On Statistical Efficiency of Robust Estimators of Multivariate Location

by

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Abstract

Univariate median is a well-known location estimator, which is $\sqrt{n}$-consistent, asymptotically Gaussian and affine equivariant. It is also a robust estimator of location with the highest asymptotic breakdown point (i.e., 50%). While there are several versions of multivariate median proposed and extensively studied in the literature, many of the aforesaid statistical properties of univariate median fail to hold for some of those multivariate medians. Among multivariate medians, the affine equivariant versions of spatial and co-ordinatewise medians have 50% asymptotic breakdown point, and they have asymptotically Gaussian distribution. The minimum covariance determinant (MCD) estimator is another widely used robust estimator of multivariate location, which is also affine equivariant, has 50% asymptotic breakdown point, and its asymptotic distribution is Gaussian. In this article, we make a comparative study of the efficiencies of affine equivariant versions of spatial and co-ordinatewise medians and the efficiencies of MCD and related estimators considered in the literature.

Keywords and phrases: Asymptotic efficiency, elliptically symmetric distributions, multivariate distributions with polynomial tails, transformation re-transformation technique.

1 Introduction

For univariate observations $X_1, X_2, \ldots, X_n$, it is well-known that the sample median $\theta = \hat{\theta}_n$ minimizes the sum of the absolute deviations $\sum_{i=1}^{n} |X_i - \theta|$. For multivariate observations $\mathbf{X}_1$, $\mathbf{X}_2$, ..., $\mathbf{X}_n$, if we use the l_1-norm $(|.|_1)$, the minimizer $\theta = \hat{\theta}_n^{(1)}$ of $\sum_{i=1}^{n} |\mathbf{X}_i - \theta|_1$ turns out to be the vector of co-ordinatewise medians (see, e.g., Babu and Rao [3], who investigated the marginal quantiles of a random vector). On the other hand, if we use the l_2-norm $(|.|_2)$, the spatial median $\theta = \hat{\theta}_n^{(2)}$ will be the minimizer of $\sum_{i=1}^{n} |\mathbf{X}_i - \theta|_2$ (see, e.g., Brown [5]). Both of these two versions of multivariate medians have 50% asymptotic breakdown point and asymptotically Gaussian distributions. However, both of them perform poorly when the components of the data vectors are correlated because of the lack of affine equivariance of these two location estimators. One may fix this problem by adopting the transformation re-transformation (TR) approach (see Chakraborty and Chaudhuri [8] and Chakraborty, Chaudhuri and Oja [10]). The transformation re-transformation technique makes the non-equivariant co-ordinatewise and spatial medians affine equivariant preserving their 50% asymptoic breakdown point and asymptotically normal distributions (see Chakraborty and

Chaudhuri [8], Chakraborty and Chaudhuri [9] and Chakraborty et al. [10]).

Some of the very well-known and extensively studied versions of multivariate median are simplicial volume median of Oja [20], Tukey's half-space median (see Donoho and Gasko [14]) and simplicial median of Liu [18] having breakdown points 0, 1/3 and $1/(d+2)$, respectively (see Oja, Niinimaa and Tableman [22], Chen [11] and Chen [12]). Further, while the asymptoic distributions of both of Oja's and Liu's medians are Gaussian (see Oja and Niinimaa [21] and Arcones, Chen and Gine [2], respectively), the asymptotic distribution of Tukey's half space median happens to be non-Gaussian (see Masse [19]).

Among other high breakdown estimators of multivariate location, minimum volume ellipsoid (MVE) estimator is widely studied, and it is defined as the center of the ellipsoid with the smallest volume covering 50% of the data (see Rousseeuw and Leroy [24]). While it has 50% asymptoic breakdown point, it is neither $\sqrt{n}$-consistent nor does it have asymptotically Gaussian distribution. To repair those undesirable asymptotic properties of MVE estimator, minimum covariance determinant (MCD) estimator was introduced by Rousseeuw, which is defined as the mean of the 50% data points for which the determinant of the empirical covariance matrix is minimum (see e.g., Rousseeuw and Leroy [24]). MCD estimator has 50% asymptoic breakdown point, and it is a $\sqrt{n}$-consistent estimator with asymptotically Gaussian distribution (see Butler, Davies and Juhn [6]).

In the next two sections, we make a comparative study of the efficiencies of TR spatial and co-ordinatewise medians and the efficiencies of MCD estimator and some of its variants considered in the literature. All of these multivariate location estimators are extensively studied and widely used. While some efficiency study already exists in the literature for co-ordinatewise and spatial medians in the case of data with multivariate Gaussian distribution, not much is reported in the literature regarding the asymptotic efficiency of those estimates for data following other elliptic distributions with heavier tails. Even for data with multivariate Gaussian distribution, we are not aware of any detailed efficiency study comparing MCD estimator with TR co-ordinatewise and spatial medians.

