SPECTRAL PROPERTIES OF RANDOM TRIANGULAR MATRICES

RIDDHIPRATIM BASU, ARUP BOSE, SHIRSHENDU GANGULY,
AND RAJAT SUBHRA HAZRA

Abstract. We provide a relatively elementary proof of the existence of the limiting spectral distribution (LSD) of symmetric triangular patterned matrices and also show their joint convergence. We also derive the expressions for the moments of the LSD of the symmetric triangular Wigner matrix using properties of Catalan words.

1. Introduction

One of the most important problems in operator theory is the *invariant subspace problem* and the *hyper invariant subspace problem*. Let $\mathcal{H}$ be a separable Hilbert space (infinite dimensional) and let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded operators on $\mathcal{H}$. An *invariant subspace* of $\mathbb{A} \in \mathcal{B}(\mathcal{H})$ is a subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that $\mathbb{A}(\mathcal{H}_0) \subset H_0$ and a *hyper invariant subspace* of $\mathbb{A}$ is a subspace $\mathcal{H}_0 \subset \mathcal{H}$ that is invariant for every operator $\mathbb{B} \in \mathcal{B}(\mathcal{H})$ that commutes with $\mathbb{A}$. The invariant subspace problem asks whether every operator in $\mathcal{B}(\mathcal{H})$ has a closed non-trivial invariant subspace. The hyper invariant subspace conjecture states that every operator in $\mathcal{B}(\mathcal{H})$ that is not a scalar multiple of the identity operator has a closed, non-trivial hyper invariant subspace. For many years there were attempts to prove this result but recently there were attempts to disprove this conjecture. Initial attempts were made by Voiculescu's circular operator but it had a large amount of invariant subspace. A natural candidate which stood out for counter examples was the DT-operators defined by Dykema and Haagerup (2004a). This is related to the following *asymmetric* matrix defined and studied in their work. This matrix T_n is a triangular version of the celebrated Wigner matrix (which is a symmetric matrix and, in its simplest form, has i.i.d. entries).

$$T_n = \begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \dots & t_{1,n-1} & t_{1,n} \\ 0 & t_{2,2} & t_{2,3} & \dots & t_{2,n-1} & t_{2,n} \\ 0 & 0 & t_{3,3} & \dots & t_{3,n-1} & t_{3,n} \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & 0 & t_{n,n} \end{bmatrix} \tag{1.1}$$

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where $(t_{i,j})_{1\le i\le j\le n}$ are i.i.d. complex Gaussian random variables having mean 0 and variance 1.

Let μ be a compactly supported measure on the complex plane $\mathbb{C}$ and D_n be a diagonal matrix with μ-distributed independent random variables. They showed that $Z_n = cn^{-1/2}T_n + D_n$ converges in $*$-moments, that is,

$$\lim_{n\to\infty} \tau_n \left(Z_n^{\epsilon(1)} Z_n^{\epsilon(2)} \dots Z_n^{\epsilon(k)}\right) \tag{1.2}$$

exists for every $k \in \mathbb{N}$ and for all $\epsilon(1), \epsilon(2), \dots, \epsilon(k) \in \{*, 1\}$. Here τ_n denotes the functional $\frac{1}{n}\,\mathrm{Tr}$. $DT(\mu, c)$ is an element Z in some $*$-non-commutative probability space whose $*$-moments are defined by the limit (1.2).

Dykema and Haagerup (2004a) showed that DT-elements are operators on some Hilbert spaces. In fact, these are elements of $(L(\mathbb{F}_2), \tau)$ where $L(\mathbb{F}_2) \subset \mathcal{B}(\ell^2(\mathbb{F}_2))$ is the von Neumann algebra generated by the left regular representations of the non-abelian free group of two generators and τ is the canonical tracial state. $DT(\mu, c)$ operator is a $DT(\mu, c)$ element of a W^* non-commutative probability space $(\mathcal{M}, \tau)$ where $\mathcal{M}$ is a von Neumann algebra and τ is a normal state. It was shown that DT-operators are decomposable (an operator K, is decomposable, if for every cover $\mathbb{C} = U \cup V$ of the complex plane of open subsets U and V, there are K-invariant closed subspaces $\mathcal{H}'$ and $\mathcal{H}''$ such that $\sigma(K|_{\mathcal{H}'}) \subset U$ and $\sigma(K|_{\mathcal{H}''}) \subset V$ and $\mathcal{H} = \mathcal{H}' + \mathcal{H}''$). Hence it follows that DT-operators whose spectra contains more than one point have non-trivial closed hyper invariant subspaces. It was also shown in Dykema and Haagerup (2004a,b) that DT-operators whose spectra are singletons have closed, non-trivial hyper invariant subspaces. So the study of the spectrum of the operator DT is of immense importance.

Few properties of the $*$-limit of $n^{-1/2}T_n$ were derived by Dykema and Haagerup (2004a,b) and Śniady (2003) and the Stieltjes transform for these kind of matrices was studied by Shlyakhtenko (1996). In particular analytical techniques and also combinatorics of operator valued free probability theory were used to derive the following:

$$\mathrm{E}\frac{1}{n}\mathrm{Tr}\left(\frac{T_n^* T_n}{n}\right)^k \to \frac{k^k}{k+1!}. \tag{1.3}$$

Moreover, the sequence $\{\frac{k^k}{k+1!}\}$ are moments of a measure ν supported on $[0, e]$ and is given by

$$d\nu(x) = \phi(x)dx \quad \text{where } \phi : (0, e) \to \mathbb{R}^+, \tag{1.4}$$

and this is the unique solution of

$$\phi\left(\frac{\sin v}{v}\exp(v\cot v)\right) = \frac{1}{\pi}\sin v \exp(-v\cot v). \tag{1.5}$$

Our goal is to relate the above limit to triangular but symmetric versions of the Wigner matrix. Define the triangular symmetric Wigner matrix as:

$$W_n^u = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1(n-1)} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \dots & x_{2(n-1)} & 0 \\ & & & \vdots & & \\ x_{1(n-1)} & x_{2(n-1)} & 0 & \dots & 0 & 0 \\ x_{1n} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \tag{1.6}$$

where $\{x_{i,j}\}$ are independent and identically distributed random variables.

The matrix W_n^u has connections with DT-operators. If one removes the symmetry from the above matrix and considers only i.i.d. entries above the anti-diagonal then one can easily relate this to the DT-operators. If we multiply the above matrix by a matrix P whose all entries on the anti-diagonal are 1 and rest are zero, then we get an upper triangular matrix with i.i.d. entries. It can also be seen that $(W_n^u)^2 = T_n T_n^*$ where T_n is an upper triangular matrix (1.1) but has dependent rows and columns now.

