Richness Orderings

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February 21, 2012

Abstract

An index of richness in a society is a measure of the extent of its affluence. This paper presents an analytical discussion on several indices of richness and their properties. It also develops criteria for ordering alternative distributions of income in terms of their richness. Given a line of richness, an income level above which a person is regarded as rich, and depending on the redistributive principle, it is shown that the ranking relation can be implemented by seeking dominance with respect to the generalized Lorenz curve of the rich or the affluence profile of the society. When the line of richness is assumed to be variable, we need to employ the stochastic dominance conditions for ordering the income distributions. *Journal of Economic Literature* Classification No.: D63.

Keywords: Richness Measurement, Poverty, Equity.

*Research supported by J.C.Bose National Fellowship, Dept. of Science and Technology, Govt. of India.

1 Introduction

The measurement of richness in a society is a somewhat less investigated area of research. Only recently it has become a focus of attention (see, among others, Medeiros, 2006; Piketty and Saez, 2006; Atkinson, 2007; Atkinson and Piketty, 2007, Brzezinski, 2010; Peichl and Pestel, 2010 and Peichl et al., 2010). Measuring richness of a society is an important issue for various reasons. In his signal contribution, Atkinson (2007) identified three major reasons why the income distribution of the rich should be a point of concern: command of the rich over resources (tax), their command over people (income as a source of power) and social significance. In the context of income taxation, the top of the income distribution is of particular interest. For instance, in Germany 50.6% (19.7%) of the entire income tax is paid by the top 10% (1%) of the taxpayers (see Merz et al., 2005). The redistributive power of income from rich to poor of a society is likely to increase if its richness increases.

However, high concentration of income in the hands of a few may lead to an increase in polarization in terms of disappearance of the middle class. This in turn may generate social conflicts and tensions. An increase in richness of a society is likely to increase the unequal political and economic power of the affluent. Barry (2002) argued that social exclusion, which is a denial of equal opportunities, exists at the top of the distribution in terms of elite separation.

Measurement of richness at the top of the distribution as a complement to poverty at the bottom is, therefore, an important issue of investigation. Following the tradition on poverty, the two steps that richness measurement must address are: (i) identification of the rich in the total population and (ii) aggregating the information on the rich into an indicator of richness. The identification problem is resolved by specifying a line of richness/affluence and a person with income above this line is regarded as rich. See, for example, Medeiros (2006) who estimated affluence line using Brazilian 1999 National Household Survey data. The aggregation step involves the construction of an index of richness using the incomes of the rich.

In a recent paper, Peichl et al. (2010) suggested sophisticated indices of richness analogous to well-known indices of poverty. Their suggested indices include the Chakravarty and Foster-Greer-Thorbecke indices. They also demonstrated that these indices provide 'extra explanatory value' in the context of empirical analysis. While their empirical application is based on German data, Brzezinski (2010) used these indices for analysing trends in income affluence in Poland.

Now, there may be arbitrariness in the choice of a particular index, which in turn implies arbitrariness of the conclusions that are derived from it. It is possible to reduce the degree of arbitrariness by choosing the family of richness indices that fulfils a set of reasonable postulates. We can then consider the possibility of ranking two income distributions by all members of this family. Evidently, for a given line of affluence, an index of richness will order two income distributions in a complete manner. But the orderings generated by two different indices satisfying these postulates may not be identical. We refer to this notion of ordering as 'richness-index ordering' since the objective of this notion of ordering is to rank distributions by richness indices belonging to a family. Depending on the principle of transfer of income from a rich to a richer rich, there will be two orderings. While one can be implemented by seeking dominance in terms of the generalized Lorenz curve of the rich, for the other we require dominance of the affluence profile of the rich.

Often the specification of the affluence line may not be accurate. It, therefore, becomes reasonable to investigate whether it is possible to order distributions by a given richness index for all affluence lines in some domain. We refer to this notion of ordering as 'richness-line ordering', since, for a given richness index, it allows variability of the line of affluence. It is shown that when income distributions are represented by distribution functions, unambiguous richness-line ordering by the head count ratio occurs if and only if one distribution dominates the other, both truncated at the line of affluence, by the first order stochastic dominance criterion. Second order dominance is implemented by seeking an inequality in terms of excess income of the rich from the line of richness. (See Foster and Shorrocks, 1988 and Zheng, 2000 for similar results in the context of poverty.)

The paper is organized as follows. The next section presents the axioms for an index of richness. Section 3 discusses the richness index orderings. Richness line orderings are analyzed in Section 4. Section 5 presents an empirical illustration of our results for Germany in the years 2002, 2006 and 2009 using the German Socio-Economic Panel (SOEP) where an additional representation of high-income households was added in 2002.

2 Axioms for an Index of Richness

For a population of size n, a typical income distribution is given by a vector $x = (x_1, x_2, \ldots, x_n)$, where $x_i \geq 0$ is the income of person i. For a fixed $n \geq 1$, the set of all income distributions is D^n, the nonnegative orthant of the n-dimensional Euclidean space $\mathbf{R}^n$ with the origin deleted. The set of all possible income distributions is $D = \bigcup_{n \in N} D^n$, where N is the set of positive integers. For any $x \in D^n$, the illfare and welfare-ranked permutations of x are denoted respectively by $\overline{x}$ and $\hat{x}$, that is, $\overline{x}_1 \leq \overline{x}_2 \ldots \leq \overline{x}_{n-1} \leq \overline{x}_n$ and $\hat{x_1} \geq \hat{x_2} \geq \ldots \geq \hat{x}_{n-1} \geq \hat{x_n}$. Sometimes it will be necessary to restrict attention to the extended domains $\Gamma^n = D^n \bigcup \{0^n\}$ and $\Gamma = \bigcup_{n \in N} \Gamma^n$ where 0^n is the n-vector of zeros.

The problem of identification of the rich requires the specification of a richness line ρ. The absolutist notion of richness where ρ is assumed to be exogenously given, contrasts with the relativist view in which the richness line is made responsive to the income distribution. For instance, a household with more than the median income may be regarded as relatively rich (see Medeiros, 2006). We assume that the richness line takes values in some subset $[\rho_0,\ \infty)$ of the real line, where $\rho_0 > 0$. For any income distribution x, person i is said to be rich if $x_i > \rho$.