2 A comparative study of Asymptotic efficiencies

For n i.i.d. observations generated from an elliptically symmetric density function $(\det \Sigma)^{-1/2} g\{(\mathbf{X}-\theta)^T \Sigma^{-1} (\mathbf{X}-\theta)\}$, where Σ is a positive definite matrix, θ is a location parameter and $g(|\mathbf{x}|_2)$ is a spherically symmetric probability density function, the $\sqrt{n}$-consistency and the asymptotic normality of TR spatial and co-ordinatewise medians follow from Chakraboty et al. [10] and Chakraborty and Chaudhuri [8], respectively. From the results in those papers, we have that the asymptotic dispersion matrices of TR spatial and TR co-ordinatewise medians are of the form $(\sigma_1^2/n)\Sigma$ and $(\sigma_2^2/n)\Sigma$, respectively, where $\sigma_1^2 = \frac{d\{\Gamma(d/2)\}^2}{g_1^2(0)\pi(d-1)^2\{\Gamma(d/2-1/2)\}^2}$ and $\sigma_2^2 = \frac{1}{4g_1^2(0)}$. Here d is the dimension, and g_1 is the marginal density function of $g(|\mathbf{x}|_2)$.

Similarly, it follows from Theorem 4 in Butler et al. [6] that for n i.i.d. observations from the elliptically symmetric density functions $(\det \Sigma)^{-1/2} g\{(\mathbf{X}-\theta)^T \Sigma^{-1}(\mathbf{X}-\theta)\}$, MCD estimator is $\sqrt{n}$-consistent and has asymptotically normal distribution. Its asymptotic dispersion is of the form $(\sigma_3^2/n)\Sigma$, where $\sigma_3^2 = \frac{d\Gamma(d/2)\int_0^m r^{d+1} g(r^2) dr}{8\pi^{d/2}\left\{\int_0^m r^{d+1} g'(r^2) dr\right\}^2}$. Here g' is the first derivative of g, and m is such that $\frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^m r^{d-1} g(r^2) dr = 1/2$.

Theorem 1: *For n i.i.d. observations from multivariate Gaussian distribution, we have $\sigma_1^2 < \sigma_2^2 < \sigma_3^2$ for all $d \geq 2$.*

In the following table, we have presented the asymptotic efficiencies of TR spatial and co-ordinatewise medians and MCD estimator relative to multivariate sample mean (i.e., the maximum likelihood estimator for the location), when data are generated from multivariate normal distribution.

Table 1: The asymptotic efficiencies of TR spatial median, TR coordinatewise (CW) median and MCD estimator relative to multivariate sample mean for data with multivariate Gaussian distribution.

	$d=2$	$d=5$	$d=10$	$d=20$	$d=50$	$d=\infty$
TR spatial median	78.5%	90.5%	95.1%	97.5%	99%	100%
TR CW median	63.6%	63.6%	63.6%	63.6%	63.6%	63.6%
MCD	15.3%	26.2%	32.7%	37.6%	42.1%	50%

It will be appropriate to note here that Arcones et al. [2] and Oja and

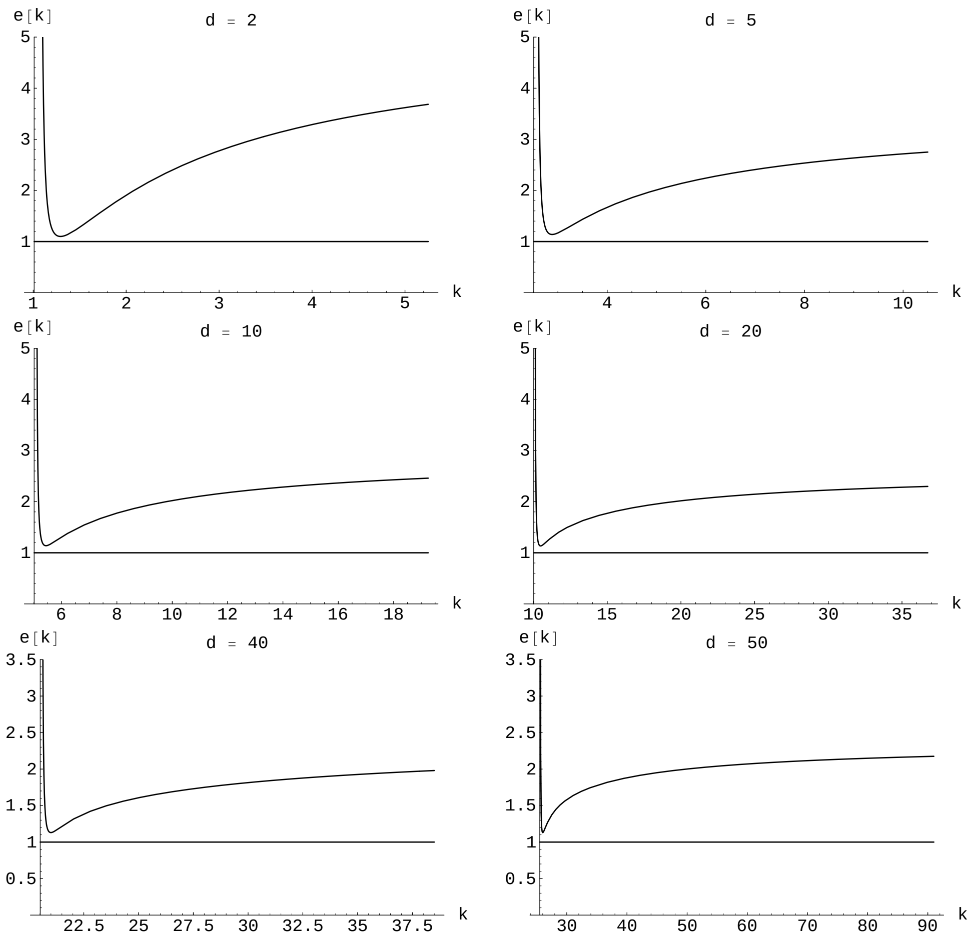
Niinimaa [21] showed that the asymptotic efficiency of Liu's median and that of Oja's median are exactly same as that of TR spatial median for data following multivariate normal distribution though both of Liu's and Oja's medians have inferior robustness properties compared to TR spatial and coordinatewise medians.