In this article we study the spectral properties of $n^{-1/2} W_n^u$ and more generally of symmetric triangular patterned random matrices. We use classical moment method to show that the limiting spectral distribution (LSD) of triangular Wigner, Topelitz, Hankel and Reverse Circulant matrices exist when the entries have mean zero and variance one and, are either independent and uniformly bounded or are i.i.d. We show by relatively easy arguments that the $(2k)$th moment of the LSD for W_n^u is given by (1.3) and the odd moments are zero. However, moment formulae for other matrices remain elusive and appear rather difficult to obtain. Our proof of the moment formula is obtained by exploiting the Catalan recursions inbuilt in the Wigner matrix.

In the next section we describe our setup and notation. For recent developments on random patterned matrices we refer the readers to Bose et al. (2010). We adopt the method developed in Bryc et al. (2006) and Bose and Sen (2008).

2. PATTERNED TRIANGULAR MATRICES AND METHOD OF MOMENTS

Let X_n be a *patterned matrix* with *link function* L and *input sequence* $\{x_i\}$ or $\{x_{ij}\}$. Existence of LSD for general patterned matrices was studied in Bose and Sen (2008) using the moment method. Throughout this article we assume that the triangular version is derived from a patterned random matrix X_n whose link function satisfies the *Property B*:

$$\Delta(L) = \sup_n \sup_t \sup_{1 \le k \le n} \#\{l : 1 \le l \le n,\ L(k,l) = t\} < \infty.$$

Let X_n^u-be the upper triangular version of the X matrix where the (i,j)-th entry of X_n^u is given by $x_{L(i,j)}$ if $(i+j) \le n+1$ and 0 otherwise. We shall often drop the subscript n, and simply denote the matrix by X^u. The triangular Wigner matrix has already been defined. Here are examples of the triangular versions of some more common patterned matrices:

1. Triangular Hankel matrix.

$$H_n^u = \begin{bmatrix} x_2 & x_3 & x_4 & \dots & x_n & x_{n+1} \\ x_3 & x_4 & x_5 & \dots & x_{n+1} & 0 \\ & & & \vdots & & \\ x_n & x_{n+1} & 0 & \dots & 0 & 0 \\ x_{n+1} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.1}$$

2. Triangular Toeplitz matrtix.

$$T_n^u = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_1 & x_0 & x_1 & \dots & x_{n-3} & 0 \\ x_2 & x_1 & x_0 & \dots & 0 & 0 \\ & & & \vdots & & \\ x_{n-1} & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.2}$$

3. Triangular Symmetric Circulant Matrix.

$$S_n^u = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \dots & x_3 & 0 \\ x_2 & x_1 & x_0 & \dots & 0 & 0 \\ & & & \vdots & & \\ x_1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{2.3}$$

It is to be noted that a triangular Reverse Circulant matrix is the same as a triangular Hankel matrix.

To prove the existence of LSD via moment method, it suffices to verify the following three conditions.

(1) For every $k \geq 1$, $\frac{1}{n}\mathrm{E}\left[\mathrm{Tr}\left(\frac{1}{\sqrt{n}}X_n^u\right)^k\right] \to \beta_k$ as $n \to \infty$ (Condition M1).

(2) For every $k \geq 1$,

$$\mathrm{E}\left[\frac{1}{n}\mathrm{Tr}\left((\frac{1}{\sqrt{n}}X_n^u)^k) - \mathrm{E}(\frac{1}{n}\,\mathrm{Tr}((\frac{1}{\sqrt{n}}X_n^u)^k)\right)\right]^4 = O(\frac{1}{n^2}) \text{ as } n \to \infty \quad \text{(Condition M4)}.$$

(3) $\sum_k \beta_{2k}^{-1/2k} = \infty$ (Carleman's Condition).

We make the following two assumptions.

Assumption A: The input sequence $\{x_i\}$ or $\{x_{ij}\}$ is independent across i, or i, j, with mean 0, variance 1, and is uniformly bounded.

Assumption B: The link function L satisfies *Property B′*, i.e.,

$$\Delta = \sup_n \sup_t \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n, k + l \leq n + 1,\ L(k, l) = t\} < \infty.$$

This condition is the restriction of Property B to the triangular matrices.

Remark 2.1. The uniform boundedness Assumption A is made for convenience. Using approximation results in Bose and Sen (2008), we can relax this to the assumption of (i) $\{x_i\}$ or $\{x_{ij}\}$ being independent with uniformly bounded moments of all order or (ii) of $\{x_i\}$ or $\{x_{ij}\}$ being i.i.d. with mean 0 and variance 1.

To verify these conditions, we use the notation of circuits, words etc. from Bose and Sen (2008). Let X_n be a symmetric patterned matrix and with link function L satisfying Property B. Let X_n^u be the corresponding upper triangular matrix. Let w be a pair-matched word of length $2k$. Let us denote

$$\Pi_X^*(w) \;=\; \{\pi : w[i] = w[j] \Rightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}$$

$$\Pi_X(w) \;=\; \{\pi : w[i] = w[j] \Leftrightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}.$$

We shall write now

$$\begin{aligned} \frac{1}{n}\mathrm{Tr}((\frac{1}{\sqrt{n}}X^u)^k) &= \frac{1}{n^{1+k/2}} \sum_{\pi:\pi \text{ circuit}} \prod_{i=1}^{k} x_{L(\pi(i-1),\pi(i))} 1_{\{\pi(i-1)+\pi(i) \le n+1\}} \\ &= \frac{1}{n^{1+k/2}} \sum_{w} \sum_{\pi \in \Pi(w)} \prod_{i=1}^{k} x_{L(\pi(i-1),\pi(i))} 1_{\{\pi(i-1)+\pi(i) \le n+1\}}. \end{aligned}$$

Let us denote

$$X_\pi * = \prod_{i=1}^{k} x_{L(\pi(i-1),\pi(i))} 1_{\{\pi(i-1)+\pi(i) \le n+1\}}.$$

For a fixed k, let us define the class Π_k^u as follows.

$$\Pi_k^u = \{\pi : \pi \text{ is circuit of length } k,\ \pi(i-1) + \pi(i) \le n+1, 1 \le i \le k\}.$$

Let us denote

$$\Pi_{1,X}(w) = \Pi_X(w) \bigcap \Pi_{2k}^u \ \text{ and } \Pi_{1,X}^*(w) = \Pi_X^*(w) \bigcap \Pi_{2k}^u.$$

For every pair-matched word w of length $2k$, define, if the limit exists,

$$p_{u,X}(w) = \lim_{n\to\infty} \frac{\#\Pi_{1,X}^*(w)}{n^{1+k}}$$

where for any set A, $\#A$ denotes the number of elements of A.