For a given population size n, a richness index R is a real valued function defined on $D^n \times [\rho_0,\ \infty)$, that is, $R : D \times [\rho_0,\ \infty) \to \mathbf{R}^1$, where $\mathbf{R}^1$ is the real line. Thus, given any income distribution $x \in D^n$, $n \in N$, and a richness line $\rho \in [\rho_0,\ \infty), R(x, \rho)$, determines the extent of richness corresponding to x. For any $n \in N$, $x \in D^n$, we denote the set of rich persons in x by $\rho(x)$, that is, $\rho(x) = \{i | x_i \geq \rho\}$. For any $n \in N$, $x \in D^n$, the income distribution of the rich is denoted by the vector x^r and D^r will stand for all such distributions.

Peichl et al. (2010) suggested several axioms for an arbitrary richness index R. In specifying these axioms and some additional ones, which hold for all $n \in N$, unless specified, we assume that the richness line ρ is given arbitrarily.

Focus Axiom (FOC): For all $x, y \in D^n$, if $\rho(x) = \rho(y)$ and $x_i = y_i$ for all $i \in \rho(x)$, then $R(x, \rho) = R(y, \rho)$.

Continuity Axiom (CON): $R(x, \rho)$ is continuous in x.

Monotonicity Axiom (MON): For all $x, y \in D^n$, if $x_j = y_j$ for all $j \neq i, i \in \rho(y)$, and $x_i > y_i, R(y, z) < R(x, z)$.

Transfer Axiom 1(TA1): For $x, y \in D^n$, if there is a pair (i, j) where $i, j \in \rho(y)$ and $x_i - y_i = y_j - x_j = c > 0$, $y_j - c \geq y_i + c$, and $x_i = y_i$ for all $l \neq i, j$, then $R(y, z) < R(x, z)$.

Transfer Axiom 2(TA2): For $x, y \in D^n$, if there is a pair (x, j) such that $i, j \in \rho(y)$ and $x_i - y_i = y_j - x_j = c > 0$, $y_j - c \geq y_i + c$ and $x_i = y_i$ for all $l \neq i, j$, then $R(y, z) > R(x, z)$.

Subgroup Decomposability Axiom (SUD): For $x^i \in D^{n_i}, i = 1, 2, \ldots, j$, we have

$$R(x, \rho) = \sum_{i=1}^{j} \frac{n_i}{n} R(x^i, \rho), \tag{1}$$

where $x = (x^1, x^2, \ldots, x^j) \in D^n$ and $\sum_{i=1}^{j} n_i = n$.

Normalization Axiom (NOM): For any $x \in D^n$ if the set $\rho(x)$ is empty, then $R(x, \rho) = 0$.

Symmetry Axiom (SYM): For all $x \in D^n$, if y is obtained from x by a permutation of the incomes, then $R(x, \rho) = R(y, \rho)$.

Population Replication Invariance Axiom (PRI): For all $x \in D^n$, $R(x, \rho) = R(y, \rho)$, where y is the l-fold replication of $x, l \geq 2$ being any integer.

First six of these axioms were suggested by Peichl et al. (2010). According to FOC, the richness index is independent of the incomes of the non-rich persons. However, it does not rule out the possibility of the dependence of the index on the number of the non-rich persons. CON demands that minor changes in incomes will generate minor changes in the richness index. MON says that the richness index increases with an increase in the income of a rich person. In TA1 and TA2, the distribution x is obtained from the distribution y by a progressive transfer of income of the amount c from the richer rich person j to the poorer rich person i such that the donor j does not become poorer than the recipient i after the transfer. But while TA1 demands that the richness index should increase under the transfer, TA2 demands the opposite. Clearly, an index satisfying MON and TA1 can be regarded as a welfare function of the rich satisfying the Strong Pareto Principle and indicating preference for egalitarian bias. See Chakravarty (1993), for an earlier treatment of TA1. A more equal distribution of the rich will make the rich persons more homogeneous and socially cohesive. Consequently, the chances of conflict and confrontation among the rich are expected to reduce. The influence of this group on social decisions with a higher equal interest in many respects is likely to increase. Many of these social decisions may benefit the non-rich community as well. For instance, an improvement in the quality/service of a public good will benefit the entire population. On the other hand, satisfaction of TA2 by an index shows that it has a clear non-egalitarian bias. Given the income distribution of the rich, an index satisfying TA2 will achieve the maximum value if all the rich persons except the richest are set at the line of richness and the richest receives the remaining of the total incomes of the rich. If we consider richness as a source of power, then the view reflected by TA2 is quite sensible (see Atkinson, 2007 and Leigh, 2009). Note that both TA1 and TA2 are formulated in terms of progressive transfers and do not allow the set of rich persons to change. If we formulate the transfer axioms using a regressive transfer, that is, a transfer from a rich to a richer rich, then the donor may become non-rich, which in turn changes the set of rich persons. In this paper, for simplicity, we consider only the versions of TA1 and TA2 stated above.

SUD says that for any partitioning of population into several subgroups with respect to some characteristic such as age, sex, region etc., the population richness is simply the population share weighted average of subgroup richness levels. Repeated application of SUD shows that we can write the richness index as

$$R(x, z) = \frac{1}{n} \sum_{i=1}^{n} R(x_i, \rho), \tag{2}$$

where $R(x_i, \rho)$ is the richness level of person i, $x \in D^n$. Therefore, $R(x_i, \rho)$ can be referred to as the individual richness function. Note that the functional form of the individual richness index $R(x_i, \rho)$ does not depend on i.

SYM means that given the richness line, all characteristics other than incomes of the individual in the population is irrelevant to the measurement of richness. SYM along with TA1 (or TA2) implies that the underlying transfers are rank preserving. Increasingness (decreasingness) of the richness index under a rank preserving progressive transfer is equivalent to its strict S-concavity (S-convexity) (see Chakravarty, 2009).[1] PRI enables us to make cross-population comparison of richness.

While all the above axioms were stated under the assumption that ρ is given a priori, in some cases it becomes necessary to allow variability of ρ. The following axioms are stated under variability of ρ.

Decreasing Richness Line Axiom (DRL): For a given $x \in D^n$, $R(x, \rho)$, is decreasing in ρ.

