Convergence of Linear Functions of Pfeifer Records

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Abstract

Probabilistic behaviour of partial sums of upper and lower records have been studied in the literature. In this article, we take a broader view and study partial sums of record like sequences. We show that such sequences converge in distribution to normal and lognormal distribution. In particular our results apply to Pfeifer records. We also show the strong convergence of partial sums of lower Pfeifer records under suitable assumptions.

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1 Introduction

Chandler (1952) initiated the study of record values and documented many of the basic properties of records. Let $\{X_0, X_1, \ldots X_n, \ldots\}$ be i.i.d. random variables with distribution function F. Let

$$C_0 = X_0,\;\; L_n = \min\{j : X_j > C_{n-1}\} \text{ and } C_n = X_{L_n}.$$

Then $\{C_n\}$ is said to be the sequence of (upper) records (from F_n). The *lower records* are defined similarly. We shall call these *classical records*. Many elegant results have been obtained for the asymptotic behaviour of classical records and their partial sums. See Arnold and Villaseñor [1] , Bose et.al. [4] [5], Resnick [10] and Resnick [11]. In the last decade or so, new models of records were introduced which generalised the classical setup in various ways. See, for example Arnold et. al [3] and Resnick [11]. The Pfeifer record is one such variation (see Pfeifer [8]). Suppose $X_{0,1}$ has the cdf F_0 and for every $n \geq 1$, $\{X_{n,j},\, j \geq 1\}$ are i.i.d. random variables with cdf F_n. Let

$$R_0 = X_{0,1},\; \Delta_n = \min\{j : X_{n,j} > R_{n-1}\} \text{ and } R_n = X_{n,\Delta_n}.$$

Then $\{R_n\}$ is said to be the sequence of (upper) *Pfeifer records* (from F_n). Pfeifer mainly concentrated on the sequence of the cdf's $\{F_n\}$ which satisfy

$$1 - F_n(x) = (1 - F_0(x))^{\alpha_n}. \tag{1}$$

We shall denote the corresponding Pfeifer records by $\{R_n(F_0)\}$. Motivated by an example of a shock model, Pfeifer assumed that $\{\alpha_n\}$ is a nondecreasing sequence of positive real numbers. However, we can think of other scenarios. For example, suppose $X_{n,j}$ is the longevity of the jth individual in the nth generation or is the production of the jth factory in the nth period. Then with advancement in science and technology over time, $\{F_n\}$ should form a stochastically increasing sequence. This would imply that $\{\alpha_n\}$ is nonincreasing. This motivates the study of Pfeifer records without imposing any specific restriction on F_n and/or $\{\alpha_n\}$.

It is easy to see that $\{R_n(F_0)\}$ has a a nice representation in terms of partial sums of independent exponential random variables. This proves very useful in studying their properties. Suppose $Y_0, Y_1, \ldots$ are independent random variables and

$$Y_i \sim Exp(\alpha_i) \tag{2}$$

and for any distribution function F,

$$\psi_{F_0}(x) = F_0^{-1}(1 - e^{-x}) \tag{3}$$

Then

$$\big(R_0, R_1, \ldots R_n\big) \stackrel{\mathcal{D}}{=} \big(\psi_{F_0}(Y_0), \psi_{F_0}(Y_0 + Y_1), \ldots \psi_{F_0}(Y_0 + Y_1 + \ldots Y_n)\big) \text{ for all } n.$$

Observe that the classical records have the above representation with $\alpha_n = 1$ for all n so that $\{Y_i\}$ are i.i.d. $Exp(1)$ random variables.

Our main objective is to study the limiting distribution of $\sum_{i=0}^{n} R_i(F_0) = \sum_{i=0}^{n} \psi_{F_0}(\sum_{j=0}^{i} Y_j)$ as $n \to \infty$. For results of this nature for classical records, see Arnold and Villaseñor [1] Bose et. al. [4], [5]. Arnold and Villaseñor [2] considered convergence of sums of Pfeifer records. Motivated by the above works we consider a general framework: suppose $\{X_i\}$ is a sequence of independent random variables (but not necessarily exponentially distributed) with mean $\{\mu_i\}$, $\{a_i\}$ is a sequence of positive reals and, ψ is any function with appropriate properties (not necessarily of the form (3)). Let

$$P_n = \sum_{k=0}^{n} a_k \psi(\sum_{j=0}^{k} X_j). \tag{4}$$

In Section 2, we first show that an appropriately normalised P_n is asymptotically normal under suitable conditions on $\{a_i\}$, ψ and $\{X_i\}$. Then we study some interesting special cases, focussing on ψ which are regularly varying. In particular, $P_n = \sum_{i=1}^n R_i(F_0)$ is one such case and we generalise the results of Arnold and Villaseñor [2]. In Section 3, we deal with rapidly varying ψ's and show that normalised $\log P_n$ is asymptotically normal. Again, $P_n = \sum_{i=1}^n R_i(F_0)$ with ψ_{F_0} rapidly varying is a special case. In Section 4 we consider lower Pfeifer records and prove a strong convergence for their partial sum.

A bit of notation: If the records $\{C_n\}$, obtained from the i.i.d. observations $\{X_0, X_1, \ldots\}$ with common distribution F, converge in distribution to G after normalization, we write $F \in D(G)$. See Resnick [11] for related results. The class of regularly varying functions or sequences with index ξ will be denoted by RV_ξ. The notation $O_\epsilon(k^\xi)$ will stand for quantities which are $O(k^{\xi \pm \epsilon})$ for arbitrarily small $\epsilon > 0$, where plus or minus sign in the power of k would be clear from the context.

2 A Central Limit Theorem

Let $\{X_j\}$ be independent with $EX_j = \mu_j$ and $VarX_j = \sigma_j^2$. Define

$$Z_n = \sum_{k=0}^n \left[a_k\{\psi(\sum_{j=0}^k X_j) - \psi(\sum_{j=0}^k \mu_j)\}\right] \tag{5}$$

where $\{a_k\}$ is a sequence of positive reals and ψ is a nonnegative function such that ψ'' exists and is monotone. This implies that ψ' is regularly varying whenever ψ is regularly varying. Let

$$Y_j = X_j - \mu_j,\ \sum_{j=0}^k \mu_j = \gamma_k,\ V(Y_j) = \sigma_j^2,\ \sum_{j=0}^k \sigma_j^2 = s_k^2,$$

$$b_j(n) = \sum_{k=j}^n a_k \psi'(\gamma_k) \text{ and } c_n^2 = \sum_{j=0}^n \sigma_j^2 b_j^2(n).$$