When there is no scope for confusion, we simply write $p_u(w)$ for $p_{u,X}(w)$. By using the arguments of Bose and Sen (2008), we immediately get the following proposition.

Proposition 2.1. *Let $\{X_n^u\}$ be a sequence of patterned triangular matrices satisfying Assumptions A and B. Suppose for every $k \ge 1$ and for every pair-matched word w of length $2k$, $p_u(w)$ exists. Then LSD of $\frac{X_n^u}{\sqrt{n}}$ exists a.s. The LSD is symmetric about 0 and is determined by the even moments*

$$\beta_{2k} = \sum_{\substack{w \text{ pair-matched,} \\ |w|=k}} p_u(w), \quad k \ge 1. \tag{2.4}$$

Proof. We provide a brief sketch of the proof. The first lemma in particular shows that the odd moments are zero and the LSD is symmetric.

Lemma 2.1. *Let w be a matched word of length k which is not pair-matched. Then,*

$$\lim_{n\to\infty} \frac{1}{n^{1+k/2}} \sum_{\pi \in \Pi_1^*(w)} \mathrm{E}[X_{\pi^*}] = 0.$$

Proof. First note that if w is non-matched then $\mathrm{E}[X_{\pi^*}] = 0$ since the entries have mean zero. Now consider a w which has more than two matches. Let $X_\pi = \prod_{i=1}^k x_{L(\pi(i-1),\pi(i))}$. Clearly then, $|X_{\pi^*}| \le |X_\pi|$ and hence $\mathrm{E}|X_{\pi^*}| \le \mathrm{E}|X_\pi|$. It is proved in Bose and Sen (2008) that,

$$\lim_{n\to\infty} \frac{1}{n^{1+k/2}} \sum_{\pi \in \Pi^*(w)} \mathrm{E}|X_\pi| = 0.$$

The Lemma follows immediately from this since $\Pi_1^*(w) \subset \Pi^*(w)$. $\square$

The next Lemma helps to verify Condition M4. The proof is a direct generalization of Lemma 2 of Bose and Sen (2008). We skip the details.

Lemma 2.2. *Let $\{X_n^u\}$ be a sequence of random patterned triangular matrices satisfying Assumptions A and B. Then*

$$\mathrm{E}\left[\frac{1}{n}\mathrm{Tr}(\frac{1}{\sqrt{n}}X_n^u)^k - \mathrm{E}(\frac{1}{n}\mathrm{Tr}(\frac{1}{\sqrt{n}}X_n^u)^k)\right]^4 = O(\frac{1}{n^2}).$$

Since there does not exist any pair-matched word of length k, for k odd, it follows from Lemma 2.1 that

$$\beta_k = \lim_{n\to\infty} \mathrm{E}(\frac{1}{n}\mathrm{Tr}(\frac{X_n^u}{\sqrt{n}})^k) = 0, \quad \text{if } k \text{ is odd.}$$

It also follows that,

$$\beta_{2k} = \lim_{n\to\infty} \mathrm{E}(\frac{1}{n}\mathrm{Tr}(\frac{X_n^u}{\sqrt{n}})^{2k}) = \sum_{\substack{w \text{ pair-matched,} \\ |w|=k}} p_u(w).$$

Hence Condition M1 also holds. From Lemma 2.2 it follows that Condition M4 is satisfied. It remains to show that the sequence $\{\beta_k\}$ satisfies Carleman's condition. It follows easily from Property B that for every pair-matched word of length $2k$,

$$\#\Pi_1^*(w) \le \#\Pi^*(w) \le n^{k+1}\Delta^k.$$

Hence $p_u(w) \le \Delta^k$ and so $\beta_{2k} \le \frac{(2k!)}{2^k k!}\Delta^k$ and Carleman's condition is thus easily verified. $\square$

We can use Proposition 2.1 to prove the existence of LSD for the triangular matrices listed earlier.

Theorem 2.1. *Let $\{X_n^u\}$ be any of the following triangular matrices with input sequence satisfying Assumption A: triangular Wigner, triangular Hankel, triangular Toeplitz and triangular Symmetric Circulant. Then LSD for $\frac{X_n^u}{\sqrt{n}}$ exists almost surely. The LSDs are symmetric with even moments given by (2.4).*

Proof. Notice that the link functions of all these matrices satisfy Property B. Thus Carleman's condition and condition M4 follow immediately if we can establish condition M1. The combinatorics to show the existence of $p_u(w)$ is done case-by-case. However, all the proofs are similar and use the volume method. We provide a sketch only for the triangular Winger and the triangular Hankel matrices. In the full matrix version of the matrices considered here, $p(w)$ is evaluated as an integral of an indicator function (see for example Bose et al. (2010)). Here also, $p_u(w)$ turns out to be an integral, but of a different indicator function corresponding to $\Pi_1^*(w)$ instead of $\Pi^*(w)$.

Triangular Wigner Matrix: Let L_W denote the Wigner link function. Now it is shown in Bose and Sen (2008) that if w is not Catalan then,

$$p(w) = \lim_{n\to\infty} \frac{\#\Pi^*(w)}{n^{1+k}} = 0.$$

As $\Pi^*(w) \supseteq \Pi_1^*(w)$, it follows that $p_u(w) = 0$ if w is non-Catalan. Thus we now focus on Catalan words of length $2k$.

Lemma 2.3. *Let w be a Catalan word of length $2k$. Let S denote the set of all generating vertices of w (including 0). Then for all $j \notin S$, there exists a unique $i \in S$ such that $i < j$ and $\pi(j) = \pi(i)$ for all $\pi \in \Pi^*(w)$.*

Proof. Let $\pi \in \Pi^*(w)$. Let j be the minimum index of a non-generating vertex of w. Clearly then, $w[j-1] = w[j]$ and hence $\pi(j-2) = \pi(j)$. Since $j > j-2 \in S$, the result is true in this case. Now let j be any non-generating vertex. Let us assume that for every non-generating vertex with index less than j, the result holds. Let $i < j$ be the index of first occurrence of $w[j]$. Let w_1 be the subword formed by letters between $w[i]$ and $w[j]$. Since w is Catalan, w_1 is also Catalan, and it can be easily shown that, $\pi(i) = \pi(j-1)$ and hence $\pi(j) = \pi(i-1)$. If $(i-1) \in S$, then we are already done. If $(i-1) \notin S$, then also by induction hypothesis the result holds.