The asymptotic efficiency of an estimate of a location parameter greatly depends on the behavior of the tail of the distribution of the data. It is of interest to investigate the asymptotic efficiency of location estimates for data having non-Gaussian distributions with heavier tails. Here we consider multivariate t-type distributions (see Fang, Kotz and Ng [15]), which have densities of the form $f(\mathbf{x}) = \frac{\Gamma(k)}{\pi^{d/2}\Gamma(k-d/2)} \frac{1}{\{1+(\mathbf{x}-\theta)^T\Sigma^{-1}(\mathbf{x}-\theta)\}^k}, \mathbf{x} \in R^d, d/2 < k < \infty$. These are elliptically symmetric densities with polynomial tails. Multivariate Cauchy density (i.e., $k = (d+1)/2$, see Fang et al. [15]) is included in this family of densities as a special case. Besides, as $k \to \infty$, $f(\mathbf{x})$ tends to multivariate Gaussian density with location θ and dispersion Σ.

Theorem 2: *Consider n i.i.d. observations from a multivariate t-type density as described above. Then, $\sigma_1^2 < \sigma_2^2$ for all $d \geq 2$ and $k > d/2$. Further, $\sigma_3^2 > \sigma_1^2$ for all $d \geq 2$ and $k > \frac{(d+1)}{(2-\sqrt{2})}$, and $\sigma_3^2 > \sigma_2^2$ for all $d \geq 2$ and $k > \frac{(d+1)\Gamma(d/2)}{2\sqrt{d}\Gamma(d/2)-\sqrt{2\pi}\Gamma((d+1)/2)}$.*

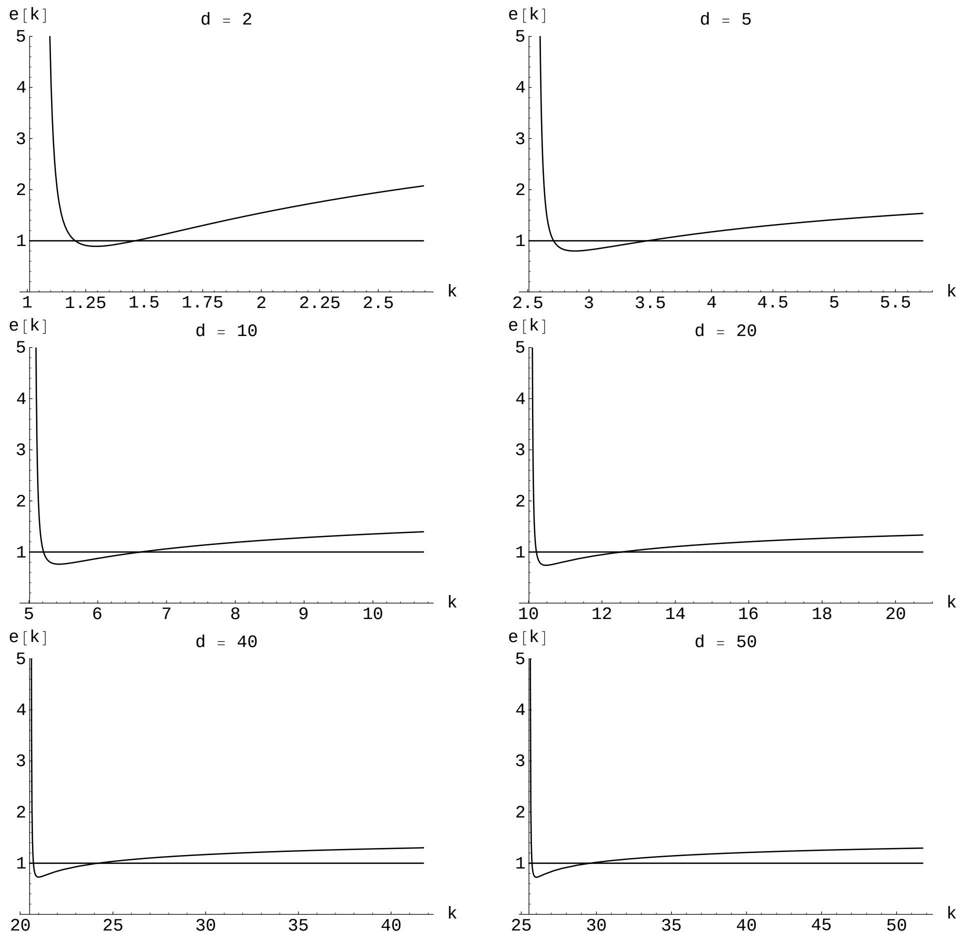
The asymptotic efficiency of TR spatial median relative to MCD estimator can be computed numerically using the software package *Mathematica* for specified values of d and k. In Figure 1, we have plotted the asymptotic efficiency of TR spatial median relative to MCD estimator for $d/2 < k \leq \frac{(d+1)}{(2-\sqrt{2})}$ and $d = 2$, 5, 10, 20, 40 and 50.

