi x+\gamma x\left(1+\alpha x\right)\right]. && (71)
\end{aligned}$$

Our objective is to find a contradiction of (71). Let:

$$\begin{aligned}
h(x) &= \left[1+\xi x\right]^{\theta}-\left[1+\alpha x\right]^{\theta-2}\left[1+\xi x+\gamma x\left(1+\alpha x\right)\right] && (72)\\
&= \left[1+\xi x\right]^{\theta-2}\left\{\left[1+\xi x\right]^2-\frac{\left(1+\alpha x\right)^{\theta-2}}{\left(1+\xi x\right)^{\theta-2}}\left[1+\xi x+\gamma x\left(1+\alpha x\right)\right]\right\}. && (73)
\end{aligned}$$

(73) implies:

$$\frac{h(x)}{\left[1+\xi x\right]^{\theta-2}} \;=\; \left[1+\xi x\right]^2-\frac{\left(1+\alpha x\right)^{\theta-2}}{\left(1+\xi x\right)^{\theta-2}}\left[1+\xi x+\gamma x\left(1+\alpha x\right)\right]. \tag{74}$$

(74) implies:

$$h(x) \overset{s}{=} [1+\xi x]^2 - \frac{[1+\alpha x]^{\theta-2}}{[1+\xi x]^{\theta-2}} [1+\xi x + \gamma x (1+\alpha x)], \tag{75}$$

provided $[1+\xi x]^{\theta-2} > 0$. This inequality holds when $\theta \geq 2$ because $1+\xi x = 1+\gamma x \left(\frac{\theta^2-\theta-1}{[\theta-1]^2}\right)$ and $\theta^2 > \theta + 1$ for all $\theta \geq 2$.

Let $t = \gamma x$. Then:

$$\xi x = \frac{\left[\theta^2-\theta-1\right]\gamma x}{[\theta-1]^2} = \frac{\left[\theta^2-\theta-1\right] t}{[\theta-1]^2}. \tag{76}$$

Also, let $y = \frac{1}{\theta-1}$. Then:

$$1+y-y^2 = 1+\frac{1}{\theta-1} - \frac{1}{[\theta-1]^2} = \frac{[\theta-1]^2 + (\theta-1) - 1}{[\theta-1]^2} = \frac{\theta^2-\theta-1}{[\theta-1]^2}. \tag{77}$$

(76) and (77) imply:

$$\xi x = \left[1+y-y^2\right] t. \tag{78}$$

Notice that:

$$1+\alpha x = 1+t[1+y] \tag{79}$$

$$\text{since} \quad \alpha x = t[1+y] \Leftrightarrow \alpha x = \gamma x[1+y]$$
$$\Leftrightarrow \alpha = \gamma[1+y] = \gamma\left[1+\frac{1}{\theta-1}\right] = \gamma\left[\frac{\theta}{\theta-1}\right]. \tag{80}$$

Also:

$$\gamma x[1+\alpha x] = \gamma\left(\frac{t}{\gamma}\right)\left[1+\gamma\left(\frac{\theta}{\theta-1}\right)\frac{t}{\gamma}\right]$$
$$= t\left[1+t\left(\frac{\theta}{\theta-1}\right)\right] = t\left[1+t\left(1+\frac{1}{\theta-1}\right)\right] = t[1+t(1+y)]. \tag{81}$$

(75), (78), (79), and (81) imply:

$$h(x) \overset{s}{=} \left[1+\left(1+y-y^2\right)t\right]^2 - \left\{\frac{[1+t(1+y)]^{\theta-2}}{[1+(1+y-y^2)t]^{\theta-2}}\right\}\left\{1+\left[1+y-y^2\right]t + t[1+t(1+y)]\right\}$$

$$= \left[1+\left(1+y-y^2\right)t\right]^2 - \frac{\left\{1+\frac{y^2 t}{1+[1+y-y^2]t}\right\}^{\frac{1}{y}}}{\left\{1+\frac{y^2 t}{1+[1+y-y^2]t}\right\}}\left\{1+\left[1+y-y^2\right]t + t[1+t(1+y)]\right\}. \tag{82}$$