Using Mean Value Theorem, we decompose the summand of (5) as

$$a_k\{\psi(\sum_{j=0}^k X_j) - \psi(\sum_{j=0}^k \mu_j)\} = a_k \sum_{j=0}^k (X_j - \mu_j)\psi'(\sum_{j=0}^k \mu_j) + \frac{1}{2}a_k\big(\sum_{j=0}^k (X_j - \mu_j)\big)^2 \psi''(S_k^*) \tag{6}$$

$$= a_k \sum_{j=0}^k Y_j \psi'(\gamma_k) + \frac{1}{2}a_k(\sum_{j=0}^k Y_j)^2 \psi''(S_k^*) \tag{7}$$

where S_k^* is a random variable lying between $\sum_{j=0}^k \mu_j$ and $\sum_{j=0}^k X_j$. Let

$$S_n = \sum_{j=0}^n Y_j b_j(n) \text{ and } E_n = \sum_{j=0}^n a_j \psi''(S_j^*)(\sum_{k=0}^j Y_k)^2.$$

Then

$$Z_n = S_n + E_n.$$

Theorem 1 *Let $\{a_k\}$ be a sequence of positive reals, ψ be a positive function, twice differentiable with monotone second order derivative and $\{X_k\}$ be a sequence of independent random variables with mean $\{\mu_k\}$ and variance $\{\sigma_k^2\}$. Let $Y_k = X_k - \mu_k$ have distribution function F_k. Let ψ, $\{Y_k\}$ and $\{a_k\}$ satisfy*

(***A***) $\dfrac{\sum_{k=0}^{n} s_k^2 a_k |\psi''(\gamma_k \pm 2(2s_k^2 \log\log s_k)^{\frac{1}{2}})|}{\left[\sum_{k=0}^{n} b_k^2(n)\sigma_k^2\right]^{\frac{1}{2}}} \to 0.$

(***B***) *The law of iterated logarithm holds for $\{Y_j\}$, that is*

$$P\{\limsup_{n\to\infty} \frac{\pm\sum_{j=0}^{n-1} Y_j}{s_n(\log\log s_n)^{\frac{1}{2}}} = \sqrt{2}\} = 1.$$

(***C***) *(Lindeberg condition)*

$$\lim_{n\to\infty} \frac{1}{c_n^2} \sum_{k=0}^{n} \int_{|x| \geq \frac{\epsilon c_n}{b_k(n)}} x^2 b_k^2(n) dF_k(x) = 0, \text{ for any } \epsilon > 0. \tag{8}$$

Then $\dfrac{\sum_{k=0}^{n} \left[a_k\{\psi(\sum_{j=0}^{k} X_j) - \psi(\sum_{j=0}^{k} \mu_j)\}\right]}{c_n} \to N(0,1)$, *in distribution.*

Proof. It is enough to show that

$$\frac{S_n}{c_n} \xrightarrow{\mathcal{D}} N(0,1) \text{ and } \frac{E_n}{c_n} \xrightarrow{P} 0.$$

We first show that (**A**), (**B**) and (**C**) imply that $\frac{|E_n|}{c_n} \xrightarrow{P} 0$. Fix $\epsilon > 0$ and $\delta > 0$. To show that there exists n_0 such that $P\{\frac{|E_n|}{c_n} > \delta\} < \epsilon$ for all $n > n_0$. Since $\{Y_j\}$ obeys the law of iterated logarithm, we have

$$P\{-2\sqrt{2}s_k(\log\log s_k)^{\frac{1}{2}} \leq \sum_{j=0}^{k} Y_j \leq 2\sqrt{2}s_k(\log\log s_k)^{\frac{1}{2}} \text{ eventually}\} = 1.$$

By Egoroff's theorem there exists a set A_ϵ of Ω such that $P(A_\epsilon) < \frac{\epsilon}{2}$ and there exists k_0 independent of ω such that for all ω in A_ϵ^c,

$$-2\sqrt{2}s_k(\log\log s_k)^{\frac{1}{2}} \leq \sum_{j=0}^{k} Y_j \leq 2\sqrt{2}s_k(\log\log s_k)^{\frac{1}{2}}, \ \forall\ k > k_0.$$

Now, $E_n = E_n I_{A_\epsilon} + E_n I_{A_\epsilon^c}$ and

$$P\{\frac{E_n}{c_n} > \delta\} \leq P\{\frac{E_n}{c_n} I_{A_\epsilon} > \frac{\delta}{2}\} + P\{\frac{E_n}{c_n} I_{A_\epsilon^c} > \frac{\delta}{2}\} = T_1 + T_2, \text{ (say)}. \tag{9}$$

Clearly $T_1 \leq P(A_\epsilon) \leq \frac{\epsilon}{2}$. We decompose $\frac{E_n}{c_n} I_{A_\epsilon^c}$ as follows:

$$\begin{aligned}\frac{E_n}{c_n} I_{A_\epsilon^c} &= \frac{\sum_{k=0}^{k_0} a_k \psi''(S_k^*)(\sum_{j=0}^{k} Y_j)^2}{c_n} I_{A_\epsilon^c} + \frac{\sum_{k=k_0+1}^{n} a_k \psi''(S_k^*)(\sum_{j=0}^{k} Y_j)^2}{c_n} I_{A_\epsilon^c} \\ &= M + N \ (say).\end{aligned}$$

$|M| \to 0$ in probability as $|M|$ is a finite sum and $c_n \to \infty$, under Assumption **C**. Using the monotonicity of ψ'' and the definition of A_ϵ^c, $\forall k \geq k_0,\ \ \forall \omega \in A_\epsilon^c$,

$$
\begin{aligned}
E|N| &\leq \frac{1}{c_n}|E[\sum_{k=k_0+1}^{n} a_k|\psi''(S_k^*)|(\sum_{j=0}^{k} Y_j)^2 I_{A_\epsilon^c}]| \\
&\leq \frac{1}{c_n}\sum_{k=k_0+1}^{n} a_k|\psi''(\gamma_k \pm 2(2s_k^2 \log\log s_k)^{\frac{1}{2}})|(\sum_{j=0}^{k}\sigma_j^2) \\
&= \frac{1}{c_n}\sum_{k=k_0+1}^{n} s_k^2 a_k|\psi''(\gamma_k \pm 2(2s_k^2 \log\log s_k)^{\frac{1}{2}})| \to 0
\end{aligned}
$$

by assumption (**A**). This shows that $\frac{E_n}{c_n} \xrightarrow{P} 0$. By Assumption **C**, Lindeberg CLT implies $S_n \to N(0,\ 1)$ and the proof is complete. □

It is well known that the asymptotic behavior of classical record sequences are quite different in the two cases: when ψ is regularly varying and when ψ is rapidly varying. In the next two subsections we assume ψ to be regularly varying and apply our CLT to two situations.