Figure 1: Graphs of asymptotic efficiency ($e[k]$) of TR spatial median relative to MCD estimator when $d/2 < k \leq \frac{(d+1)}{(2-\sqrt{2})}$. Each diagram in the figure corresponds to a specific dimension.



In Figure 2, we have plotted the asymptotic efficiency of TR co-ordinatewise median relative to MCD estimator for $d/2 < k \leq \frac{(d+1)\Gamma(d/2)}{2\sqrt{d}\Gamma(d/2)-\sqrt{2\pi}\Gamma((d+1)/2)}$ and $d = 2$, 5, 10, 20, 40 and 50.

Figure 2: Graphs of asymptotic efficiency ($e[k]$) of TR co-ordinatewise median relative to MCD estimator when $d/2 < k \leq \frac{(d+1)\Gamma(d/2)}{2\sqrt{d}\Gamma(d/2)-\sqrt{2\pi}\Gamma((d+1)/2)}$. Each diagram in the figure corresponds to a specific dimension.



It follows from Figures 1 and 2 that while TR spatial median outperforms MCD estimator in terms of asymptotic efficiency in all the cases considered by us, MCD estimator outperforms TR co-ordinatewise median for some values of d and k.

3 Finite sample simulation study

In this section, we investigate finite sample performance of TR spatial and co-ordinatewise medians, and MCD estimator. It has been claimed by Willems et al. [25] that a reweighted version of MCD estimator has the same robustness but better efficiency properties, and we have included this estimator also in our simulation study. We generated $m = 1000$ samples from standard multivariate normal and Cauchy distributions (see Fang et al. [15]) for dimensions d =2 and 3 and sample sizes n = 10, 20 and 30. For an equivariant location estimator T, we have calculated its empirical mean squared error (EMSE) as $(1/m)\sum_{i=1}^{m} |T_i - \theta|_2^2$, where θ is the zero vector in R^d, and T_i is the estimate based on the i-th sample. The finite sample efficiency of any location estimator relative to another estimator is obtained by the ratio of their EMSEs. Simulation results are summarized in Table 2.

As indicated by the figures in Table 2, the TR spatial median performs better than both of MCD estimator and its reweighted version in all the cases considered in our simulation study. It is interesting to observe here that the reweighted MCD estimator (see Willems et al. [25]) performs better than the usual MCD estimator for data generated from multivariate Gaussian distribution but the performance of the reweighted MCD estimator is worse than that of the usual MCD estimator when data follow multivariate Cauchy distribution.

Table 2: Relative efficiencies of TR co-ordinatewise (CW) median, MCD estimator and its reweighted version relative to TR spatial median.

	Multivariate normal					
	$d = 2$			$d = 3$		
	n=10	n=20	n=30	n=10	n=20	n=30
TR CW median	75%	77%	78%	66%	68%	70%
MCD	27%	28%	29%	21%	23%	25%
Reweighted MCD	56%	58%	58%	50%	50%	53%
	Multivariate Cauchy					
	$d = 2$			$d = 3$		
	n=10	n=20	n=30	n=10	n=20	n=30
TR CW-median	74%	74%	76%	65%	67%	68%
MCD	69%	70%	76%	67%	67%	69%
Reweighted MCD	45%	43%	46%	41%	41%	42%

The procedure given in Chakraborty et al. [10], which uses minimum covariance determinant estimator of the dispersion matrix, has been used for computing TR medians in this section as well as in the next section. The optimization problem described in Chakraborty et al. [10, p.770] has been solved by complete enumeration. The computation of spatial and co-ordinatewise medians are done by the "depth" package in the software *R*.

For computing MCD estimator and its and reweighted version, we have used the "rrcov" package, which is also available in the software R.

4 Analysis of real data

In this section, we investigate the performance of TR spatial and co-ordinatewise medians, MCD estimator and its reweighted version when applied on two real data sets, namely, the Iris data and the diabetes data. The Iris data is available from http://archive.ics.uci.edu/ml, and the diabetes data can be obtained from the "mclust" package in the software R. In each case, we have used $m = 1000$ bootstrap replications to estimate the standard error of each real valued component of a multivariate location estimate. Further, the overall bootstrap mean square error (BMSE) of a multivariate location estimator can be computed as $\frac{1}{m}\sum_{i=1}^{m}|T_i - \bar{T}|_2^2$, where T_i is the estimate of the location parameter in the i-th bootstrap replication, and $\bar{T} = (1/m)\sum_{i=1}^{m} T_i$. Then, the bootstrap efficiency of a multivariate location estimator relative to another can be obtained by the ratio of their BMSEs. The results obtained for this two data sets are reported below.