Since $(1+a)^p \geq 1+ap$ for all $a \geq 0$ and $p \geq 1$:

$$\left\{1+\frac{y^2 t}{1+\left[1+y-y^2\right] t}\right\}^{\frac{1}{y}} \geq 1+\frac{1}{y}\left\{\frac{y^2 t}{1+\left[1+y-y^2\right] t}\right\}$$

$$\Rightarrow \left\{1+\frac{y^2 t}{1+\left[1+y-y^2\right] t}\right\}^{\frac{1}{y}} \geq 1+\frac{y t}{1+\left[1+y-y^2\right] t} \tag{83}$$

$$\Rightarrow \frac{\left\{1+\frac{y^2 t}{1+[1+y-y^2] t}\right\}^{\frac{1}{y}}}{1+\frac{y^2 t}{1+[1+y-y^2] t}} \geq \frac{1+\left\{\frac{y t}{1+[1+y-y^2] t}\right\}}{1+\frac{y^2 t}{1+[1+y-y^2] t}}$$

$$= \frac{\frac{1+\left[1+y-y^2\right] t+y t}{1+[1+y-y^2] t}}{\frac{1+[1+y-y^2] t+y^2 t}{1+[1+y-y^2] t}} = \frac{1+\left[1+y-y^2\right] t+y t}{1+\left[1+y-y^2\right] t+y^2 t} = \frac{1+\left[1+y-y^2\right] t+y t}{1+[1+y] t}. \tag{84}$$

(84) implies:

$$\frac{\left\{1+\frac{y^2 t}{1+[1+y-y^2] t}\right\}^{\frac{1}{y}}}{1+\frac{y^2 t}{1+[1+y-y^2] t}} \geq \frac{1+\left[1+y-y^2\right] t+y t}{1+[1+y] t}. \tag{85}$$

Using (85) in (82) provides:

$$\left[1+\left(1+y-y^2\right) t\right]^2 - \frac{\left\{1+\frac{y^2 t}{1+(1+y-y^2) t}\right\}^{\frac{1}{y}}}{1+\frac{y^2 t}{[1+(1+y-y^2)] t}}\left\{1+\left[1+y-y^2\right] t+t[1+t(1+y)]\right\}$$

$$\leq \left[1+\left(1+y-y^2\right) t\right]^2 - \left\{\frac{1+\left[1+y-y^2\right] t+y t}{1+[1+y] t}\right\}\left\{1+\left[1+y-y^2\right] t+t[1+t(1+y)]\right\}$$

$$= \frac{1}{1+[1+y] t}\left\{[1+(1+y) t]\left[1+\left(1+y-y^2\right) t\right]^2\right.$$
$$\left. - \left[1+\left[1+y-y^2\right] t+y t\right]\left[1+\left(1+y-y^2\right) t+t(1+t[1+y])\right]\right\} \tag{86}$$

$$= \left[\frac{1}{1+[1+y] t}\right] s(x), \quad \text{where}$$

$$s(x) = [1+(1+y) t]\left[1+\left(1+y-y^2\right) t\right]^2\left\{1+\left[1+y-y^2\right] t+y t\right\}\left\{1+\left[1+y-y^2\right] t\right.$$
$$\left. +t[1+t(1+y)]\right\} \tag{87}$$

$$= 1+2 t\left[1+y-y^2\right]+\left[1+y-y^2\right]^2 t^2+[1+y] t+2 t^2[1+y]\left[1+y-y^2\right]$$
$$+[1+y]\left[1+y-y^2\right]^2 t^3-\left\{1+\left[2+y-y^2\right] t+t^2[1+y]\right\}$$
$$-\left\{\left[1+2 y-y^2\right] t+\left[1+2 y-y^2\right]\left[2+y-y^2\right] t^2+\left[1+2 y-y^2\right][1+y] t^3\right\}. \tag{88}$$