If the i corresponding to a fixed j is not unique then we have a non-trivial relation between two generating vertices which implies $\#\Pi^*(w) = O(n^k)$, and hence contradicting the fact that $\lim_{n\to\infty} \frac{\#\Pi^*(w)}{n^{1+k}} = 1$. This proves the uniqueness. □

Definition 2.1. For any j (not necessarily in S), let us denote by $\phi(j)$, the *unique* vertex such that

$$\phi(j) \in S, \phi(j) \leq j \ \text{ and } \ \pi(j) = \pi(\phi(j)) \ \text{ for all } \ \pi \in \Pi^*(w).$$

Note that if $j \in S$, then $j = \phi(j)$. Next we note that among the $2k$ equations, $\pi(i-1) + \pi(i) \leq n+1, 1 \leq i \leq 2k$, each equation is repeated twice, as $w[i] = w[j] \Rightarrow \pi(i-1) + \pi(i) = \pi(j-1) + \pi(j)$. So we can write,

$$\begin{aligned} \Pi_1^*(w) &= \{\pi : w[i] = w[j] \Rightarrow L_W(\pi(i-1), \pi(i)) = L_W(\pi(j-1), \pi(j)), \\ &\qquad \pi(i-1) + \pi(i) \leq n+1 \ \forall i \in S - \{0\}\} \\ &= \{\pi : \pi(j) = \pi(\phi(j)) \ \ \forall j \notin S, \pi(\phi(i-1)) + \pi(\phi(i)) \leq n+1 \ \ \forall i \in S - \{0\}\}. \end{aligned}$$

Now we use the standard volume method arguments. Let us define,

$$v_i = \frac{\pi(i)}{n}, \ U_n = \left\{\frac{1}{n}, ..., \frac{n-1}{n}, 1\right\} \text{ and } v_S = \{v_i : i \in S\}. \tag{2.5}$$

Then,

$$\begin{aligned} \#\Pi_1^*(w) &= \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \ \forall 0 \leq i \leq 2k, v_i = v_{\phi(i)} \ \ \forall i \notin S, \\ &\qquad v_{\phi(i-1)} + v_{\phi(i)} \leq 1 + 1/n \ \ \forall i \in S - \{0\}, v_0 = v_{2k}\} \\ &= \#\{v_S : v_i \in U_n \ \ \forall i \in S, v_{\phi(i-1)} + v_{\phi(i)} \leq 1 + 1/n \ \ \forall i \in S - \{0\}\}. \end{aligned}$$

From the above equation it follows that, $\frac{\#\Pi_1^*(w)}{n^{1+k}}$ is nothing but the Riemann sum for the function $I(v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\})$ over $[0,1]^{k+1}$. Since the function is clearly Riemann integrable, the Riemann sum converges to the integral

$$\lim_{n\to\infty} \frac{1}{n^{1+k}} \#\Pi_1^*(w) = \int_{[0,1]^{k+1}} I(v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}) dv_S. \tag{2.6}$$

It follows that $p_u(w) = \lim_{n\to\infty} \frac{1}{n^{1+k}} \#\Pi_1^*(w)$ exists.

Triangular Hankel Matrix: The Hankel link function is $L(i,j) = i+j$. Here

$$\#\Pi_1^*(w) = \{\pi : w[i] = w[j] \Rightarrow \pi(i-1)+\pi(i) = \pi(j-1)+\pi(j), \pi(i-1)+\pi(i) \le n+1\}.$$

Let S denote the set of all generating vertices of w. For every $i \in S - \{0\}$, let j_i denote the index such that $w[i] = w[j_i]$. Let us define, v_i, U_n, v_S as in (2.5). Then,

$$\begin{aligned}\#\Pi_1^*(w) &= \#\{(v_0, ..., v_{2k}) : v_i \in U_n \forall 0 \le i \le 2k, v_{(i-1)} + v_i = v_{(j_i-1)} + v_{j_i} \forall i \in S, \\ &\qquad v_{(i-1)} + v_i \le 1 + 1/n \forall i \in S - \{0\}, v_0 = v_{2k}\}.\end{aligned}$$

It can easily be seen from the above equations (other than $v_0 = v_{2k}$) that each of the $\{v_i : i \notin S\}$ can be written uniquely as an integer linear combination $L_i(v_S)$. Moreover, $L_i(v_S)$ only contains generating vertices of index less than i with non-zero coefficients. For all $i \in S$, let us define $L_i(v_s) = v_i$.

Clearly,

$$\begin{aligned}\#\Pi_1^*(w) &= \#\left\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \le i \le 2k, v_0 = v_{2k}, v_i = L_i(v_S) \ \ \forall i \notin S, \right. \\ &\qquad \left. v_{i-1} + v_i \le 1 + 1/n \ \ \forall i \ \ \in S - \{0\}\right\}.\end{aligned}$$

Integer linear combinations of elements of U_n are again in U_n iff they are between 0 and 1. Hence,

$$\begin{aligned}\#\Pi_1^*(w) &= \#\left\{v_S : v_i \in U_n \ \ \forall i \ \ \in S, v_0 = L_{2k}(v_S), 0 < L_i^l(v_S) \le 1 \ \ \forall i \ \ \notin S, \right. \\ &\qquad \left. L_{i-1}(v_S) + L_i(v_s) \le 1 + 1/n \ \ \forall i \ \ \in S - \{0\}\right\}.\end{aligned} \tag{2.7}$$

From (2.7) it follows that, $\frac{\#\Pi_1^*(w)}{n^{1+k}}$ is nothing but the Riemann sum for the function $I(0 \le L_i^l(v_S) < 1, i \notin S, v_0 = L_{2k}^l(v_S), L_{i-1}(v_S) + L_i(v_s) \le 1 \ \ \forall i \in S - \{0\})$ over $[0,1]^{k+1}$. Since the function is clearly Riemann integrable, the Riemann sum converges to the integral

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^{1+k}} \#\Pi_1^*(w) &= \int_{[0,1]^{k+1}} I(0 \le L_i^l(v_S) < 1, i \notin S, v_0 = L_{2k}^l(v_S), \\ &\qquad L_{i-1}(v_S) + L_i(v_s) \le 1 \ \ \forall i \in S - \{0\}) dv_S.\end{aligned}$$

Hence $p_u(w)$ exists for every pair-matched word w.

Incidentally, for a Hankel matrix $p(w) = 0$ if w is not a symmetric word. Hence it also follows that $p_u(w) = 0$ for every asymmetric word w.

□

3. The LSD of Triangular Wigner Matrix

Note that unlike the (full) Wigner matrix, contribution of all Catalan words is not same for the triangular Wigner matrix. This leads to the LSD being non-semicircular. In the following table we list the values of $p_u(w)$ for some Catalan words of small lengths. It may be observed that the contributions are equal within certain isomorphic classes. Though it does not seem easy to obtain the individual $p_u(w)$'s for different Catalan words, it is, however possible to calculate their total contribution in a relatively simple way. We turn to this direction now.