2.1 Asymptotic normality of Pfeifer records

Proposition 1 *Suppose $\{Z_j\}$ are i.i.d. with distribution $Exp(1)$ and $a_k \equiv 1$ for all k. Let $\psi \in RV_\alpha$, $\alpha > 0$, be nondecreasing with monotone second derivative. Let $\{\beta_j\} \in RV_\delta$, $\delta > -\frac{1}{2}$ be a positive sequence. Let $\{X_j = \beta_j Z_j\}$. Then all conditions of Theorem 1 are satisfied.*

Remark 1. It will be clear from the proof that the above Proposition continues to hold if $\{Z_j\}$ has distribution F with mean $\mu(\neq 0)$, variance $\sigma^2(\neq 0)$ and finite $(2+\epsilon)$th order moment for some $\epsilon > 0$. This is an extension of the result of Arnold and Villaseñor [2].

Remark 2: Consider Pfeifer records $\{R_n(F_0)\}$ with parameters $\{\alpha_j\}$. Let $\psi^{-1}(x) = -\log(1 - F_0(x))$ and $\beta_j = 1/\alpha_j$. Suppose $\psi \in RV_\alpha$, $\alpha > 0$, with monotone second derivative. Suppose $\{\beta_j\} \in RV_\delta$, $\delta > -\frac{1}{2}$ is a positive sequence. Then from Proposition 1, $\sum_{i=1}^{n} R_i(F_0)$ is asymptotically normal.

Proof of Proposition 1. Here,

$$
Y_j = Z_j - 1,\ s_n^2 = \sum_{j=0}^{n}\beta_j^2,\ b_j(n) = \beta_j \sum_{k=j}^{n}\psi'(\gamma_k)
$$

where $\gamma_k = \sum_{i=1}^{k}\beta_j$ and $c_n^2 = \sum_{j=0}^{n} b_j^2(n)$. We first check that Assumption (**A**) is satisfied. From our assumptions on ψ, $\psi'(x) \in RV_{\alpha-1}$ and $(sgn(\alpha-1))\psi'' \in RV_{\alpha-2}$, (see Resnick [11], p.21). Since $\beta_j \in RV_\delta$, and $\delta > -\frac{1}{2}$, $\gamma_k \sim O_\epsilon(k^{\delta+1})$ and $s_k \sim O_\epsilon(k^{\delta+\frac{1}{2}})$, where the former dominates. Therefore, order of $\gamma_k \pm 2(2s_k^2 \log\log s_k^2)^{\frac{1}{2}}$ is same as order of γ_k.

So it is enough to show

$$
\lim_{n\to\infty} \frac{\sum_{k=0}^{n} s_k^2|\psi''(\gamma_k)|}{(\sum_{k=0}^{n} b_k^2(n))^{\frac{1}{2}}} \to 0.
$$

By our hypothesis on ψ, $|\psi''(x)| \sim O_\epsilon(x^{\alpha-2})$ as $x \to \infty$. Then easy calculation, using regular variation property of β_j as a function of j gives us

$$O(\sum_{k=0}^{n} s_k^2|\psi''(\gamma_k)|) \leq O_\epsilon(n^{2\delta+2+(\alpha-2)(\delta+1)})$$

and

$$O(\sum_{k=0}^{n} b_k^2(n))^{\frac{1}{2}} \geq O_\epsilon(n^{(\alpha-1)(\delta+1)+\delta+\frac{3}{2}}).$$

Therefore,

$$O(\frac{\sum_{k=0}^{n} s_k^2|\psi''(\gamma_k)|}{(\sum_{k=0}^{n} b_k^2(n))^{\frac{1}{2}}}) \leq O_\epsilon(n^{-\frac{1}{2}}) \to 0 \text{ as } n \to \infty.$$

Theorem **4** on LIL for weighted sums $\sum_{j=0}^{n} \beta_j Y_j$ of Chow and Teicher, p. 378 [6]) implies **(B)** holds. We omit the details. Routine calculation of the order of the relevant quantities implies that **(C)** is satisfied. See Villaseñor and Arnold [2] for details. Hence the Proposition is proved. □

2.2 Further Examples

Proposition 2 *Let $\{X_j\}$ be independent with $EX_j = \mu_j \in RV_{\delta_1}$, $\delta_1 > -1$, $V(X_j) = \sigma_j^2 \in RV_{2\delta_2}$, $\delta_2 > -\frac{1}{2}$. Suppose $E|X_k|^{\theta_k} = O(k^p)$ for some $2 < \theta_j \leq 3$. Suppose $\psi \in RV_\alpha$, $\alpha > 0$ and ψ'' is monotone. Let $\{a_k\} \in RV_{\delta_3}$. Then Theorem 1 holds if $\delta_3 > \delta_1 - \alpha(\delta_1 + 1)$ and either of the following holds:*

(i) there exists $\epsilon > 0$ such that $\theta_k \geq 2 + \epsilon$ and $p < 2\delta_2 < 2\delta_1 + 1$ or,

(ii) As $k \to \infty$, $\theta_k - 2 = k^{-\lambda}$ for some $\lambda > 0$ and $p + \lambda < 2\delta_2 < 2\delta_1 + 1$.

Example 1: Let X_n have the p.d.f. $f_n(x) = \frac{\alpha_n^q}{\Gamma(q)} \exp(-\alpha_n x) x^{q-1}$, for $0 < x < \infty$. Then $E(X_n^r) = \frac{q(q+1)...(q+r-1)}{\alpha_n^r}$. If $\alpha_n = n^{\frac{1}{6}}$ then $\delta_1 = \delta_2 = -\frac{1}{6} > -\frac{1}{2}$. Choose $\theta_n \equiv 3$. So $p = -\frac{1}{2}$ and $\lambda = 0$. Choose $\alpha > 0$ and $\delta_3 > \frac{\alpha}{6}$. Then all the conditions of Proposition 2 are satisfied.

Example 2: Let X_n have lognormal distribution with parameters $-\infty < \kappa_n < \infty$ and $\zeta > 0$ so that $\log X_n$ are normally distributed with mean κ_n and variance ζ^2. In this case the rth moment about zero is $\exp(r\kappa_n + \frac{1}{2}r^2\zeta^2)$. If we take $\kappa_n = -\frac{1}{6}\log n$ and ζ fixed for all n then $\delta_1 = -\frac{1}{6}$, $2\delta_2 = -\frac{1}{3}$ and $E|X_n|^3 \sim n^{-\frac{1}{2}}$. So $p = -\frac{1}{2}$ and $\lambda = 0$. If we choose $\alpha > 0$ and $\delta_3 > -\frac{1}{6}(1 + 5\alpha)$ then it is easy to see that all the conditions of Proposition 2 are satisfied.