The first data set actually consists of three multivariate data sets corresponding to three different varieties of Iris, namely, *Iris setosa*, *Iris virginica* and *Iris versicolor*, each of size 50. In each data point, there are four measurements, namely, sepal length, sepal width, petal length and petal width. In Table 3, we have reported the values of different location estimates and their standard errors estimated by the bootstrap for different variables in different Iris species. In Table 4, we have reported the bootstrap efficiencies of TR co-ordinatewise median, MCD estimator and its reweighted version relative to TR spatial median for the three Iris species.

The second data set was originally analysed by Reven and Miller [23]. This data set contains measurements on three variables, namely, glucose level, insulin area and steady state plasma glucose level for 145 individuals. These individuals are classified into three classes according to some clinical criteria. These classes consist of normal individuals, chemical diabetes cases and overt diabetic individuals. There are 76, 36 and 33 individuals in these three classes, respectively. In Table 5, we have reported the values of different location estimates and their standard errors estimated by the bootstrap for different variables associated with individuals in different classes. In Table 6, we have reported the bootstrap efficiencies of TR co-ordinatewise median, MCD estimator and its reweighted version relative to TR spatial median for the three classes.

It is clear from the figures reported in Tables 3 through 6 that, for both the data sets, TR spatial median outperforms the other estimators in terms of their bootstrap standard errors and efficiencies. Further, TR co-ordinatewise median also outperforms MCD estimator and its reweighted version. It will be appropriate to note also that re-weighted version of MCD estimator performs better than the original MCD estimator in both the data sets.

Table 3: Values of location estimates and their bootstrap standard errors (given within parentheses) for sepal length, sepal width, petal length and petal width of three Iris species.

		TR spatial median	TR CW median	MCD	Reweighted MCD
Iris setosa	sepal length	5.04 (0.052)	5.01 (0.128)	4.89 (0.294)	4.93 (0.137)
	sepal width	3.41 (0.011)	3.39 (0.013)	3.37 (0.062)	3.39 (0.058)
	petal length	1.47 (0.001)	1.46 (0.004)	1.46 (0.007)	1.46 (0.004)
	petal width	0.25 (0.001)	0.22 (0.002)	0.20 (0.009)	0.20 (0.005)
Iris virginica	sepal length	6.54 (0.019)	6.53 (0.037)	6.46 (0.184)	6.48 (0.084)
	sepal width	2.99 (0.009)	3.01 (0.016)	2.94 (0.080)	2.96 (0.41)
	petal length	5.50 (0.018)	5.56 (0.031)	5.56 (0.053)	5.43 (0.043)
	petal width	2.04 (0.008)	1.99 (0.015)	2.01 (0.043)	2.02 (0.035)
Iris versicolor	sepal length	5.91 (0.014)	5.91 (0.027)	5.91 (0.126)	5.92 (0.071)
	sepal width	2.80 (0.002)	2.81 (0.005)	2.81 (0.007)	2.81 (0.007)
	petal length	4.28 (0.008)	4.33 (0.012)	4.33 (0.101)	4.24 (0.059)
	petal width	1.33 (0.001)	1.31 (0.002)	1.31 (0.011)	1.31 (0.028)

Table 4: Bootstrap efficienceis of TR co-ordiantewise median and MCD estimator and its reweighted version relative to TR spatial median for Iris data.

	Iris setosa	*Iris virginica*	*Iris versicolor*
TR CW median	57.2%	62.4%	65.8%
MCD	24.4%	19.5%	33.0%
Reweighted MCD	49.4%	41.0%	43.3%

Table 5: Values of location estimates and their bootstrap standard errors (given within parentheses) for glucose levels (g.l.), insulin areas (i.a.) and steady state plasma glucose levels (s.s.p.g.) for the three classes of individuals.

	Normal			Chemical			Overt		
	g.l.	i.a.	s.s.p.g.	g.l.	i.a.	s.s.p.g.	g.l.	i.a.	s.s.p.g.
TR spatial	91.37	349.92	165.77	99.10	480.27	265.17	199.15	988.71	100.49
median	(0.763)	(1.389)	(4.753)	(1.01)	(3.29)	(10.74)	(5.23)	(17.64)	(8.23)
TR CW	90.35	352.33	158.57	99.53	477.25	251.53	208.77	969.34	87.42
median	(1.123)	(3.111)	(7.235)	(1.39)	(10.11)	(19.33)	(10.32)	(30.21)	(16.14)
MCD	91.94	346.84	165.17	99.63	477.15	243.30	226.71	1091.75	86.75
	(1.556)	(6.265)	(9.915)	(2.46)	(14.07)	(41.22)	(23.33)	(74.71)	(32.70)
Reweighted	92.09	343.86	159.92	99.38	482.16	259.83	226.71	1090.77	86.42
MCD	(1.055)	(4.017)	(8.813)	(1.47)	(14.68)	(23.49)	(14.62)	(58.38)	(27.27)

Table 6: Bootstrap efficiencies of TR co-ordiantewise median and MCD estimator and its reweighted version relative to TR spatial median for the three classes.