TABLE 1. $p_u(w)$ for Catalan words for W_n^u

Word	$p_u(w)$
aa	1/2
aabb	1/3
abba	1/3
aabbcc	1/4
abbcca	1/4
abbacc	5/24
aabccb	5/24
abccba	5/24

Theorem 3.1. *Let W_n^u be a triangular Wigner matrix with an input sequence satisfying Assumption A. Then almost surely*

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left(\frac{W_n^u}{\sqrt{n}}\right)^{2k}=\frac{k^k}{(k+1)!}.$$

Proof. To begin with, we need the following the simple lemma whose proof is omitted.

Lemma 3.1. *(i) Let w be a Catalan word of length $2k$. Let π be a function $\pi:\{0,1,...,k\}\to\{1,2,...,n\}$ such that $w[i]=w[j]\Rightarrow(\pi(i-1),\pi(i))=(\pi(j),\pi(j-1))$. Then $\pi(0)=\pi(2k)$ and hence $\pi\in\Pi^*(w)$.*

(ii) Let $\phi(j)$ be as in Definition 2.1. If $w=w_1w_2$ where w_1 and w_2 are Catalan words of length $2k_1$ and $2k_2$ respectively. Then for every vertex $i>2k_1$, $\phi(i)>2k_1$ or $\phi(i)=\phi(2k_1)$.

Now let w be a Catalan word of length $2k$. Let S be the set of generating vertices for w. It has already been shown that (see (2.6))

$$p_u(w)=\underset{[0,1]^{k+1}}{\int\cdots\int}I(v_{\phi(i-1)}+v_{\phi(i)}\le 1, i\in S-\{0\})dv_S. \tag{3.1}$$

This integral can be evaluated as an iterated integral. It is clear that integrating out all the variables other that v_0 leaves a polynomial in v_0, say Q_w. It follows that

$$p_u(w)=\int_0^1 Q_w(v_0)dv_0 \tag{3.2}$$

for some *polynomial* $Q_w(\cdot)$. Let us illustrate with a few examples. Let $w=aa$. Then

$$p_u(w)=\int_{v_0+v_1\le 1}dv_0dv_1=\int_0^1(1-v_0)dv_0$$

and hence

$$Q_{aa}(x)=1-x.$$

Now let $w=abba$. Then

$$p_u(w) \;=\; \underset{[0,1]^3}{\int\cdots\int}I(v_0+v_1\le 1, v_1+v_2\le 1)dv_0dv_1dv_2$$

$$
\begin{aligned}
&= \int_0^1 \int_0^{1-v_0} \int_0^{1-v_1} dv_2 dv_1 dv_0 \\
&= \int_0^1 \int_0^{1-v_0} (1-v_1) dv_1 dv_0 = \int_0^1 \frac{(1-v_0^2)}{2} dv_0.
\end{aligned}
$$

Hence

$$Q_{abba}(x) = \frac{1-x^2}{2}.$$

The following two Lemma collect the required properties of $Q_w(\cdot)$.

Lemma 3.2. *(i) Let $w = w_1 w_2$ be a Catalan word of length $2k$ where w_1 and w_2 are both Catalan. Then $Q_w(x) = Q_{w_1}(x) Q_{w_2}(x)$.*

(ii) Let $w = a w_1 a$ be a Catalan word with w_1 Catalan. Then

$$Q_w(x) = \int_0^{1-x} Q_{w_1}(y) dy.$$

Proof. (i) Let w_1 and w_2 be Catalan words of length $2k_1$ and $2k_2$ respectively. We divide the set of inequalities in the indicator function in (3.1) above into two classes:

(1) $v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}, i \leq 2k_1$; i.e. the inequalities corresponding to generating vertices in w_1.

(2) $v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}, i > 2k_1$; i.e. the inequalities corresponding to generating vertices in w_2.

From Lemma 3.1 (i) it follows that $\phi(2k_1) = 0$ and from Lemma 3.1 (ii) that the inequalities in items 1 and 2 do not have any common variable except v_0. Hence in (3.1) integrating w.r.t the variables corresponding to the generating vertices in w_1 we get $Q_{w_1}(v_0)$ and integrating w.r.t. the variables corresponding to the generating vertices in w_2, we get $Q_{w_2}(v_0)$. The result now follows from the definition of Q_w.

(ii) Proof of this is similar and we only give a sketch. Note that the variable v_0 does not occur anywhere in the equations corresponding to the generating vertices in w_1. Integrating w.r.t. all these variables (other than v_1), we get $Q_{w_1}(v_1)$. Hence the last step of evaluating the iterated integral is by (3.1)

$$p_u(w) = \int_{v_0+v_1 \leq 1} Q_{w_1}(v_1) dv_0 dv_1 = \int_0^1 \left[\int_0^{1-v_0} Q_{w_1}(v_1) dv_1 \right] dv_0$$

and the result follows. □

Let

$$G_0(x) = 1 \text{ and } G_{2n}(x) = \sum_{\substack{w \text{ Catalan,} \\ |w|=n}} Q_w(x). \tag{3.3}$$

Lemma 3.3. *Then*
(i) for $n \geq 1$ we have,

$$G_{2n}(x) = \sum_{k=1}^{n} G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y) dy.$$

(ii)

$$G_{2n}(x) = \frac{(1-x)(n+1-x)^{n-1}}{n!}$$

for all $n, n \geq 0$.

Proof. (i) For $n = 1$, the only Catalan word of length $2k$ is aa. Hence $G_2(x) = Q_{aa}(x) = 1 - x$ and the result is true for $n = 1$. Let $n \geq 2$ and let $G_{2n,k}(x)$ be the sum of $Q_w(x)$ over all Catalan words w such that the first letter is repeated at the $2k$-th place. Clearly, for such a w, $w = aw_1aw_2$ where w_1 is a Catalan word of length $2(k-1)$ and w_2 is a Catalan word of length $2(n-k)$.
Using the previous lemma, it follows that

$$\begin{aligned}
G_{2n,k}(x) &= \sum_{|w_1|=(k-1),|w_2|=(n-k)} Q_{aw_1aw_2}(x) \\
&= \sum_{|w_1|=(k-1),|w_2|=(n-k)} Q_{aw_1a}(x)Q_{w_2}(x) \\
&= \sum_{|w_1|=(k-1)} Q_{aw_1a}(x) \sum_{|w_2|=(n-k)} Q_{w_2}(x) \\
&= \sum_{|w_1|=(k-1)} \int_0^{1-x} Q_{w_1}(y)dy \sum_{|w_2|=(n-k)} Q_{w_2}(x) \\
&= \sum_{|w_2|=(n-k)} Q_{w_2}(x) \int_0^{1-x} (\sum_{|w_1|=(k-1)} Q_{w_1}(y))dy \\
&= G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y)dy.
\end{aligned}$$

As $G_{2n}(x) = \sum_{k=1}^{n} G_{2n,k}(x)$, part (i) follows.