Scale Invariance Axiom (SCI): For all $x \in D^n, \rho \in [\rho_0, \ \infty), R(x, \rho) = R(cx, c\rho)$, where $c > 0$ is any scalar such that $c\rho \in [\rho_0, \ \infty)$.

[1] A function $W : D^n \to \mathbf{R^1}$ is called S-concave if $W(xB) \geq W(x)$ for all bistochastic matrices B of order n. An $n \times n$ nonnegative matrix B is called bistochastic if each of its rows and columns sums to unity. For strict S-concavity of W, the weak inequality is to be replaced by a strict inequality whenever xB is not a permutation of x. W is S-convex (strictly S-convex) if $-W$ is S-concave (strictly S-concave). All S-concave and S-convex functions are symmetric.

Since an increase in the richness line may reduce the number of rich persons and also worsen the relative positions of the rich persons in comparison with the richness line, DRL is a quite reasonable requirement. SCI means that the richness index is homogeneous of degree zero in its arguments—it is a relative index. In the next section we show that these axioms are consistent in the sense that there exist indices that satisfy all of them.

While a relative index measures richness in terms of the proportionate gap between incomes of the rich and the line of richness ρ, often we may be interested in measuring richness using absolute excesses of incomes of the rich over ρ. Such indices are absolute indices. More formally, a richness index is called an absolute index if it remains invariant under equal absolute changes in all incomes and the richness line, that is, if it satisfies the following postulate:

Translation Invariance Axiom (TRA): For all $x \in D^n$, $\rho \in [\rho_0,\ \infty)$, $R(x, \rho) = R(x + c1^n, \rho + c)$, c being any scalar such that $\rho + c \in [\rho_0,\ \infty)$ and $x + c1^n \in D^n$, where 1^n is the n-coordinated vector of ones.

3 Richness-Index Orderings

The objective of this section is to rank alternative income distributions in terms of richness. The ranking criteria are developed assuming first that the richness line is fixed and then we allow variability of the richness line. The ranking of income distributions using strictly S-concave richness indices can be implemented via the generalized Lorenz curve. For any given income distribution x, its generalized Lorenz curve represents the cumulative income, expressed as an average of the population size, enjoyed by the bottom t $(0 \leq t \leq 1)$ proportion of the population. Formally, for $x \in D^n$, the generalized Lorenz curve of x is a plot of $GL(x, j/n) = \sum_{i=1}^{j} \overline{x}_i/n$ against j/n, where $GL(x, 0) = 0$.

For any $x, y \in D^n$, we say that x generalized Lorenz dominates y, we write $x \succeq_{GL} y$, if $GL(x, j/n) \geq GL(y, j/n)$ for all $j = 1, 2, \ldots, n$, with $>$ for at least one j. That is, the generalized Lorenz curve of x is nowhere below and at least at some place(s) above that of y. Note that the generalized Lorenz curve is population replication invariant. Thus, the generalized Lorenz curves of $x \in D^n$ and $x^l \in D^{nl}$ coincide, where x^l is the l-fold replication of x, with $l \geq 2$ being any positive integer. Therefore, $\succeq_{GL}$ also remains invariant under any desired replications of the concerned distributions.

The following result can now be stated:

Theorem 1: Let $x \in D^n$ and $y \in D^m$ be arbitrary. Then the following conditions are equivalent:

(i) $x^r \succeq_{GL} y^r$.

(ii) $W(x^r) > W(y^r)$ for all social welfare functions $W : D_R \to \mathbf{R^1}$ that are strictly S-concave, population replication invariant and increasing, where D_R is the set of all possible income distributions of the rich.

(iii) $R(x, \rho) > R(y, \rho)$ for all richness indices $R : D_R \times [\rho_0,\ \infty) \to \mathbf{R^1}$ that satisfy FOC, MON, TA1, PRI and SYM.

(iv) $\frac{1}{|\rho(x)|} \sum_{i \in \rho(x)} U(x_i) > \frac{1}{|\rho(y)|} \sum_{i \in \rho(y)} U(y_i)$, where U is increasing and strictly concave and $|\rho(x)|(|\rho(y)|)$ stands for the number of rich persons in $x(y)$.

Theorem 1 is quite general in the sense that it involves comparisons of richness of distributions over variable population sizes. Condition (i) of the theorem means that the generalized Lorenz curve of the distribution of the rich individuals in x is never below and at least at some place(s) above that of the rich individuals in y. Condition (ii) says that all efficiency preferring (that is, showing preference for higher total) and equity-oriented (as indicated by strict S-concavity) welfare functions of the rich make x^r socially better than y^r. Condition (iii) clearly shows that all richness indices satisfying the specified axioms regard x as richer than y. Condition (iv) is essentially a restatement of condition (ii) when the welfare function is additive. Equivalence between conditions (i) and (ii) was established by Shorrocks (1983). Marshall and Olkin (1979, p.12) demonstrated that conditions (i) and (iv) are equivalent. Theorem 2.1 of Chakravarty (2009) shows that all these three conditions are equivalent to (iii). Since the dominance condition $\succeq_{GL}$ is very easy to implement, the novelty of Theorem 1 is that, of two income distributions if the generalized Lorenz curve of incomes of the rich in one distribution dominates that in the other distribution we can unambiguously rank them by all richness indices that satisfy the conditions laid down in statement (iii). Calculation of any particular index for the purpose of ranking is not necessary. However, if the two curves intersect, the results of the theorem do not apply. Hence the ordering provided by the theorem is not complete, although it is transitive.