Proof of Proposition 2: We verify the three Assumptions **A, B , C** of Theorem 1.

(**A**): Observe that

$$\frac{\sum_{k=0}^{n} s_k^2 a_k[\psi''(\gamma_k \pm 2(2s_k^2 \log\log s_k^2)^{\frac{1}{2}})]}{\left[\sum_{k=0}^{n} b_k^2(n)\sigma_k^2\right]^{\frac{1}{2}}} \sim O(\frac{n^{2\delta_2+\delta_3+2+[(\alpha-2)(\delta_1+1)\bigvee(\alpha-2)(\delta_2+\frac{1}{2})]}}{n^{\delta_2+\delta_3+\frac{3}{2}+(\alpha-1)(\delta_1+1)}}) \tag{10}$$

The detailed calculations are given in Appendix. So, if $\delta_1 > \delta_2 - \frac{1}{2}$ then (**A**) is satisfied.

(**B**): Now LIL holds for $\{Y_k\}$ if the following two conditions are satisfied (see Chow and Teicher [6] p.377, Cor. 4):

(a)

$$\frac{1}{s_n^2}\sum_{j=1}^{n}\int_{|x|>\frac{s_j}{\sqrt{\log\log s_j^2}}} x^2 dF_j(x) \;=\; o(1) \tag{11}$$

(b) For some $\rho \in (0, 2]$,

$$\sum_{n=1}^{\infty}\frac{1}{\left(s_n^2\log\log s_n^2\right)^{\frac{\rho}{2}}}\int_{|x|>\frac{\epsilon s_n}{\sqrt{\log\log s_n^2}}}|x|^{\rho}dF_n(x) < \infty. \tag{12}$$

With our assumptions on the order of s_k and the moment condition on F_k it is easy to see that conditions (a) and (b) are satisfied. We omit the details.

(**C**):

$$\sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}\int_{|x|\geq\frac{\epsilon c_n}{b_k(n)}} x^2 dF_k(x) = \sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}E(Y_k^2 I_{|Y_k|\geq\frac{\epsilon c_n}{b_k(n)}}) \to 0 \text{ as } n\to\infty \tag{13}$$

If we set $\frac{\epsilon c_n}{b_k(n)} = t$, the right side of Equation (13) becomes

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}\Big[\int_0^{t^2}P(Y_k^2 I_{|Y_k|>t}>y)dy+\int_{t^2}^{\infty}P(Y_k^2 I_{|Y_k|>t}>y)dy\Big]\\
&=\sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}\Big[P(|Y_k|>t)t^2+\int_{t^2}^{\infty}P(Y_k^2>y)dy\Big]\\
&=\sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}\Big[P(|Y_k|>\frac{\epsilon c_n}{b_k(n)})\frac{\epsilon^2 c_n^2}{b_k^2(n)}+\int_{\frac{\epsilon^2 c_n^2}{b_k^2(n)}}P(|Y_k|^2>y)dy\Big] = T_1+T_2 \text{ (say).}
\end{aligned}$$

Using the facts $\frac{b_0(n)}{c_n} < 1$ and $\theta_k > 2$, for some constant C, T_1 is bounded above by

$$\epsilon^2\sum_{k=0}^{n}E|Y_k|^{\theta_k}\big(\frac{b_k(n)}{\epsilon c_n}\big)^{\theta_k} \leq \epsilon^2\sum_{k=0}^{n}E|Y_k|^{\theta_k}\big(\frac{b_0(n)}{\epsilon c_n}\big)^{\theta_k} \leq C\sum_{k=0}^{n}E|Y_k|^{\theta_k}\big(\frac{b_0(n)}{c_n}\big)^2. \tag{14}$$

Now, T_2 is bounded above by

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{b_k^2(n)}{c_n^2}\int_{\frac{\epsilon^2 c_n^2}{b_k^2(n)}}\frac{E|Y_k|^{\theta_k}}{y^{\frac{\theta k}{2}}}dy\\
&\leq \epsilon^2\sum_{k=0}^{n}E|Y_k|^{\theta_k}\big(\frac{2}{\theta_k-2}\big)\big(\frac{b_0(n)}{\epsilon c_n}\big)^{\theta_k} \leq \epsilon^2\sum_{k=0}^{n}E|Y_k|^{\theta_k}\big(\frac{2}{\theta_k-2}\big)\big(\frac{b_0(n)}{\epsilon c_n}\big)^2.
\end{aligned}$$

Since, by our assumption $\delta_3+(\alpha-1)(\delta_1+1) > -1$ i.e. $\delta_3 > \delta_1-\alpha(\delta_1+1)$, using simple algebra of regularly varying functions we get

$$O(b_0(n)) \leq O_\epsilon(n^{\delta_3+(\alpha-1)(\delta_1+1)+1}) \tag{15}$$

and

$$O(c_n) \geq O_\epsilon(n^{\delta_2+\delta_3+(\alpha-1)(\delta_1+1)+\frac{3}{2}}). \tag{16}$$

See Appendix for the details. Hence $(\frac{c_n}{b_0(n)})^2 \geq O_\epsilon(n^{2\delta_2+1})$.

From the above estimates for T_1 and T_2 and using the relations between p, δ_2 and δ_3, it is easy to see that $T_1+T_2 \to 0$ if θ_k is as in (i) or (ii). This completes the proof of the Proposition. □

3 Asymptotic log normality of Pfeifer records for rapidly varying ψ

Suppose $\psi_{F_0}(x) = F_0^{-1}(1-e^{-x}) \sim C_1 \exp(C_2\sqrt{x})$ as $x \to \infty$ where $C_1 > 0, C_2 > 0$. Then the corresponding classical records is well-studied (see Resnick [11] and Arnold et. al. [3]) and in particular, $F_0 \in D(G)$ where G is lognormal. Here we will study its analogue for Pfeifer records. As before, we will formulate the problem in a general setting. Let $\{Y_n\}$ and $\{W_n\}$ be independent and

$$X_n = Y_n \psi(\sum_{j=1}^{n} \beta_j W_j), \; S_n = \sum_{i=1}^{n} X_i. \tag{17}$$

Let

$$a_n = |f'(\sum_{j=1}^{n} \beta_j)|(\sum_{j=1}^{n} \beta_j^2)^{-\frac{1}{2}} \text{ and } b_n = -a_n f(\sum_{j=1}^{n} \beta_j) \text{ where } f = \log \psi.$$

Theorem 2 *Suppose* $\{\beta_j\} \in RV_\delta,\ \delta > -\frac{1}{2}$ *and* $\psi(x) \sim \exp(x^\rho),\ \rho > 0$ *as* $x \to \infty$. *Assume* ψ *is increasing* ψ' *exists and* $(\log \psi)'(= \frac{\psi'}{\psi})$ *is monotone. Suppose* $\{W_n\}$ *are i.i.d. with distribution* $Exp(1)$. *Let* $\{Y_n\}$ *be independent positive random variables such that*

(i) $\forall\ u > 0,\ \exists\ M$ *so that* $P(\frac{1}{M} < Y_n < M) > 1-u\ \forall\ n$.