(ii) We prove this by induction. The case $n = 0, 1$ is clear. Now let us suppose that the result is true for all $j < n$. Then by the last lemma and induction hypothesis

$$\begin{aligned}
G_{2n}(x) &= \sum_{k=1}^{n} G_{2(n-k)}(x) \int_0^{1-x} G_{2(k-1)}(y)dy \\
&= \sum_{k=1}^{n} \frac{(1-x)(n-k+1-x)^{n-k-1}}{(n-k)!} \int_0^{1-x} \frac{(1-y)(k-y)^{k-2}}{(k-1)!}dy \\
&= \sum_{k=1}^{n} \frac{(1-x)(n-k+1-x)^{n-k-1}}{(n-k)!} \frac{(1-x)(k-1+x)^{k-1}}{k!} \\
&= \frac{(-1)^{n-1}}{n!} \sum_{k=1}^{n} \binom{n}{k} p_k(z)p_{n-k}(-z)
\end{aligned}$$

where $z = 1 - x$ and

$$p_n(x) = x(x-n)^{n-1}$$

is the Abel Polynomial of degree n. It is a well known fact (Riordan, 1979) that Abel polynomials satisfy the following combinatorial identity:

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} p_k(x) p_{n-k}(y).$$

It follows that

$$\begin{aligned} G_{2n}(x) &= \frac{(-1)^{n-1}}{n!} \sum_{k=0}^{n} \binom{n}{k} p_k(z) p_{n-k}(-z) + \frac{(-1)^n}{n!} p_0(z) p_n(-z) \\ &= \frac{(-1)^{n-1}}{n!} p_n(0) + \frac{(-1)^n}{n!} p_n(-z) \\ &= \frac{(-1)^n(-z)(-z-n)^{n-1}}{n!} \\ &= \frac{z(z+n)^{n-1}}{n!} \\ &= \frac{(1-x)(n+1-x)^{n-1}}{n!} \end{aligned}$$

and then the proof of Lemma 3.3 is complete. □

Now

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}\Big(\frac{W_n^u}{\sqrt{n}}\Big)^{2k} &= \sum_{\substack{w \text{ Catalan,} \\ |w|=k}} p_u(w) \\ &= \int_0^1 G_{2k}(x)dx \qquad \text{(see equation(3.3))} \\ &= \int_0^1 \frac{(1-x)(k+1-x)^{k-1}}{k!} dx \quad \text{(Lemma 3.3(ii))} \\ &= \frac{k^k}{(k+1)!}, \end{aligned}$$

proving Theorem 3.1 completely. □

4. Some comments on other variants and joint convergence

(1) It can be shown that the limit which appears in the triangular Wigner matrix can appear in a variety of other matrices which satisfy the so called *Property P* introduced in Banerjee and Bose (2011). The Property P states that

$$M^* = \sup_n \sup_{i,j \le n} |\{1 \le k \le n : k+i \le n+1,\ k+j \le n+1,\ L(k,i) = L(k,j)\}| < \infty.$$

The proof of this fact is similar to the proofs in in Banerjee and Bose (2011) and hence we skip it.

(2) The joint convergence of patterned random matrices was initiated in Bose et al. (2011) and was further studied by Basu et al. (2011) for multiple copies of different independent patterned random matrices. We can so study the joint convergence of independent patterned triangular matrices along the same

lines. It can be shown that if the entries have uniformly bounded moments and if $p_u(w)$ exists then independent copies of patterned random matrices jointly converge with respect to $\frac{1}{n}$ Tr for any of the matrices considered in this article. The behavior in the limit though, is a highly non-trivial problem.

(3) Let $W^u_{1,n}, W^u_{2,n}$ be two triangular Wigner matrices having uniformly bounded moments. Let $(\{a_1, a_2\}, \phi)$ denote the joint limit of $(\{\frac{W^u_{1,n}}{\sqrt{n}}, \frac{W^u_{2,n}}{\sqrt{n}}\}, \phi_n)$ where $\phi_n = \frac{1}{n}\mathrm{E}[\mathrm{Tr}(\cdot)]$. Then a_1 and a_2 are not free. In fact,

$$\phi(a_1^2) = \phi(a_2^2) = \lim_{n\to\infty} \frac{1}{n}\mathrm{E}[\mathrm{Tr}(W^u_{1,n}/\sqrt{n})^2] = p_{u,W}(aa) = 1/2.$$

Also, from Table 1,

$$\begin{aligned} \phi(a_1^2 a_2^2) &= \lim_{n\to\infty} \frac{1}{n}\mathrm{E}[\mathrm{Tr}((W^u_{1,n}/\sqrt{n})^2 (W^u_{2,n}/\sqrt{n})^2)] \\ &= p_{u,W}(aabb) = 1/3. \end{aligned}$$

As $\phi(a_1^2 a_2^2) \neq \phi(a_1^2)\phi(a_2^2)$, a_1 and a_2 are not free.

(4) Another variant of the triangular Wigner matrix is

$$W^l_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & x_{1n} \\ 0 & 0 & 0 & \dots & x_{2(n-1)} & x_{2n} \\ & & & \vdots & & \\ 0 & x_{2(n-1)} & x_{3(n-1)} & \dots & x_{(n-1)(n-1)} & x_{(n-1)n} \\ x_{1n} & x_{2n} & x_{3n} & \dots & x_{(n-1)n} & x_{nn} \end{bmatrix}.$$

It should be noted that we can always delete the diagonal from the above matrix without changing the limiting spectral distribution. Let P be the following orthogonal matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 1 & 0 \\ & & & \vdots & & \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{4.1}$$

Then we have

$$PW^u_n P' = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & x_{1n} \\ 0 & 0 & 0 & \dots & x_{2(n-1)} & x_{1(n-1)} \\ & & & \vdots & & \\ 0 & x_{2(n-1)} & 0 & \dots & x_{22} & x_{12} \\ x_{1n} & x_{1(n-1)} & x_{1(n-2)} & \dots & x_{12} & x_{11} \end{bmatrix} \tag{4.2}$$

which can be seen to be equal to W^l_n upon renaming of the variables. Since conjugation by orthogonal matrices preserves eigenvalues, we can conclude that the limiting spectral distribution of $W^l_n/\sqrt{n}$ is same as that of W^u_n.

(5) Interestingly, $W_n^u/\sqrt{n}$ and $W_n^l/\sqrt{n}$ jointly converge. If we call the limiting variables as a_1, a_2, then again it can be shown that they are not free. But it easily follows that $a_1 + a_2$ obeys the the semicircular law.