In order to illustrate the theorem, we now provide some examples of richness indices identified in condition (iii) of the theorem, assuming that there are rich persons. All these indices are based on the essential idea that they are increasing functions of the relative incomes $\frac{\overline{x}}{\rho}$ (Peichl et al., 2010). Increasingness is a necessary condition for MON to be fulfilled. The first example we consider is

$$R_\theta(x,\rho) = \begin{cases} \left[\frac{1}{n}\sum_{i=n-q+1}^{n}\left(\frac{\overline{x_i}}{\rho}-1\right)^\theta\right]^{1/\theta}, & \theta < 1, \quad \theta \neq 0 \\ \prod_{n-q=1}^{n}\left(\frac{\overline{x_i}}{\rho}-1\right)^{1/n}, & \theta = 0. \end{cases} \tag{3}$$

R_θ parallels the Atkinson (1970) index of inequality. It can be interpreted as the symmetric average of the relative excesses $\left(\frac{\overline{x}_i}{\rho}-1\right)$ of the rich incomes over the richness line ρ, where the averaging is done using the total population size. In addition to the axioms specified in Condition (iii) of Theorem 1, R_θ also satisfies CON, PRI, DRL and SCI. For any given income distribution a progressive transfer will increase the value of the index by a larger amount the lower is the value of θ. For any of $\theta < 1$, the richness contour will be strictly convex to the origin and it becomes increasingly convex as the value of θ decreases. When $\theta = 1$, the contour becomes a straight line. For $\theta = 1, R_\theta$ coincides with $R_H R_{RAG}$, the product of the head-count ratio $R_H = \frac{q}{n}$ and the relative affluence gap $R_{RAG} = \frac{1}{q}\sum_{i=n-q+1}^{n}\left(\frac{\overline{x_i}}{\rho}-1\right)$.

An alternative of interest arises from the following specification:

$$R_G(x,\rho) = \frac{1}{n^2}\sum_{i=n-q+1}^{n}\left(\frac{\overline{x}_i}{\rho}-1\right)(2(n-i)+1) = \frac{1}{n^2}\sum_{i=1}^{n}\left(\frac{x_i^*}{\rho}-1\right)(2(n-i)+1), \tag{4}$$

where it is assumed that there are q rich persons, $x_i^* = \rho$ if $i < n-q+1$ and $x_i^* = \overline{x_i}$ if $i \geq n-q+1$. Evidently, we can rewrite $R_G(x,\rho)$ as

$$R_G(x,\rho) = \frac{1}{n^2}\sum_{i=1}^{n}\frac{x_i^*}{\rho}(2(n-i)+1) - 1 = W_G\left(\frac{x_1^*}{\rho}, \frac{x_2^*}{\rho}, \ldots, \frac{x_n^*}{\rho}\right) - 1,$$

where $W_G\left(\frac{x_1^*}{\rho}, \frac{x_2^*}{\rho}, \ldots, \frac{x_n^*}{\rho}\right)$ is the Gini welfare function based on the relative income distribution $\left(\frac{x_1^*}{\rho}, \frac{x_2^*}{\rho}, \ldots, \frac{x_n^*}{\rho}\right)$. Since this index employs Gini-type aggregation, we refer to it as the Gini index of richness.

The above indices do not satisfy SUD. The following index suggested by Peichl et al. (2010) satisfies SUD:

$$R(x,\rho) = \frac{1}{n}\sum_{i\in\rho(x)} h\left(\frac{x_i}{\rho}\right), \tag{5}$$

where $h : [1,\ \infty) \to \mathbf{R}^1$ is continuous, increasing and strictly concave. Strict concavity of h is the requirement that R satisfies TA1. It also meets PRI, DRL and SCI. Clearly, the function h is the individual richness function defined on relative incomes of the rich. If $h(t) = \left(1 - \frac{1}{t}\right)^{\alpha}, 0 < \alpha < 1$ then the corresponding index in (5) reduces to the Foster-Greer-Thorbecke concave richness index. An increase in the value of α makes the index more sensitive to transfers at the lower end. On the other hand, for $h(t) = \left(1 - \frac{1}{t^{\beta}}\right),\ \beta > 0$ the index in (5) reduces to the Chakravarty index and in this case the sensitivity of the index to transfers at lower level increases as β increases (see Peich et al., 2010, for further discussion).

In order to present an ordering involving TA2, we define the affluence profile of any $x \in D^n$ as a plot of $AP\left(x, \frac{j}{n}\right) = \frac{1}{n}\sum_{i=1}^{j}\left(\frac{x_i'}{\rho} - 1\right)$ against $\frac{j}{n}$, $1 \le j \le n$, where $x_i' = \hat{x}_t$ if $\hat{x}_i > \rho$ and $x_i' = \rho$ if $\hat{x}_t \le \rho$. We write x' for the vector $(x_1', x_2', \ldots, x_n')$. The affluence profile, which is affluence counterpart to the Shorrocks (1998) deprivation profile, is non-decreasing and concave. The curve coincides with the horizontal axis if all the rich persons are at the line of richness. The larger the deviation of the curve from the horizontal axis, the greater is the extent of richness. The curve becomes flat in the region where x_i''s coincide with ρ. The head count ratio is the population proportion at which the curve becomes flat. The non-flat portion of the curve indicates existence of inequality among the rich. The maximum height of the curve is the product of the head-count ratio and the relative affluence gap. Thus, the curve provides three important information on richness – the head count ratio, the relative affluence gap and existence of inequality in the income distribution of the rich. See Jenkins and Lambert (1997, 1998a, 1998b) for a similar discussion on poverty.

For any $x, y \in D^n$, we say that x affluence profile dominates y and we write $x \succeq_{AP} y$, if $AP(x', j/n) \ge AP(y', j/n)$ for all $j = 1, 2, \ldots, n$, with $>$ for at least one j. Clearly, the ordering $\succeq_{AP}$ remains invariant under replications of the concerned distributions. The following theorem can now be stated :

Theorem 2: Let $x \in D^n$, $y \in D^m$ be arbitrary. Then the following conditions are equivalent:

(i) $x \succeq_{AP} y$,

(ii) $H\left(\left(\frac{x_i'}{\rho}-1\right),\left(\frac{x_2'}{\rho}-1\right),\ldots,\left(\frac{x_n'}{\rho}-1\right)\right) > H\left(\left(\frac{y_1'}{\rho}-1\right),\left(\frac{y_2'}{\rho}-1\right),\ldots,\left(\frac{y_m'}{\rho}-1\right)\right)$ for all richness indices $H : \Gamma \to \mathbf{R}^1$ that are increasing, population replication invariant and strictly S-convex.

(iii) $H\left(\left(\frac{x_i'}{\rho}-1\right),\left(\frac{x_2'}{\rho}-1\right),\ldots,\left(\frac{x_n'}{\rho}-1\right)\right) > H\left(\left(\frac{y_1'}{\rho}-1\right),\left(\frac{y_2'}{\rho}-1\right),\ldots,\left(\frac{y_m'}{\rho}-1\right)\right)$ for all richness indices $H : \Gamma \to \mathbf{R}^1$ that satisfy MON, TA2, PRI and SYM.