	Normal	Chemical	Overt
TR CW median	39.6%	56.5%	51.8%
MCD	16.5%	14.3%	21.9%
Reweighted MCD	26.4%	26.5%	30.4%

5 Some concluding remarks

For univariate data, both of spatial and co-ordinatewise medians reduce to usual sample median, and MCD estimator coincides with the 1/2-least trimmed squares (LTS) estimator. Dhar and Chaudhuri [13] showed that the sample median is asymptotically more efficient relative to the 1/2-LTS estimator for large classes of symmetric univariate distributions with exponential and polynomial tails.

The use of both of TR spatial and co-ordinatewise medians can be extended to linear regression problems, where we have data : $(\mathbf{y_1}, \mathbf{x_1})$, $(\mathbf{y_2}, \mathbf{x_2})$, $\ldots$, $(\mathbf{y_n}, \mathbf{x_n})$ satisfying the regression model: $\mathbf{y}_i = \mathbf{x}_i^T\theta + \mathbf{e}_i$, $i = 1, \ldots, n$. Here $\mathbf{y}_i \in R^d$ denotes the response vector, $\mathbf{x}_i \in R^p$ represents the vector of explanatory variables, $\mathbf{e}_i$ are i.i.d. random vectors with a common elliptically symmetric distribution with scatter matrix Σ, and $\theta \in R^{p\times d}$ is the regression parameter to be estimated. A natural extension of the MCD estimator for such regression problems has been considered by Agullo, Croux and Aelst [1], and they called it the multivariate least trimmed squares estimator. Suppose that Σ_1, Σ_2 and Σ_3 are the asymptotic dispersion matrices of the regression analogues of TR spatial median, TR co-ordinatewise median and MCD estimator, respectively. Then, it follows from Bai et al. [4] and Chakraborty [7] that $\Sigma_1 = \sigma_1^2 \Sigma \otimes \Gamma^{-1}$ and $\Sigma_2 = \sigma_2^2 \Sigma \otimes \Gamma^{-1}$, respectively. Further, we have $\Sigma_3 = \sigma_3^2 \Sigma \otimes \Gamma^{-1}$, which follows from Agullo et al. [1]. Here $\Gamma = E[\mathbf{x}\mathbf{x}^T]$, and $\otimes$ denotes the Kroneker product. Hence, comparing Σ_1, Σ_2 and Σ_3 is equivalent to comparing σ_1^2, σ_2^2 and σ_3^2, and consequently, the efficiency results derived and discussed in Section 2 can be extended from multivariate location problems to regression problems with multivariate response. However, it is important to note that the regression analogues of TR medians will have asymptotic breakdown point 0% while that of the regression analogue of MCD estimator, namely, the multivariate least trimmed squares estimator, is 50%.

A different affine equivariant version of spatial median has been considered by Isogai [17], who suggested spatial median based on observations transformed by the square root of the usual variance co-variance matrix. Isogai [17] derived its asymptotic distribution, which is same as that of TR spatial median. However, this affine equivariant version of spatial median is not robust because the variance co-variance matrix used in its construction has 0% breakdown point.

6 Appendix: Proofs

Proof of Theorem 1: First, we show that $\sigma_1^2 < \sigma_2^2$ for all $d \geq 2$. Using the expressions of σ_1^2 and σ_2^2 given in the first paragraph of section 2, it follows that in order to prove $\sigma_1^2 < \sigma_2^2$, it is enough to show that

$$\frac{d(\Gamma(d/2))^2}{\pi(d-1)^2\{\Gamma((d-1)/2)\}^2} < \frac{1}{4} \Leftrightarrow \frac{d(\Gamma(d/2))^2}{\{\Gamma((d+1)/2)\}^2} < \pi. \tag{1}$$

Define $f(d) = \frac{d\{\Gamma(d/2)\}^2}{\{\Gamma((d+1)/2)\}^2}$. Then, we have

$$f(d+2) = \frac{(1+2/d)^2(d/2)^2}{\{(d+1)/2\}^2} f(d) = \frac{d^2+2d}{d^2+2d+1} f(d) < f(d),$$

and $f(1) = \pi$ and $f(2) = 8/\pi < \pi$. The inequality is now proved using induction on d. Note that we have not used the normal distribution of the data anywhere in the proof, and the inequality $\sigma_1^2 < \sigma_2^2$ has been established for any elliptic distribution.

Next, we have to show that $\sigma_2^2 < \sigma_3^2$ for all $d \geq 2$. For multivariate normal distribution, it follows from Butler et al. [6] that

$$\sigma_3^2 = \frac{d\Gamma(d/2) \int_0^{m_1^2/2} e^{-y} y^{d/2} dy}{2\left\{\int_0^{m_1^2/2} e^{-y} y^{d/2} dy\right\}^2} = \frac{d\Gamma(d/2)}{2\int_0^{m_1^2/2} e^{-y} y^{d/2} dy},$$

and we have $\sigma_2^2 = \frac{\pi}{2}$. Here $m_1^2/2$ is the median of gamma distribution with scale parameter $= 1$ and shape parameter $= d/2$. It follows from a result stated in Groeneveld and Meeden [16, p.121] that $m_1^2/2$(= the median) $<$ $d/2 =$ the expected value.