(6) As is well known, for the full version of the matrices, the contribution of every Catalan word to the limiting moment equals one for all the common patterned matrices. Further, for the Symmetric Circulant every word contributes one. For the Reverse Circulant, only the so called symmetric words contribute, each contributing one. For the Hankel matrices also, only the symmetric words contribute but all of them do not contribute one or even contribute equally. See for example Bose et al. (2011). It is an interesting question to study the individual contribution of each word to the limiting moment for the triangular matrices. This does not appear to be easy, even for the Wigner case.

(7) The following facts on word contribution can be proved for the triangular Wigner matrices. The proof is given in Section 5.

(i) For any word w of length $2k$, $p_u(w) \leq 1/(k+1)$.

(ii) If w is of the form $w = aabbcc\ldots$ then $p_u(w) = 1/(k+1)$.

(iii) If w and w' are Catalan words of the form $w = aw_1aw_2$ and $w' = aaw_1w_2$ then $p_u(w) \leq p_u(w')$.

(iv) If w and w' are Catalan words of the form $w = abbaw_1w_2$ and $w' = abw_1baw_2$ are Catalan words then $p_u(w) \geq p_u(w')$.

(8) We wish to point out some histogram plots for the patterned triangular matrices. In Figures 1 and 2 we have plotted the histograms of the empirical spectral distribution of the symmetric triangular Wigner, Toeplitz, Hankel and Symmetric Circulant matrices.

It can be easily checked from the density equation given earlier in 1.5 that the density is unbounded at zero for the triangular Wigner. Further, from the moment formula, one can easily check that the support of the LSD is contained in $[-\sqrt{e},\ \sqrt{e}]$. We can see evidence of both these facts in the histogram. It is known that the maximimum eigenvalue of the full Wigner matrix converges almost surely to 2. We believe that likewise, for the triangular Wigner, the maximum eigenvalue converges to $\sqrt{e}$ almost surely.

As pointed out earlier, obtaining moment properties of the LSD for the other three matrices does not seem to be easy. However, it appears from the simulations that all the other three LSDs also have densities and these are also unbounded at zero. Moreover all these LSDs have unbounded support, just like for their full matrix versions.

Interestingly, it is known that the LSD of the full Hankel is not unimodal (simulations results show that it is bimodal but there is no formal proof). However, the simulation evidence implies that the triangular Hankel LSD is unimodal. Recall that for the full Hankel matrix, the entries in the upper triangle and the entries in the lower triangle (ignoring the main antidiagonal part) are independent of each other. One wonders whether this particular nature has anything to do with the bimodality of the LSD of the full Hankel.

(9) Now suppose we consider triangular patterned matrices but we drop the assumption of symmetry. Let A_n be any such matrix. It can be shown by the moment method that the LSD of the symmetric matrix A_nA_n'/n exists. A natural (and hard) question is what happens to the LSD for the asymmetric matrix $A_n/\sqrt{n}$. The moment method cannot be directly applied and one has to resort to the Stieltjes transform.

(10) Let $X^{(s)}$ be a symmetric matrix with link function L and let X be it asymmetric version as defined in Bose et al. (2010a). A general question that was raised there was the following. Suppose $X^{(s)}/\sqrt{n}$ has an LSD identified by a random variable say X. Suppose XX'/n has an LSD denoted by Y say. When do X^2 and Y have the same distribution?

The answer is affirmative when $X^{(s)}$ is a symmetric Wigner matrix and X is its asymmetric version (the matrix with i.i.d. entries) from the relation between Marčenko-Pastur law and the Wigner law. In Bose et al. (2010a) it was shown that this is true for the Toeplitz matrix but not for Hankel or Reverse Circulant matrices.

The result in this article gives another important example in the form of the triangular Wigner matrix.

FIGURE 1. Histogram plots of empirical distribution of triangular matrices ($n = 1000$) with entries $N(0,1)$ of (i) Wigner (left) (ii) Toeplitz (right)

FIGURE 2. Histogram plots of empirical distribution of triangular matrices ($n = 1000$) with entries $N(0,1)$ of (i) Symmetric Circulant (left) (ii) Hankel (right)

5. Appendix

5.1. **Contribution of different Catalan Words for triangular Wigner Matrix.**

It appears difficult, even for triangular Wigner matrix, to determine $p_u(w)$ for each Catalan word w. However, $p_u(w)$ can be determined for some special classes of words. As we have already seen, unlike what happens for a full Wigner matrix, $p_u(w)$ is not same for all Catalan words. Here we record some observations about contributions of different Catalan words for triangular Wigner matrices. This might be useful in obtaining more information about the individual $p_u(w)$'s. We start with a simple lemma.

Lemma 5.1. *Let w be the Catalan word of length $2k$, $w = a_1a_1a_2a_2...a_ka_k$. Then $p_u(w) = \frac{1}{k+1}$.*

Proof. Clearly, $\pi \in \Pi^*(w)$ implies

$$(\min(\pi(0), \pi(1)), \max(\pi(0), \pi(1))) = (\min(\pi(1), \pi(2)), \max(\pi(1), \pi(2)))$$

and hence $\pi(0) = \pi(2)$. Arguing similarly, it follows that,

$$\Pi^*(w) = \{\pi : \pi(0) = \pi(2) = \pi(4) = \ldots = \pi(2k)\}.$$

Hence,

$$|\Pi_1^*(w)| = \#\{\pi : \pi(0) + \pi(1) \le n, \pi(0) + \pi(2) \le n, \ldots, , \pi(0) + \pi(k) \le n\}.$$

Now

$$\begin{aligned} p_u(w) &= \lim_{n\to\infty} \frac{|\Pi_1^*(w)|}{n^{1+k}} \\ &= \int\cdots\int_{[0,1]^{k+1}} I(v_0 + v_1 \le 1, v_0 + v_2 \le 2, v_0 + v_3 \le 1, ..., v_0 + v_k \le 1) dv_0 dv_1 ... dv_k \\ &= \int_0^1 \left[\int_0^{1-v_0} \cdots \int_0^{1-v_0} dv_1 \cdots dv_k\right] dv_0 \\ &= \int_0^1 (1 - v_0)^k dv_0 \\ &= \frac{1}{k+1} \end{aligned}$$

which completes the proof. □

Now we show that words in some classes contribute more than words in some other classes. Before that we need the following two lemmas.