(ii) $E|\log Y_n| \sim O(n^p)$.

If $p < \rho(\delta+1) - \frac{3}{2}$ *and* $\rho(\delta+1) > \frac{1}{2}$ *then* $a_n \log S_n + b_n \xrightarrow{\mathcal{D}} N(0,1)$.

Remark 3: Let $\rho = \frac{1}{2}$ and $\delta > 2$. Suppose $S_n = \sum_{i=1}^{n} R_n(F_0)$ where $F_0 \in D(G)$, G being log-normal and suppose $\beta_j = \frac{1}{\alpha_j} \in RV_\delta$ and $Y_n \equiv C,\ \forall\ n$ where C is a positive constant. Then with $p = 0$ all conditions of Theorem 3 are satisfied. Consequently, there exists sequences of reals $\{a_n\}$ and $\{b_n\}$ such that $a_n \log S_n + b_n$ converges in distribution to standard normal.

Remark 4: We emphasize that $\{W_i\}$ need not be exponential. If $\psi(x) = \exp(x)$, i.e. $\rho = 1$, (17) represents a model proposed by Todorovic and Gani [12] (see also Puri [9]) for the effect of soil erosion on crop production.

Remark 5: If we take $Y_n = r_n$, where r_n's are nonrandom positive reals then Theorem 2 gives us conditions for the convergence of linear functions of Pfeifer records with rapidly varying ψ.

Proof of Theorem 2. Let $U_n = X_{n+1}$. Then $a_n \log S_n + b_n = a_n \log U_n + a_n \log \frac{S_n}{U_n} + b_n$. We shall show that $a_n \log U_n + b_n \xrightarrow{\mathcal{D}} N(0,\ 1)$ and $a_n \log \frac{S_n}{U_n} \xrightarrow{P} 0$. Observe that $a_n \log U_n + b_n = a_n \log Y_{n+1} + a_n f(\sum_{j=1}^{n+1} \beta_j W_j)$ where $f(x) = \log \psi(x) \in RV_\rho$. Hence using the arguments of Sections 1 and 2 and using our conditions on $\{\beta_j\}$, $\{W_j\}$ and f,

$$\frac{1}{c_n}[f(\sum_{j=1}^{n+1} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j)] \to N(0,1)$$

where

$$c_n^2 = \sum_{j=1}^{n} \beta_j^2 [f'(\gamma_n)]^2 \text{ and } \gamma_n = \sum_{j=1}^{n} \beta_j. \tag{18}$$

Note that $c_n \to \infty$ by our condition $\rho(\delta+1) > \frac{1}{2}$. Next we will show

$$\frac{1}{c_n} \log Y_{n+1} \to 0 \text{ in probability.} \tag{19}$$

Using condition (ii) and $p < \rho(\delta+1) - \frac{3}{2}$,

$$P(\frac{1}{c_n}|\log Y_{n+1}| > \epsilon) \leq \frac{E|\log Y_{n+1}|}{\epsilon c_n} \leq O(n^{p-(\delta+(\rho-1)(\delta+1)+\frac{1}{2})}) = o(1).$$

To show $a_n \log \frac{S_n}{U_n} \xrightarrow{P} 0$, observe that $\frac{S_n}{U_n} = \frac{1}{Y_{n+1}} \sum_{i=1}^n Y_i \Big(\exp(f(\sum_{j=1}^i \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)) \Big)$. Since f is increasing and $\beta_j \geq 0$ and $W_j \geq 0$ $a.s.$,

$$f(\sum_{j=1}^{i} \beta_j W_j) \leq f(\sum_{j=1}^{n+1} \beta_j W_j) \;\; \forall\, i \leq n. \tag{20}$$

Also, by our assumption, $Y_i > 0 \; \forall \; i$. Therefore we have,

$$Y_n \exp(f(\sum_{j=1}^{n} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)) \leq \sum_{i=1}^{n} Y_i \exp(f(\sum_{j=1}^{i} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)) \leq \sum_{i=1}^{n} Y_i \tag{21}$$

Using this inequality we can write,

$$\begin{aligned}
&P(\frac{1}{c_n}|\log \frac{S_n}{U_n}| > \epsilon) \\
&\leq P(\frac{1}{c_n}|\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}}| > \epsilon) + P(\frac{1}{c_n}|\log \frac{Y_n}{Y_{n+1}} + f(\sum_{i=1}^{n} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)| > \epsilon) \\
&\leq P(|\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}}| > \epsilon c_n) + P(|\log \frac{Y_n}{Y_{n+1}}| > \frac{\epsilon}{2} c_n) \\
&\qquad + P(|f(\sum_{j=1}^{n} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)| > \frac{\epsilon}{2} c_n) = L_1 + L_2 + L_3 \;\; \text{(say)}.
\end{aligned}$$

Fix any $u > 0$ and choose M satisfying condition (i). Then

$$\begin{aligned}
L_1 &= P(|\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}}| > \epsilon c_n) \\
&\leq P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} > \epsilon c_n) + P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} < -\epsilon c_n) \\
&= P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} > \epsilon c_n, Y_{n+1} > \frac{1}{M}) + P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} > \epsilon c_n, Y_{n+1} \leq \frac{1}{M}) \\
&\qquad + P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} < -\epsilon c_n, Y_{n+1} < M) + P(\log \frac{\sum_{i=1}^n Y_i}{Y_{n+1}} < -\epsilon c_n, Y_{n+1} \geq M) \\
&\leq P(\log(\sum_{i=1}^{n} Y_i) + \log M > \epsilon c_n) + P(Y_{n+1} \leq \frac{1}{M}) + P(\log \frac{\sum_{i=1}^n Y_i}{M} < -\epsilon c_n) + P(Y_{n+1} \geq M) \\
&\leq P(\sum_{i=1}^{n} Y_i > \frac{1}{M} \exp(\epsilon c_n)) + P(\sum_{i=1}^{n} Y_i < M \exp(-\epsilon c_n)) + u \\
&\leq \sum_{i=1}^{n} P(Y_i > \frac{\frac{1}{M}\exp(\epsilon c_n)}{n}) + \sum_{i=1}^{n} P(Y_i < \frac{M \exp(-\epsilon c_n)}{n}) + u \\
&= \sum_{i=1}^{n} P(\log Y_i > \log \frac{1}{M} + \epsilon c_n - \log n) + \sum_{i=1}^{n} P(\log Y_i < \log M - \epsilon c_n - \log n) + u.
\end{aligned}$$