Condition (iii) of the theorem says that an arbitrary richness index defined on individual relative affluence gaps satisfying MON, TA2, PRI and SYM regards x richer than y. Since TA2 and SYM is same as strict S-convexity and MON is same as increasingness, condition (ii) is a restatement of condition (iii). This is equivalent to the condition that the individual relative affluence gaps x affluence profile dominates y (condition (i)). The proof of equivalence between conditions (i) and (ii) can be found in Marshall and Olkin (1979, p.12 and p.60).

The Atkinson index of richness in this case is obtained by replacing $\overline{x}_i$ by x_i' in (3) and the parametric restriction is $\theta > 1$. Likewise, in the Gini index given by (4) we replace x_i^* by x_i'. An additive index considered by Peich et al. (2010) that satisfies TA2 is the Foster-Greer-Thorbecke index for which $h(t) = \left(1 - \frac{1}{t}\right)^{\alpha}$, where $\alpha > 1$.

Note that condition (iii) of Theorem 1 holds for both relative and absolute indices. Thus, Theorem 1 applies to both relative and absolute indices. We may therefore present now some examples of absolute indices. The absolute counterpart to the Atkinson index is

$$RA_\theta(x,\rho) = \begin{cases} \left[\frac{1}{n}\sum_{i=n-q+1}^{n}(\overline{x}_i-\rho)^\theta\right]^{1/\theta}, \ \theta < 1, \ \theta \neq 0. \\ \prod_{i=n-q+1}^{n}(\overline{x}_i-\rho)^{1/n}, \theta = 0. \end{cases} \tag{6}$$

For $\theta = 1$, nRA_θ determines the total amount of excess income of the rich over the line of richness. Therefore, if a policy maker desires to put all the rich persons at the line of richness and use their excess incomes over the line of richness for some socially beneficial investment, then for $\theta = 1$, nRA_θ provides the necessary information on the available fund. This is quite useful from policy perspective.

The Gini absolute index of richness is obtained by multiplying $R_G(x,\rho)$ by ρ. The subgroup decomposable absolute indices will be of the form

$$RA(x, \rho) = \frac{1}{n} \sum_{i \in \rho(x)} h(x_i - \rho), \tag{7}$$

where $h : [0,\ \infty) \to \mathbf{R^1}$ is continuous, increasing and strictly concave (or convex). Thus, the Foster-Greer-Thorbecke absolute richness index satisfying $TA1$ ($TA2$) is given by

$$\frac{1}{n} \sum_{i \in p(x)} (x_i - \rho)^{\alpha}, \tag{8}$$

where $0 < \alpha < 1 (\alpha > 1)$.

The affluence profile considered in Theorem 2 relies on relative excesses of incomes over ρ. We can similarly define the absolute affluence profile using absolute excesses over ρ. This is obtained by multiplying the affluence profile by ρ. More formally, the absolute affluence profile of any $x \in D^n$ is a plot of $AAP\left(x, \frac{i}{q}\right) = \frac{1}{n} \sum_{i=1}^{j} (x'_i - \rho)$ against $\frac{j}{n}$, $1 \geq j \geq n$. We then say that for any $x, y \in D^n$, x absolute affluence profile dominates y (we then write $x \succeq_{AAP} y$), if $AAP(x', j/n) \geq AAP(y', j/n)$ for all $j = 1, 2, \ldots, n$, with $>$ for at least one j. The following theorem, which is the absolute version of Theorem 2, can then be stated.

Theorem 3: Let $x \in D^n$ $y \in D^m$ be arbitrary. Then the following conditions are equivalent:

(i) $x \succeq_{AAP} y$,

(ii) $H((x'_1 - \rho), (x'_2 - \rho), \ldots, (x'_n - 1)) > H((y'_1 - \rho), (y'_2 - \rho), \ldots, (y'_m - \rho))$ for all richness indices $H : \Gamma \to \mathbf{R^1}$ that are increasing, population replication invariance and strictly S-convex.

(iii) $H((x'_1 - \rho), (x'_2 - \rho), \ldots, (x'_n - 1)) > H((y'_1 - \rho), (y'_2 - \rho), \ldots, (y'_m - \rho))$ for all richness indices $H : \Gamma \to \mathbf{R^1}$ that satisfy MON, $TA2$, PRI and SYM.

The theorem provides a way of ranking income distributions by all absolute richness indices satisfying certain axioms.

4 Richness-Line Orderings

In this section we assume that there is a continuum of population. Let $F : [0,\ \infty) \rightarrow [0,\ 1]$ be the cumulative distribution function of income. Then $F(v)$ is the cumulative proportion of persons whose incomes do not exceed the level v. We assume that F is non-decreasing, continuously differentiable and $F(0) = 0$. Write f for the derivative of F, that is, f is the density function associated with F.

For any two distribution functions $F, G : [0,\ \infty) \rightarrow [0,\ 1]$, we say that F first order stochastic dominates G over $[\rho_0,\ \infty)$ if

$$F^*(\rho) \leq G^*(\rho) \tag{9}$$

for all $\rho \in [\rho_0,\ \infty)$ with $>$ for some $\rho \in [\rho_0,\ \infty)$, where

$$F^*(\rho) = \frac{F(\rho) - F(\rho_0)}{1 - F(\rho_0)}, \quad \rho \in [\rho_0,\ \infty)$$

and G^* is defined similarly. We can rewrite this inequality as

$$\frac{1 - F(\rho)}{1 - F(\rho_0)} \geq \frac{1 - G(\rho)}{1 - G(\rho_0)}$$

for all $\rho \in [\rho_0,\ \infty)$ with $>$ for some $\rho \in [\rho_0,\ \infty)$. This means that for each line of affluence the proportion of rich persons under G is not higher than that under F and for at least one line of affluence G has a lower proportion of rich. In other words, F dominates G by the head-count ratio. This is same as the requirement that the graph of $\frac{1-F(\rho)}{1-F(\rho_0)}$ lies nowhere above, and somewhere below, that of its G-counterpart over $[\rho_0,\ \infty)$. Equivalently, average utility under F^* is higher than that under G^*, that is,

$$\int_{\rho}^{\infty} U(\rho) \frac{f(\rho)}{1 - F(\rho_0)} d\rho > \int_{\rho_0}^{\infty} U(\rho) \frac{g(\rho)}{1 - G(\rho_0)} d\rho, \tag{10}$$

where the utility function $U : [\rho_0,\ \infty) \rightarrow \mathbf{R}^1$ is increasing (see Hanoch and Levy, 1969). That is, all utilitarians who approve of efficiency (since U is increasing) as the only distinguishing criterion between two distributions prefer F to G.