In order to prove $\sigma_3^2 > \sigma_2^2$, we have to show that

$$\begin{aligned}
& \frac{d\Gamma(d/2)}{2\int\limits_0^{m_1^2/2} e^{-y} y^{d/2} dy} > \frac{\pi}{2} \\
\Leftrightarrow \quad & \frac{d\Gamma(d/2)}{2\int\limits_0^{m_1^2/2} e^{-y} y^{d/2-1} y dy} > \frac{\pi}{2} \\
\Leftarrow \quad & \frac{d\Gamma(d/2)}{2 \times \frac{1}{2}\Gamma(d/2)(d/2)} > \frac{\pi}{2} \text{ (since } y \le m_1^2/2 < d/2 \text{ and } \int\limits_0^{m_1^2/2} e^{-y} y^{d/2-1} y dy = \tfrac{1}{2}\Gamma(d/2)) \\
\Leftrightarrow \quad & 2 > \frac{\pi}{2}.
\end{aligned}$$

This completes the proof. □

Lemma 1: *For the random variable R with density function $g(r) = \frac{2\Gamma(k)}{\Gamma(d/2)\Gamma(k-d/2)} \frac{r^{d-1}}{(1+r^2)^k}$, we have $m =$ median of $R < E(R) = \frac{\Gamma(k-d/2-1/2)\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(k-d/2)}$. Here $R \in [0, \infty)$, and $k > d/2$.*

Proof of Lemma 1: It follows from Groeneveld and Meeden [16, p.120] that $M =$ the mode of $R < m =$ the median of R if $h(y) := g(M-y)/g(M+y) < 1$ for at least one $y \in (0, M)$. Now,

$$h(y) = \left\{\frac{(M-y)}{(M+y)}\right\}^{d-1} \times \left\{\frac{1+(M+y)^2}{1+(M-y)^2}\right\}^k,$$

so that $h(0) = 1$, and $h(M) = 0$. This implies that $h(y) < 1$ for at least one

$y \in (0, M)$ using the continuity of h, and consequently $M < m$.

Next, we have to show that $k(y) := g(m-y)/g(m+y) = 1$ for at least one $y \in (0, m)$ in order to prove that $m =$ the median of $R < \mu = E(R)$ (see Groeneveld and Meeden [16, p.120]). Here also $k(0) = 1$, $k(m) = 0$, and we already have $M < m$. Now, there exists $y_1 > 0$ such that $M < (m - y_1)$, and $g(m-y_1) > g(m+y_1) \Leftrightarrow k(y_1) = g(m-y_1)/g(m+y_1) > 1$ as g is a unimodal density function. So, there exists $y \in (0, m)$ such that $k(y) = 1$ as $k(0) = 1$, $k(m) = 0$, and k is a continuous function. Hence, $m =$ the median of $R < E(R) = \frac{\Gamma(k-d/2-1/2)\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(k-d/2)}$. □

Proof of Theorem 2: Note that we have already proved the inequality $\sigma_1^2 < \sigma_2^2$ for any elliptic distribution while proving Theorem 1.

Next, we consider the inequality $\sigma_1^2 < \sigma_3^2$. From Fang et al. [15, p.83], we have

$$g_1(0) = \frac{\Gamma(k-(d-1)/2)}{\sqrt{\pi}\Gamma(k-d/2)}.$$

Using this expression of $g_1(0)$ along with the expression of σ_1^2 in the first paragraph of section 2, we have

$$\sigma_1^2 = \frac{d\{\Gamma(d/2)\}^2\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2\{\Gamma((d+1)/2)\}^2}$$

for any multivariate t-type density. Further, for any multivariate t-type distribution, it follows from Butler et al. [6] that

$$\sigma_3^2 = \frac{d\Gamma(d/2)\Gamma(k-d/2)\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^k}dy}{4k^2\Gamma(k)\left\{\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^{k+1}}dy\right\}^2},$$

where m is such that

$$\frac{2\Gamma(k)}{\Gamma(d/2)\Gamma(k-d/2)}\int\limits_0^{m} \frac{r^{d-1}}{(1+r^2)^k}dr = 1/2. \tag{2}$$

In other words, m is the median of the distribution with density $g(r) = \frac{2\Gamma(k)}{\Gamma(d/2)\Gamma(k-d/2)}\frac{r^{d-1}}{(1+r^2)^k}$, where $r \in [0,\infty)$, and $k > d/2$. It follows from Lemma 1 that $m < \frac{\Gamma(k-d/2-1/2)\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(k-d/2)}$.