Now by our assumption, $c_n \sim O_\epsilon(n^{\delta+(\rho-1)(\delta+1)+\frac{1}{2}}) > O(\log n)$. Therefore the right side of the last equality is smaller than

$$\sum_{i=1}^{n} P(\log Y_i > \frac{\epsilon c_n}{2}) + \sum_{i=1}^{n} P(\log Y_i < -\frac{\epsilon c_n}{2}) + u \tag{22}$$

$$= \sum_{i=1}^{n} P(|\log Y_i| > \frac{\epsilon c_n}{2}) + u \leq \frac{\sum_{i=1}^{n} E|\log Y_i|}{\frac{\epsilon c_n}{2}} + u. \tag{23}$$

Since $\frac{\sum_{i=1}^{n} E|\log Y_i|}{\frac{\epsilon c_n}{2}}$ converges to 0 by our condition $p < \rho(\delta+1) - \frac{3}{2}$, $L_1 \to 0$ follows. By similar argument $L_2 \to 0$. To tackle L_3, observe that

$$L_3 = P(|f(\sum_{j=1}^{n} \beta_j W_j) - f(\sum_{j=1}^{n+1} \beta_j W_j)| > \frac{\epsilon c_n}{2}) = P(|\beta_{n+1} W_{n+1} f'(\Sigma_n)| > \frac{\epsilon c_n}{2}),$$

where Σ_n lies between $\sum_{j=1}^{n} \beta_j W_j$ and $\sum_{i=1}^{n+1} \beta_j W_j$. In Section 2 we have already checked that LIL holds for $\{\beta_j W_j\}$ and by our assumption f' is monotone. Therefore if f' is nondecreasing we have the following inequalities for all large n:

$$f'(\gamma_n - 2\sqrt{2s_n^2 \log\log s_n^2}) \leq f'(\sum_{j=1}^{n} \beta_j W_j) \leq f'(\gamma_n + 2\sqrt{2s_n^2 \log\log s_n^2}) \tag{24}$$

$$f'(\gamma_{n+1} - 2\sqrt{2s_{n+1}^2 \log\log s_{n+1}^2}) \leq f'(\sum_{j=1}^{n+1} \beta_j W_j) \leq f'(\gamma_{n+1} + 2\sqrt{2s_{n+1}^2 \log\log s_{n+1}^2}). \tag{25}$$

Obviously, if f' is decreasing the inequalities in (24) and (25) are reversed. So without loss of generality, we assume f' is nondecreasing. Also f' is positive as ψ is increasing. Therefore,

$$\{f'(\gamma_n - 2\sqrt{2s_n^2 \log\log s_n^2}) \bigwedge f'(\gamma_{n+1} - 2\sqrt{2s_{n+1}^2 \log\log s_{n+1}^2})\} \tag{26}$$

$$\leq f'(\Sigma_n) \leq \{f'(\gamma_n + 2\sqrt{2s_n^2 \log\log s_n^2}) \bigvee f'(\gamma_{n+1} + 2\sqrt{2s_{n+1}^2 \log\log s_{n+1}^2})\} \tag{27}$$

for all large n. Hence

$$f'(\Sigma_n) \sim O_\epsilon(n^{(\rho-1)(\delta+1)}). \tag{28}$$

Now, using (28), note that

$$\beta_{n+1}|f'(\Sigma_n)| \sim O_\epsilon(n^{\delta+(\rho-1)(\delta+1)}) \text{ and } c_n \sim O_\epsilon(n^{\delta+(\rho-1)(\delta+1)+\frac{1}{2}}).$$

Hence

$$\frac{\epsilon c_n}{2\beta_{n+1}|f'(\Sigma_n)|} \to \infty \text{ and } L_3 = P\left(W_{n+1} > \frac{\epsilon c_n/2}{\beta_{n+1}|f'(\Sigma_n)|}\right) \to 0.$$

The proof of Theorem 2 is complete. □.

4 Almost sure behaviour of sums of lower Pfeifer records

Let for each fixed $j \geq 0, X_{n,j}$ be independent with distribution F_n and let

$$R_0^L = X_{0,1},\ \Delta_n = \min\{j \geq 1 : X_{n,j} < R_{n-1}^L\} \text{ and } R_n^L = X_{n,\Delta_n}.$$

Then $\{R_n^L\}$ are the lower Pfiefer records from F_n. Instead of (1) assumed earlier for upper records, we now assume,

$$X_{nj} \sim F_n(x) = F_0^{\alpha_n}(x). \tag{29}$$

and write $\{R_n^L(F_0)\}$ as the corresponding lower Pfiefer records. Let $Z_{n,j} = -\log(X_{n,j})$. For each $n \geq 0$, let G_n be the common cdf of $\{Z_{n,j},\ j \geq 1\}$. Then $G_n = 1 - F_n(e^{-x})$. Clearly, the upper Pfeifer records $\{R_n(G_0)\}$ are related to the lower Pfeifer records $\{R_n^L(F_0)\}$ as:

$$R_n^L(F_0) = e^{-R_n(G_0)}.$$

Now, it is easy to see $\{G_n\}$ obeys the pattern (1). Therefore,

$$\{R_k(G_0) :\ k \leq n\} \stackrel{\mathcal{D}}{=} \{\psi_{G_0}(\sum_{i=1}^{k} Y_i) :\ k \leq n\}$$

where Y_i's are as in (2) and $\psi_{G_0} = G_0^{-1}(1 - e^{-x})$. Therefore, for every $n \geq 1$,

$$\sum_{k=1}^{n} R_k^L(F_0) \stackrel{\mathcal{D}}{=} \sum_{k=1}^{n} F_0^{-1}\left(e^{-\sum_{i=1}^{k} Y_i}\right) \stackrel{\mathcal{D}}{=} \sum_{k=1}^{n} F_0^{-1}\left(e^{-\sum_{i=1}^{k} \beta_i W_i}\right) \tag{30}$$

where $\beta_i = \frac{1}{\alpha_i}$ and $\{W_i\}$ is i.i.d. with distribution $Exp(1)$.

Theorem 3 *Suppose support of $F_0 \subset [0, \infty)$ and $\beta_n = \frac{1}{\alpha_n} \in RV_\delta$.*

a) Suppose $\delta + 1 > 0$. If for all positive constants C_1, C_2 and ζ, $\sum_{n=1}^{\infty}\{F_0^{-1}(C_1 \exp(-C_2 n^{\zeta}))\} < \infty$, then $\sum_{k=1}^{\infty} R_k^L(F_0)$ converges almost surely to a finite limit.

b) Suppose $\delta + 1 < 0$. If $F_0^{-1}(x) > 0\ \forall\ x > 0$, then $\sum_{k=1}^{\infty} R_k^L(F_0)$ diverges almost surely.