While the first order dominance is expressed in terms of the head count ratio, we now look at implication of the second order stochastic dominance. For distribution functions $F, G : [0,\ \infty) \rightarrow [0,\ 1]$, F is said to second order stochastic dominate G over $[\rho_0,\ \infty)$ if

$$\int_{\rho_0}^{\rho} F^*(t) dt \leq \int_{\rho_0}^{\rho} G^*(t) dt \tag{11}$$

for all $\rho \in [\rho_0,\ \infty)$ with $<$ for some $\rho \in [\rho_0,\ \infty)$. This is equivalent to the condition that

$$\int_{\rho_0}^{\infty} U(\rho)\frac{f(\rho)}{1-F(\rho_0)}d\rho > \int_{\rho_0}^{\infty} U(\rho)\frac{g(\rho)}{1-G(\rho_0)}d\rho, \tag{12}$$

where the utility function $U : [\rho_0,\ \infty) \to \mathbf{R}^1$ is increasing and strictly concave. In other words, F is preferred to G by all utilitarians who have likings for both efficiency (since U is increasing) and equity (since U is strictly concave).

Now,

$$\int_{\rho_0}^{\rho} F^*(t)dt = \int_{\rho_0}^{\rho}\int_{\rho_0}^{t} \frac{f(v)}{1-F(\rho_0)}dv\ dt,\ \text{where } \rho_0 \leq \rho.$$

Interchanging the order of integration, we get

$$\begin{aligned}
\int_{\rho_0}^{\rho}\int_{\rho_0}^{t} \frac{f(v)}{1-F(\rho_0)}dv\ dt &= \frac{1}{1-F(\rho_0)}\int_{\rho_0}^{\rho} f(v)\int_{v}^{\rho} dt\ dv \\
&= \frac{1}{1-F(\rho_0)}\int_{\rho_0}^{\rho} f(v)(\rho - v)dv \qquad (13)\\
&= \rho\left(\frac{1}{1-F(\rho_0)}\right)\int_{\rho_0}^{\rho} f(v)\left(1-\frac{v}{\rho}\right)dv \\
&= \rho\left(\frac{1}{1-F(\rho_0)}\right)\int_{\rho_0}^{\infty} f(v)\left(1-\frac{v}{\rho}\right)^{+}dv \\
&= \rho E_{F^*}\left(1-\frac{X}{\rho}\right)^{+} \qquad (14)
\end{aligned}$$

where X is a random variable with distribution function F^* and E stands for expectation. So, second order stochastic dominance of F over G is equivalent to

$$E_{F^*}\left(1-\frac{X}{\rho}\right)^{+} \leq E_{G^*}\left(1-\frac{Y}{\rho}\right)^{+} \tag{15}$$

where $X \sim F^*$ and $Y \sim G^*$.

The expression in (14) represents the expected value of the income shortfall of the population proportion in the income interval $[\rho_0,\ \rho]$ from the maximum income level ρ multiplied by ρ, where expectation is taken under F^*. Thus, second order stochastic dominance of F over G is the requirement that this shortfall under F^* is not higher than that under G^* at all lines of richness $\rho \geq \rho_0$ and for some values of ρ it is strictly lower.

We can rewrite (13) as

$$\frac{\rho\int_{\rho_0}^{\rho} f(v)dv - \int_{\rho_0}^{\rho} vf(v)dv}{1-F(\rho_0)} = \frac{\rho[F(\rho)-F(\rho_0)] - \int_{\rho_0}^{\rho} vf(v)dv}{1-F(\rho_0)}. \tag{16}$$

Now, the quantity $\frac{[F(\rho)-F(\rho_0)]}{1-F(\rho_0)}$ is the cumulative proportion of persons whose incomes lie between ρ_0 and $\rho, \rho_0 \leq \rho$. Thus, $\frac{\rho[F(\rho)-F(\rho_0)]}{1-F(\rho_0)}$ is the total income of this proportion of population when everybody belonging to this proportion enjoys the income level ρ. On the other hand, $\frac{\int_{\rho_0}^{\rho} vf(v)dv}{1-F(\rho_0)}$ is the average income of the individuals with incomes in the interval $[\rho_0,\ \rho]$. Hence the gap $\frac{\rho\left[F(\rho)-F(\rho_0)\right]-\int_{\rho_0}^{\rho} vf(v)dv}{1-F(\rho_0)}$ represents the excess of the income of the population proportion in the interval $[\rho_0,\ \rho]$ when everybody in the proportion possesses the maximum income ρ over the proportion's average income. Thus, the second order stochastic dominance condition defined in (11) is equivalent to the condition that for every $\rho \geq \rho_0$, the excess income under F cannot be higher than that under G and in at least one case the excess is lower.

When F does not necessarily have a density, the discussion above remains valid–at all places, one simply needs to replace $f(v)dv$ by $dF(v)$ and $g(v)dv$ by $dG(v)$

5 An Empirical Illustration

This section provides an empirical application of the orderings and indices introduced above. We base the analysis on German data on the years 2002, 2006 to 2009. The German Socio-Economic Panel (SOEP) is an ongoing panel survey with a yearly re-interview design (see http://www.diw.de/gsoep). The starting sample in 1984 was almost 6,000 households based on a random multi-stage sampling design. A sample of about 2,200 East German households was added in June 1990, half a year after the fall of the Berlin wall and of the German currency, social and economic unification which happened on July 1, 1990. In 1994/95 an additional subsample of 500 immigrant households was included to capture the massive influx of immigrants since the late 1980s. Finally, in 1998 and 2000 two more random samples were added which increased the overall number of interviewed households in 2000 to about 13,000 with approximately 24,000 individuals aged 17 and over. In the year 2002, SOEP added an additional representation of high-income households, initially covering 1,224 households with 2,671 respondents. This sample was chosen by defining *rich* someone whose monthly net household income was more than 4,500 Euro (7,500 DM). For this reason our period of analysis starts in 2002 and as richness line for yearly household income we use 54,000 Euro, the (approximately) yearly equivalent of 4,500 Euro. The income measure we investigate is yearly net household income. In order to compare income over time, all income measures are deflated to 2000 prices, also accounting for purchasing power differences between East and West Germany. Our sample is composed of 31232 individuals in 2002, 29193 in 2006 and 27217 in 2009. The number of rich individuals is 7139 in 2002, 5577 in 2006 and 4729 in 2009.