Re-write $s(x)$ as:

$$s(x) = a_0+a_1 t+a_2 t^2+a_3 t^3, \quad \text{where} \tag{89}$$

$$
\begin{aligned}
a_0 &= 1 - 1 = 0; \\
a_1 &= 2\left[1 + y - y^2\right] + 1 + y - \left[2 + y - y^2\right] - \left[1 + 2y - y^2\right] \qquad (90) \\
&= 2 + 2y - 2y^2 + 1 + y - 2 - y + y^2 - 1 - 2y + y^2 = 0; \qquad (91) \\
a_2 &= \left[1 + y - y^2\right]^2 + 2\left[1 + y\right]\left[1 + y - y^2\right] - \left[1 + y\right] - \left[1 + 2y - y^2\right]\left[2 + y - y^2\right] \\
&= \left[1 + y - y^2\right]\left[1 + y - y^2 + 2 + 2y\right] - \left[1 + y\right] - \left[1 + 2y - y^2\right]\left[2 + y - y^2\right] \\
&= 2 + 5y - y^2 - 4y^3 + y^4 - \left(2 + 5y - y^2 - 3y^3 + y^4\right) = -y^3 < 0; \quad \text{and} \qquad (92) \\
a_3 &= \left[1 + y\right]\left[1 + y - y^2\right]^2 - \left[1 + 2y - y^2\right]\left[1 + y\right] = \left[1 + y\right]\left\{\left[1 + y - y^2\right]^2 - \left[1 + y - y^2\right] - y\right\} \\
&= \left[1 + y\right]\left\{\left[1 + y - y^2\right]\left[1 + y - y^2 - 1\right] - y\right\} = \left[1 + y\right]\left\{\left[1 + y - y^2\right]\left[y - y^2\right] - y\right\} \\
&= y\left[1 + y\right]\left[-2y^2 + y^3\right] = y^3\left[1 + y\right]\left[y - 2\right] = y^3\left[1 + y\right]\left[y - 2\right] < 0. \qquad (93)
\end{aligned}
$$

(89) - (93) imply $s(x) < 0$, which provides a contradiction of $g\left(\widetilde{p}^A\right) \geq 0$. Hence, $h(x) < 0$ from (82), and so $g\left(\widetilde{p}^A\right) < 0$ from (64) and (71). □

Conclusion 3. Suppose $k^A = k^B$. Then $\pi^A < \pi^B$ at the solution to [P-SQ]$'$.

Proof. From Conclusion 2, agent A is paid less than agent B when the project succeeds. Furthermore, $p^A > p^B$ from Conclusion 1. Therefore, agent A incurs higher effort costs than agent B, since $k^A = k^B$. □

Conclusion 4. Suppose $k^A = k^B$. Then $\Pi^{SQ} > \Pi^S$.

Proof. Consider the solution to problem [P-S]$'$, which is the generalization of problem [P-S] in which agent i's cost of delivering contribution p^i is $\frac{1}{\theta}k^B(\widetilde{p}^B)^\theta$ for $i \in \{A, B\}$. It is readily verified that at the solution to [P-S]$'$:

$$
T_S^A = \frac{k^A\left(p^A\right)^{\theta-1}}{1 + \left[\frac{\gamma\theta}{\theta-1}\right]p^B} \qquad \text{and} \qquad T_S^B = \frac{k^B\left(p^B\right)^{\theta-1}}{1 + \gamma p^A}. \qquad (94)
$$

Therefore, [P-S]$'$ can be written as:

$$
\underset{\{p^A,\, p^B\}}{Maximize}\ \left[p^A + p^B + \gamma p^A p^B\right]\left[V_S - \frac{k^A\left(p^A\right)^{\theta-1}}{1 + \left[\frac{\gamma\theta}{\theta-1}\right]p^B} - \frac{k^B\left(p^B\right)^{\theta-1}}{1 + \gamma p^A}\right] \qquad (95)
$$

subject to:

$$
0 \leq p^i \leq 1 \ \text{ for } i = A, B; \quad \text{and} \quad 0 \leq p^A + p^B + \gamma p^A p^B \leq 1.
$$