In order to prove $\sigma_1^2 < \sigma_3^2$, we now have to show that

$$\frac{d\Gamma(d/2)\Gamma(k-d/2)\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^k}dy}{4k^2\Gamma(k)\left\{\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^{k+1}}dy\right\}^2} > \frac{d\{\Gamma(d/2)\}^2\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2\{\Gamma((d+1)/2)\}^2}$$

$$\Leftarrow \quad \frac{d\Gamma(d/2)\Gamma(k-d/2)\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^k}dy}{4k^2\Gamma(k)\left\{\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^{k}}dy\right\}^2} > \frac{d\{\Gamma(d/2)\}^2\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2\{\Gamma((d+1)/2)\}^2}$$

(since $1+y \geq 1$)

$$\Leftrightarrow \quad \int\limits_0^{m^2} \frac{y^{d/2-1}}{(1+y)^k}ydy < \frac{\{\Gamma(k-(d-1)/2)\}^2\{\Gamma((d+1)/2)\}^2}{k^2\Gamma(k)\Gamma(d/2)\Gamma(k-d/2)}$$

$$\Leftarrow \quad \frac{1}{2}\frac{\Gamma(d/2)\Gamma(k-d/2)}{\Gamma(k)}m^2 < \frac{\{\Gamma(\frac{k-(d-1)}{2})\}^2\{\Gamma(\frac{d+1}{2}\}^2}{k^2\Gamma(k)\Gamma(d/2)\Gamma(k-d/2)}$$

(using (2)and $y \le m^2$)

$$\Leftarrow \quad \frac{1}{2}\frac{\Gamma(d/2)\Gamma(k-d/2)}{\Gamma(k)}\left\{\frac{\Gamma(k-d/2-1/2)\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(k-d/2)}\right\}^2$$

$$< \quad \frac{\{\Gamma(k-(d-1)/2)\}^2\{\Gamma((d+1)/2)\}^2}{k^2\Gamma(k)\Gamma(d/2)\Gamma(k-d/2)} \text{ (using Lemma 1)}$$

$$\Leftrightarrow \quad k^2 < 2(k-d/2-1/2)^2 \text{ (after cancellation of terms)}$$

$$\Leftrightarrow \quad k > \frac{(d+1)}{(2-\sqrt{2})}.$$

This completes the proof of $\sigma_1^2 < \sigma_3^2$.

Next, we consider the inequality $\sigma_2^2 < \sigma_3^2$. Using the expression of $g_1(0)$ at the beginning of the proof along with the expression of σ_2^2 in the first paragraph of section 2, we have

$$\sigma_2^2 = \frac{\pi\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2}$$

for any multivariate t-type density. In order to prove $\sigma_2^2 < \sigma_3^2$, we have to show that

$$\frac{d\Gamma(d/2)\Gamma(k-d/2)\int_0^{m^2}\frac{y^{d/2}}{(1+y)^k}dy}{4k^2\Gamma(k)\left\{\int_0^{m^2}\frac{y^{d/2}}{(1+y)^{k+1}}dy\right\}^2} > \frac{\pi\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2}$$

$$\Leftarrow \quad \frac{d\Gamma(d/2)\Gamma(k-d/2)\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^k}dy}{4k^2\Gamma(k)\left\{\int\limits_0^{m^2} \frac{y^{d/2}}{(1+y)^k}dy\right\}^2} > \frac{\pi\{\Gamma(k-d/2)\}^2}{4\{\Gamma(k-(d-1)/2)\}^2} \text{ (since } 1+y \geq 1)$$

$$\Leftrightarrow \quad \int\limits_0^{m^2} \frac{y^{d/2-1}}{(1+y)^k}(y)dy < \frac{d\Gamma(d/2)\{\Gamma(k-(d-1)/2)\}^2}{\pi k^2\Gamma(k)\Gamma(k-d/2)}$$

$$\Leftarrow \quad \frac{1}{2}\frac{\Gamma(d/2)\Gamma(k-d/2)}{\Gamma(k)}m^2 < \frac{d\Gamma(d/2)\{\Gamma(k-(d-1)/2)\}^2}{\pi k^2\Gamma(k)\Gamma(k-d/2)} \text{ (using (2)and } y \leq m^2)$$

$$\Leftarrow \quad \frac{1}{2}\frac{\Gamma(d/2)\Gamma(k-d/2)}{\Gamma(k)}\left\{\frac{\Gamma(k-d/2-1/2)\Gamma((d+1)/2)}{\Gamma(d/2)\Gamma(k-d/2)}\right\}^2$$

$$< \quad \frac{d\Gamma(d/2)\{\Gamma(k-(d-1)/2)\}^2}{\pi k^2\Gamma(k)\Gamma(k-d/2)} \text{ (using Lemma 1)}$$

$$\Leftrightarrow \quad \frac{(k-d/2-1/2)^2}{k^2} > \frac{\pi\{\Gamma((d+1)/2)\}^2}{2d\{\Gamma(d/2)\}^2} \text{ (after cancellation of terms)}$$

$$\Leftrightarrow \quad k > \frac{(d+1)\Gamma(d/2)}{2\sqrt{d}\Gamma(d/2)-\sqrt{2\pi}\Gamma((d+1)/2)}.$$

This completes the proof of $\sigma_2^2 < \sigma_3^2$. □

Acknowledgments: The first author is supported by a grant from Council of Scientific and Industrial Research, Government of India.

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