Results for Germany in the analyzed years are quite striking: with the richness-index orderings we offered we are able to rank unambiguously the distributions of the three years. Following the results of Theorem 1, the distribution of the rich individuals in 2006 dominates all others; while 2009 dominates 2002. In these cases all richness indices that satisfy the conditions laid down in statement (iii) will agree and calculation of any particular index for the purpose of ranking is not necessary. Similarly according to the results of Theorems 2 and 3, the affluence profile and absolute affluence profile of 2002 dominate those of the other two years, while 2006 dominates 2009.

To apply the richness-line dominance results we fit adaptive kernel densities and distributions for each of our selected years. We then check for dominance for all $\rho \in [54,000,\ 648,000]$, where the first value is the line of affluence we adopted and the second value is the extreme value of the common support of the three years. The results are plotted in Figures 3 and 4. There is no dominance according to the head-count ratio as the differences between the proportion of rich persons over the years assumes both positive and negative values. The distribution functions of the various years do cross when the affluence line varies over the interval. Results are more neat when we check for second order stochastic dominance. 2009 is dominated by both years while no definite conclusions can be drawn for 2002 and 2006. As opposed to the first two years, 2009 is during the Great Recession and the reduction of the excess income of the rich is expected.

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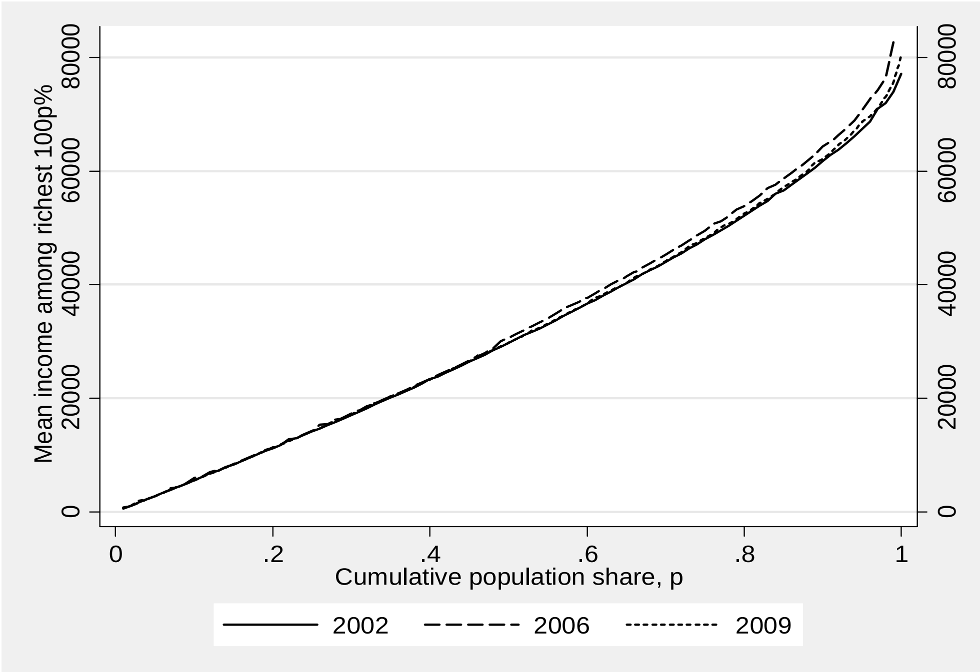

Figure 1: Generalized Lorenz curves among the rich in Germany for the years 2002, 2006 and 2009.

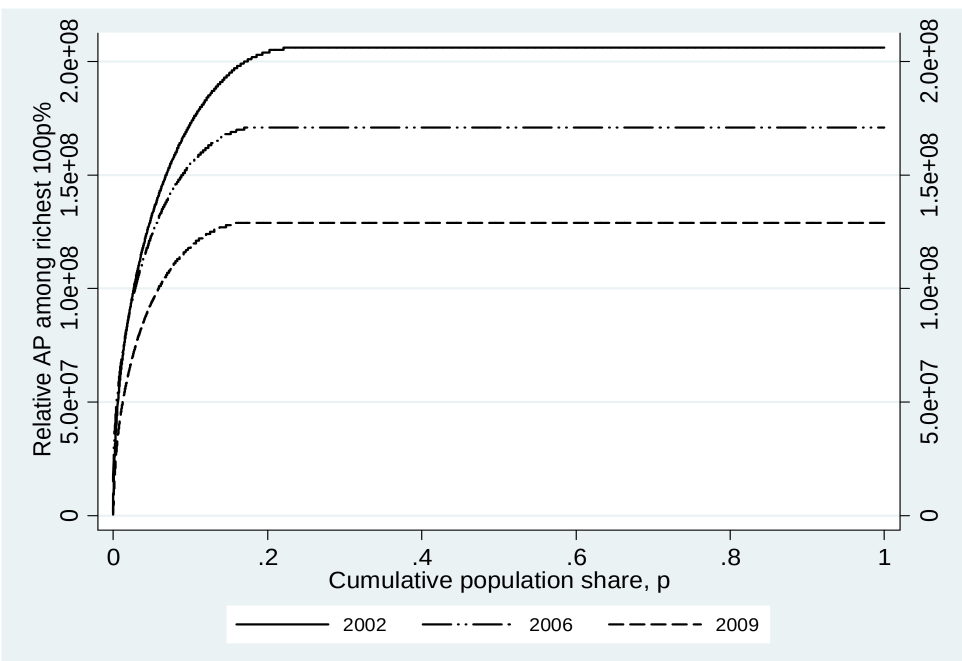

Figure 2: Affluence profiles in Germany for the years 2002, 2006 and 2009.