Let $\widehat{p}^{A*}$ and $\widehat{p}^{B*}$ denote the values of p^A and p^B that solve [P-S]$'$. Then, the principal's expected profit at the solution to [P-S]$'$ is:

$$\Pi^S = \left[\widehat{p}^{A*} + \widehat{p}^{B*} + \gamma \widehat{p}^{A*} \widehat{p}^{B*}\right] \left[V_S - \frac{k^A \left(\widehat{p}^{A*}\right)^{\theta - 1}}{1 + \gamma \widehat{p}^{B*}} - \frac{k^B \left(\widehat{p}^{B*}\right)^{\theta - 1}}{1 + \gamma \widehat{p}^{A*}} \right]. \quad (96)$$

The principal can implement $\widehat{p}^{A*}$ and $\widehat{p}^{B*}$ when the agents choose their efforts sequentially. From (9), the principal's profit in this case is:

$$\Pi^{q*} = \left[\widehat{p}^{A*} + \widehat{p}^{B*} + \gamma \widehat{p}^{A*} \widehat{p}^{B*}\right] \left[V_S - \frac{k^A \left(\widehat{p}^{A*}\right)^{\theta - 1}}{1 + \left[\frac{\gamma\theta}{\theta - 1}\right] \widehat{p}^{B*}} - \frac{k^B \left(\widehat{p}^{B*}\right)^{\theta - 1}}{1 + \gamma \widehat{p}^{A*}} \right]. \quad (97)$$

$\frac{k^A \left(\widehat{p}^{A*}\right)^{\theta-1}}{1+\left[\frac{\gamma\theta}{\theta-1}\right]\widehat{p}^{B*}} < \frac{k^A \left(\widehat{p}^{A*}\right)^{\theta-1}}{1+\gamma\widehat{p}^{B*}}$ since $\gamma > 0$ and $\theta > 1$. Therefore, since $\Pi^{SQ} \geq \Pi^{q*}$, (96) and (97) imply that $\Pi^{SQ} > \Pi^S$ when $\gamma > 0$. □

Conclusions 1 - 4 complete the proof of the proposition. ■

Proof of Proposition 1.

It follows from (8) and (94) that when $\gamma = 0$ and $k^A = k^B$, agents A and B will receive the same payment for success at the solution to [P-S] and at the solution to [P-SQ]. Furthermore, the payment for success will be the same at the solutions to the two problems. It is readily verified that the identical payment induces the identical agents to deliver identical contributions to project success. Consequently, the two agents secure the same expected profit at the solution to [P-S] and at the solution to [P-SQ]. Furthermore, $\Pi^{SQ} = \Pi^S$ when $\gamma = 0$. ■

Proof of Proposition 3.

Let p^{A*} and p^{B*} solve [P-SQ]′. Then, (9) implies that:

$$\Pi^{SQ} = \left[p^{A*} + p^{B*} + \gamma p^{A*} p^{B*}\right] \left[V_S - \frac{k^A \left(p^{A*}\right)^{\theta - 1}}{1 + \left[\frac{\theta}{\theta - 1}\right] \gamma p^{B*}} - \frac{k^B \left(p^{B*}\right)^{\theta - 1}}{1 + \gamma p^{A*}} \right]. \quad (98)$$

The principal can implement p^{A*} and p^{B*} when the agents choose their efforts simultaneously. From (96), the principal's expected profit in this case would be:

$$\widetilde{\Pi} = \left[p^{A*} + p^{B*} + \gamma p^{A*} p^{B*}\right] \left[V_S - \frac{k^A \left(p^{A*}\right)^{\theta - 1}}{1 + \gamma p^{B*}} - \frac{k^B \left(p^{B*}\right)^{\theta - 1}}{1 + \gamma p^{A*}} \right]. \quad (99)$$

Since $\gamma < 0$ and $\frac{\theta}{\theta-1} > 1$:

$$1 + \gamma p^{B*} > 1 + \left[\frac{\theta}{\theta - 1}\right] \gamma p^{B*} \Rightarrow \frac{1}{1 + \gamma p^{B*}} < \frac{1}{1 + \left[\frac{\theta}{\theta - 1}\right] \gamma p^{B*}}. \quad (100)$$

(98) – (100) imply that $\widetilde{\Pi} > \Pi^{SQ}$, which proves the proposition (since $\Pi^S \geq \widetilde{\Pi}$). ■

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