Example 3: Let $F_0(x) = 1 - \exp(-x^{\alpha})$, $\alpha > 0$ for $x > 0$ with $F_0(0) = 0$. Then it is easy to verify that this F_0 satisfies the conditions both in (a) and (b) on F_0 of Theorem 3.

Proof of Theorem 3. The first term in equation (30) is a partial sum of positive random variables. Hence this sequence is an increasing sequence almost surely (the limit may equal ∞). If the last term in equation (30) converges almost surely to a finite limit then the first term converges to the same finite limit in distribution as they are distributionally same and this must coincide with its almost sure limit. Thus it is enough to show that $\sum_{k=1}^{\infty} F_0^{-1}(e^{-\sum_{i=1}^{k} \beta_i W_i}) < \infty$ almost surely.

Let $Y_n = \beta_n W_n$. Then $EY_n = \beta_n$ and $\sigma_n^2 = VarY_n = \beta_n^2$. Let $s_n^2 = \sum_{k=1}^{n} \sigma_k^2 = \sum_{k=1}^{n} \beta_k^2$. Then $s_k \in RV_{\delta + \frac{1}{2}}$.

First we will consider the case $\delta > -\frac{1}{2}$. If $\delta = -\frac{1}{2}$, (i.e. $\beta_n \in RV_{-\frac{1}{2}}$) and $\sum_{i=1}^{n} \beta_i^2 \to \infty$ then the following argument remains valid for $\delta = -\frac{1}{2}$ also. In this case, $s_k \to \infty$ as $k \to \infty$. Easy calculation (see Appendix) yields the following:

$$\sum_{n=1}^{\infty} P(|Y_n| > \epsilon s_n \log s_n) < \infty. \tag{31}$$

Under this condition, a version of SLLN for independent random variables (see Chow and Theicher [6], p.362, Cor. 4) holds and using this we can write

$$\frac{\sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} \beta_k}{s_n \log s_n} \to 0, \quad a.s.[\omega].$$

So for each ω there exists an $N(\omega)$, such that

$$|\sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} \beta_k| < \frac{1}{2} s_n \log s_n, \quad \forall\ n \geq N(\omega).$$

Therefore, $e^{-\sum_{i=1}^{k} Y_i(\omega)} \leq e^{-\sum_{i=1}^{k} \beta_i + \frac{1}{2} s_n \log s_n}$, for all $k \geq N(\omega)$ for almost all ω. Now, by our assumption on $\{\beta_n\}$,

$$O(\sum_{i=1}^{k} \beta_i + \frac{1}{2} s_n \log s_n) \geq O_\epsilon(n^{\delta+1}).$$

Therefore, for any $\epsilon > 0$,

$$\exp(-\sum_{i=1}^{n} Y_i) \leq \exp(-Cn^{\delta+1-\epsilon}) \tag{32}$$

for all n large enough. Next consider the case $\delta < -\frac{1}{2}$. Since s_k now converges to a finite limit, $\sum_{k=1}^{n} Y_k - \sum_{k=1}^{n} \beta_k \to V,\ a.s.$ where V is a random variable. Therefore,

$$\exp(-\sum_{i=1}^{n} \beta_i - V(\omega) - \frac{1}{2}) \leq \exp(-\sum_{i=1}^{n} Y_i(\omega)) \leq \exp(-\sum_{i=1}^{n} \beta_i - V(\omega) + \frac{1}{2}) \tag{33}$$

for almost all ω and for n large enough. Set $K_1(\omega) = \exp(-V(\omega) + \frac{1}{2})$. Suppose $-1 < \delta < -\frac{1}{2}$. Then $\beta_i \in RV_\delta$ and $\delta + 1 > 0$ imply $\sum_{i=1}^{n} \beta_i \in RV_{\delta+1}$. So, for any $\epsilon > 0$ there is a positive constant K_2 such that for almost all ω,

$$\exp(-\sum_{i=1}^{n} Y_i(\omega)) \leq K_1(\omega) \exp(-K_2 n^{\delta+1-\epsilon}) \tag{34}$$

for all n sufficiently large.

Now, for the proof of part (a), since $\delta + 1 > 0$, we choose $\epsilon > 0$ such that $\delta + 1 - \epsilon > 0$. Then (32), (34) and our condition on F_0 give the result.

However, if $\delta + 1 < 0$, then $\sum_{i=1}^{n} \beta_i$ converges to a positive finite limit. Then from left side of (33) it is evident that $\exp(-\sum_{i=1}^{n} Y_i(\omega))$ is greater than a positive constant depending on ω and hence part (b) of the Theorem follows. □

Remark 6. For $\{\beta_k\}$ and F_0 which do not satisfy the conditions of the above theorem, the sum of lower records may either converge or diverge (a.s.). For $\alpha > 0$, let

$$F_0(x) = \exp(-\frac{1}{x^\alpha}) \text{ for } x > 0.$$

Then

$$F_0^{-1}(C_1 e^{-C_2 n^\zeta}) = \frac{1}{(C_2 n^\zeta - \log C_1)^{\frac{1}{\alpha}}}. \tag{35}$$

If $0 < \zeta < \alpha$ then (35) is not summable and hence condition (a) does not hold. However, if $\delta + 1 > \alpha$ then choosing $\epsilon < \delta + 1 - \alpha$ we see that $F_0^{-1}(K_1(\omega) \exp(-K_2 n^{\delta+1-\epsilon}))$ is summable for any fixed ω and hence the sum of lower records converge a.s. by the arguments in the proof of Theorem 3. On the other hand, if $\delta + 1 < \alpha$ then it is easy to see that the sum of lower records diverge. We omit the details.

Remark 7. In the classical setup of records of i.i.d. observations, it is known that the infinite sum of lower records (under suitable assumptions) is infinitely divisible (see Bose et.al. [5]). It would be interesting to know if the limit obtained for sums of lower Pfeifer records is also so.

5 Appendix

Here we provide some of the details of calculations that we have left out in the main body of the paper. The following result from Resnick [11], p.22 will be repeatedly used while dealing with regularly varying functions: Let $V \in RV_\kappa$, then for any $\epsilon > 0$, there exists x_0 such that,

$$(1-\epsilon)\Big(\frac{x}{x_0}\Big)^{\kappa-\epsilon} < \frac{V(x)}{V(x_0)} < (1+\epsilon)\Big(\frac{x}{x_0}\Big)^{\kappa+\epsilon} \ \forall\ x \geq x_0.$$

Therefore, for any $\epsilon > 0$, there exists positive constants C_1 and C_2 such that $C_1.x^{\kappa-\epsilon} < V(x) < C_2.x^{\kappa+\epsilon}$, for all x large enough.