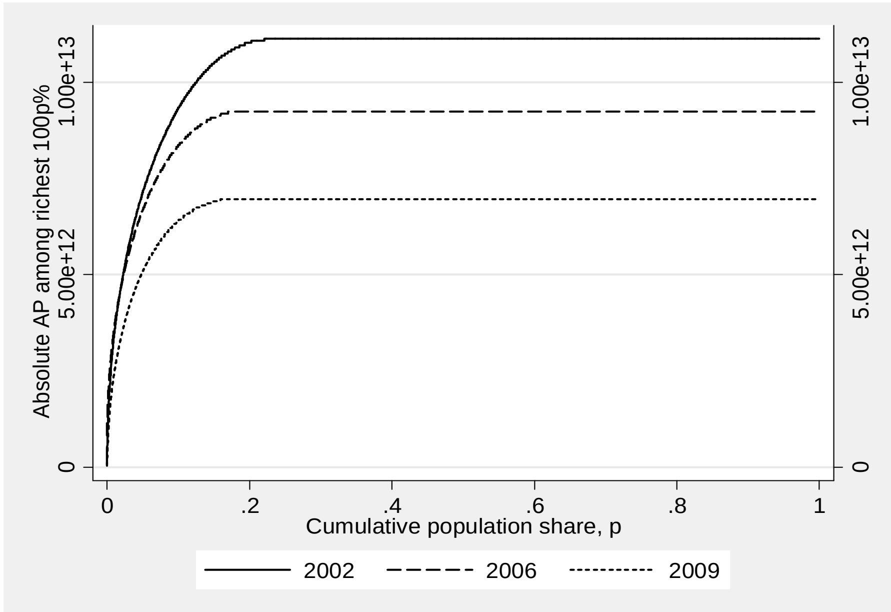

Figure 3: Absolute affluence profiles in Germany for the years 2002, 2006 and 2009.

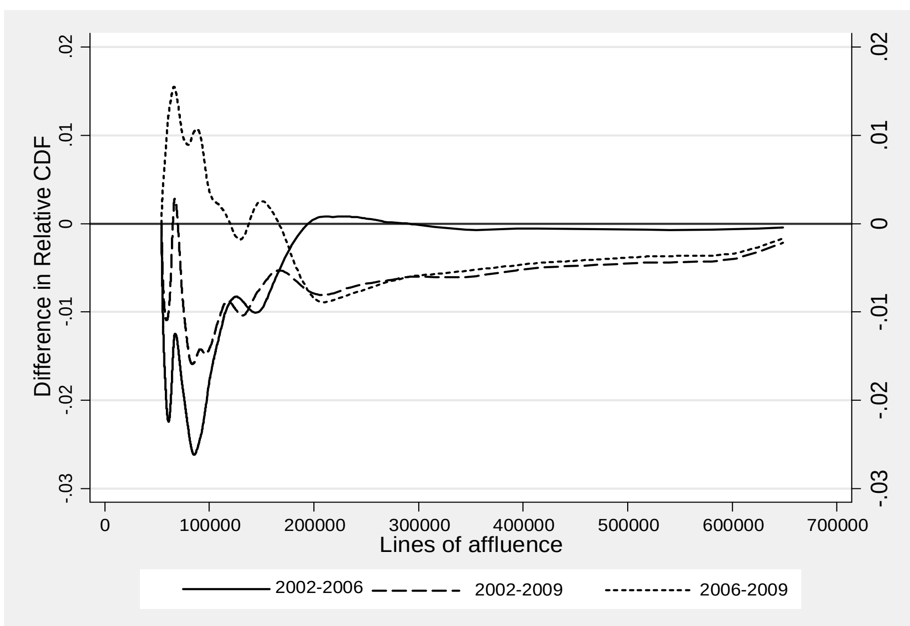

Figure 4: Differences in the proportion of rich persons for different lines of affluence in Germany for the years 2002, 2006 and 2009.

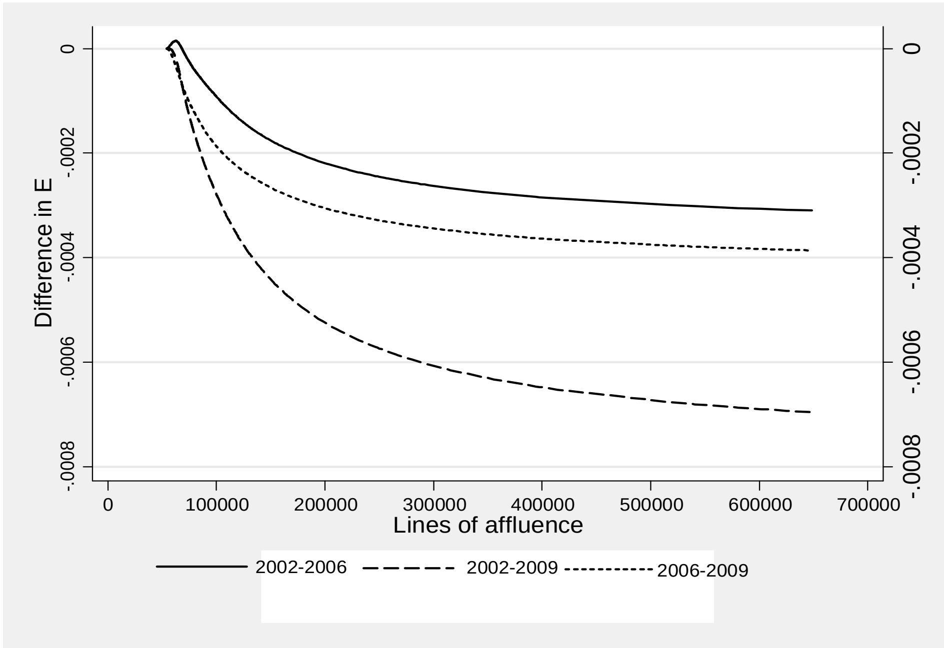

Figure 5: Differences in the expected value of the income shortfall of the population for different lines of affluence in Germany for the years 2002, 2006 and 2009.