Section 2.2. To verify Lindeberg condition we need to calculate the order of $\frac{c_n}{b_0(n)}$. To do this we use the following Lemma.

Lemma 1 *Let $\psi \in RV_\alpha$, $\alpha > 0$ be a nondecreasing function. Let $\gamma_j \in RV_{\delta_1+1}$, $\sigma_j \in RV_{\delta_2}$ $a_j \in RV_{\delta_3}$ be three sequences. Define $b_k(n) = \sum_{j=k}^n a_j \psi'(\gamma_j)$. Then order of $c_n^2 = \sum_{k=0}^n b_k^2(n)\sigma_k^2 \geq O_\epsilon(n^{2\delta_2+2\delta_3+2(\alpha-1)(\delta_1+1)+3})$ and order of $b_0^2(n) \leq O_\epsilon(n^{2(\delta_3+(\alpha-1)(\delta_1+1)+1)})$.*

Proof. Using Cauchy-Schwartz inequality we have,

$$c_n^2 = \sum_{k=0}^n b_k^2(n)\sigma_k^2 \geq \Big(\sum_{k=0}^n b_k(n)\sigma_k\Big)^2 \frac{1}{n}. \tag{36}$$

Now, $\sum_{k=0}^n b_k(n)\sigma_k = \sum_{k=0}^n \sigma_k \sum_{j=k}^n a_j\psi'(\gamma_j) = \sum_{j=0}^n (\sum_{k=0}^j \sigma_k) a_j \psi'(\gamma_j)$. Using the algebra of regularly varying functions we have

$$O\Big(\sum_{k=0}^n b_k(n)\sigma_k\Big) \geq O(n^{\delta_2+1+\delta_3+(\alpha-1)(\delta_1+1)+1-\epsilon}).$$

Using inequality (36) yields the first claim. Another use of algebra of regularly varying functions yields the second claim. □

To verify the order estimate in (10), observe that by our assumption $\gamma_j \sim O_\epsilon(j^{\delta_1+1})$, $s_k^2 \sim O_\epsilon(n^{2\delta_2+1})$ and $|\psi''(x)| \sim O_\epsilon(x^{\alpha-2})$. Then using the usual algebra of regularly varying functions we get the order.

Section 3: The order of normalising constant in (18), $c_n^2 = \sum_{j=1}^n \beta_j^2 (f'(\gamma_n))^2$, is obtained by using algebra of regularly varying function which gives

$$O(c_n^2) \sim O_\epsilon(n^{2\delta+2(\rho-1)(\delta+1)+1}).$$

So to make $c_n \to \infty$ we need $\rho(\delta+1) > \frac{1}{2}$.

Now, we verify the Lindeberg condition. The condition takes the form:

$$\lim_{n\to\infty}\frac{1}{c_n^2}\sum_{j=1}^{n}\int_{|\beta_j f'(\gamma_n)(W_j-1)|>\epsilon c_n}\beta_j^2 f'(\gamma_n)^2(W_j-1)^2dP=0.$$

for all $\epsilon>0$. To calculate the order of $\frac{c_n}{\beta_j f'(\gamma_n)}$, for $1\le j\le n$, observe that $\beta_j\in RV_\delta$ and hence, for any $\epsilon>0$ there exists j_0 and constant $C>0$ depending on ϵ, such that $\beta_j\le C.j^{\delta+\epsilon}\le C.n^{\delta+\epsilon}$, for all j, $j_0\le j\le n$. Therefore, $\frac{c_n}{\beta_j f'(\gamma_n)}\ge\frac{c_n}{Cn^{\delta+\epsilon}f'(\gamma_n)}$ for all j such that $j_0\le j\le n$.

It is easy to see that $O\big(\frac{c_n}{Cn^{\delta+\epsilon}f'(\gamma_n)}\big)\sim O_\epsilon(n^{\frac{1}{2}})$. Now, to verify Lindeberg condition:

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{c_n^2}\sum_{j=1}^{n}\int_{|\beta_j f'(\gamma_n)(W_j-1)|>\epsilon c_n}\beta_j^2 f'(\gamma_n)^2(W_j-1)^2dP\\
&=\lim_{n\to\infty}\frac{1}{c_n^2}\sum_{j=1}^{j_0}\int_{|W_j-1|>\frac{\epsilon c_n}{\beta_j f'(\gamma_n)}}\beta_j^2 f'(\gamma_n)^2(W_j-1)^2dP\\
&\quad+\lim_{n\to\infty}\frac{1}{c_n^2}\sum_{j=j_0+1}^{n}\int_{|W_j-1|>\frac{\epsilon c_n}{\beta_j f'(\gamma_n)}}\beta_j^2 f'(\gamma_n)^2(W_j-1)^2dP\\
&\sim\lim_{n\to\infty}\frac{1}{c_n^2}\sum_{j=1}^{j_0}\int_{|W_j-1|>\frac{\epsilon c_n}{\beta_j f'(\gamma_n)}}\beta_j^2 f'(\gamma_n)^2(W_j-1)^2dP\\
&\quad+\lim_{n\to\infty}\frac{1}{c_n^2}\Big(\sum_{j=j_0+1}^{n}\beta_j^2 f'(\gamma_n)^2\Big)\Big(\int_{|W-1|>n^{\frac{1}{2}}}(W-1)^2dP\Big).
\end{aligned}$$

The first term of the last inequality is a limit of a finite sum divided by c_n^2, hence is 0 as $c_n^2\to\infty$ and the second term is $\lim_{n\to\infty}\int_{|W-1|>n^{\frac{1}{2}}}(W-1)^2dP=0$

Section 4: The calculation we need to verify (31) is as follows:

$$\begin{aligned}
\sum_{n=k}^{\infty}P(|U_n|>\epsilon s_n\log s_n)&=\sum_{n=k}^{\infty}P\Big(W_n>\frac{\epsilon s_n\log s_n}{\beta_n}\Big)\\
&=\sum_{n=k}^{\infty}\exp\Big(-\frac{\epsilon s_n\log s_n}{\beta_n}\Big)\\
&\sim\sum_{n=k}^{\infty}\exp\Big(-\frac{\epsilon n^{\delta+\frac{1}{2}}(\delta+\frac{1}{2})\log n}{n^\delta}\Big)\\
&\le\sum_{n=k}^{\infty}\exp(-C\sqrt{n}),\ \text{ for large } k
\end{aligned}$$

where C is a positive constant. (Note, by our assumption $\delta+\frac{1}{2}>0$). Hence the result follows.

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