Lemma 5.2. *Let w be a Catalan word of length $2k$. Let π be a function $\pi : \{0, 1, ..., k\} \to \{1, 2, ..., n\}$ such that $w[i] = w[j] \Rightarrow (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))$. Then $\pi(0) = \pi(2k)$ and hence $\pi \in \Pi^*(w)$.*

Proof. The proof is easy by induction and we omit the details. □

Lemma 5.3. *Let w be a Catalan word of length $2k$ such that $w = aw_1aw_2$, where w_1 and w_2 are Catalan words. Let w' denote the Catalan word aaw_1w_2. Then $p_u(w) \leq p_u(w')$.*

Proof. Let us denote, for any set $U \in \mathbb{R}^{k+1}$, by $Vol(U)$, the Lebesgue measure of the set $U \bigcap [0,1]^{k+1}$. From what we have already proved, it follows that for every Catalan word w_0 of length $2k$, $p_u(w_0) = Vol(U_0)$ where U_0 is the region in $\mathbb{R}^{k+1}$ given by the set of all v's determined by the set of inequalities

$$N = \{v_{\phi(i-1)} + v_{\phi(i)} \leq 1, i \in S - \{0\}\}. \tag{5.1}$$

Now let $w = aw_1aw_2$ and let w_1 and w_2 be Catalan words of lengths $2k_1$ and $2k_2$. Now we partition the set N into three classes.

(1) $N_1 = \{v_0 + v_1 \leq 1\}$: This is the first inequality since $\phi(0) = 0, \phi(1) = 1$ and $1 \in S$.

(2) $A_{w_1}(v_1)$: this is the set of all inequalities $\{v_{\phi(i-1)} + v_{\phi(i)} \leq 1 : 2 \leq i \leq 2k_1 + 1, i \in S - \{0\}\}$; i.e. the inequalities corresponding to the generating vertices in w_1. By Lemma 3.1 it follows that these inequalities do not involve the variable v_0. Also, the inequalities only depend on w_1 and v_1.

(3) $A_{w_2}(v_0)$: this is the set of all the inequalities $\{v_{\phi(i-1)} + v_{\phi(i)} \leq 1 : 2k_1 + 3 \leq i \leq 2k, i \in S - \{0\}\}$; i.e. the class of inequalities corresponding to the generating vertices in w_2. From Lemma 5.2, it follows that $\pi(2k+2) = \pi(0)$ for every $\pi \in \Pi^*(w)$ and hence $\phi(2k+2) = 0$. Now Lemma 3.1 implies that all the variables occurring in these inequalities are independent of the variables that occurred previously, apart from v_0. Also, the inequalities only depend on w_1 and v_0.

It follows that $p_u(w) = Vol(B_w)$ where

$$B_w = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0).$$

Similarly, for the word $w' = aaw_1w_2$ the three classes of inequalities are:

(1) N_1:

(2) $A_{w_1}(v_0)$: in this case $\phi(2) = 0$. By renaming the variables other than v_0, if necessary, the class of inequalities corresponding to the generating vertices in w_1 in this case is same as that in the previous case with v_0 replaced by v_1. By Lemma 3.1 it follows that these inequalities do not involve the variable v_1.

(3) $A_{w_2}(v_0)$: the class of inequalities corresponding to the generating vertices in w_2 in this case is same as that in the previous case after renaming the variables other than v_0. Lemma 3.1 implies that all the variables occurring in these inequalities are independent of the variables occurred previously, apart from v_0.

It follows that $p_u(w) = Vol(B_{w'})$ where

$$B_{w'} = N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0).$$

Now

$$Vol(B_w) \quad = \quad Vol(B_w^1) + Vol(B_w^2) \text{ and}$$

$$Vol(B_{w'}) \;=\; Vol(B^1_{w'}) + Vol(B^2_{w'})$$

where

$$\begin{aligned}B^1_w &= N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_0 \le v_1\},\\ B^2_w &= N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_1 \le v_0\}\end{aligned}$$

and

$$\begin{aligned}B^1_{w'} &= N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \le v_1\},\\ B^2_{w'} &= N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_1 \le v_0\}.\end{aligned}$$

It is easy to see that $B^1_{w'} \supseteq B^1_w$ and $B^2_w \supseteq B^2_{w'}$.

Now,

$$\begin{aligned}B^1_{w'} - B^1_w &= N_1 \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \le v_1\} \cup (A_{w_1}(v_1))';\\ B^2_w - B^2_{w'} &= n_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_0) \cup \{v_1 \le v_0\} \cup (A_{w_1}(v_0)'.\end{aligned}$$

Now by interchange of variables v_0 and v_1, it follows that $Vol(B^1_{w'} - B^1_w) = Vol(C_{w,w'})$ where

$$C_{w,w'} = N_1 \cup A_{w_1}(v_1) \cup A_{w_2}(v_1) \cup \{v_1 \le v_0\} \cup (A_{w_1}(v_0))'.$$

Again, $v_1 \le v_0$ and $A_{w_2}(v_0)$ together imply $A_{w_2}(v_1) \le 1$ and hence $C_{w,w'} \supseteq B^2_w - B^2_{w'}$. It follows that,

$$\begin{aligned}Vol(C_{w,w'}) &\ge Vol(B^2_w - B^2_{w'})\\ \Rightarrow Vol(B^1_{w'} - B^1_w) &\ge Vol(B^2_w - B^2_{w'})\\ \Rightarrow Vol(B^1_{w'}) + Vol(B^1_{w'}) &\ge Vol(B^1_w) + VolB^2_w)\end{aligned}$$

and this completes the proof.

□

The next Lemma is similar to the previous one.

Lemma 5.4. *Let $w = abbaw_1w_2$ and $w' = abw_1baw_2$ be two Catalan words where w_1 and w_2 are Catalan words. Then $p_u(w) \ge p_u(w')$.*

Proof. We use the same notations. As in the previous Lemma, now $p_u(w) = Vol(B_w)$ and $p_u(w') = Vol(B_{w'})$ where

$$\begin{aligned}B_w &= N_1 \cup \{v_1 + v_2 \le 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \text{ and}\\ B_{w'} &= N_1 \cup \{v_1 + v_2\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0).\end{aligned}$$

Now

$$\begin{aligned}Vol(B_w) &= Vol(B^1_w) + Vol(B^2_w) \text{ and}\\ Vol(B_{w'}) &= Vol(B^1_{w'}) + Vol(B^2_{w'})\end{aligned}$$

where

$$\begin{aligned}B^1_w &= N_1 \cup \{v_1 + v_2 \le 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_0 \le v_2\},\\ B^2_w &= N_1 \cup \{v_1 + v_2 \le 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0) \cup \{v_2 \le v_0\},\end{aligned}$$

and

$$\begin{aligned} B^1_{w'} &= N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0) \cup \{v_0 \leq v_2\}, \\ B^2_{w'} &= N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0) \cup \{v_2 \leq v_0\}. \end{aligned}$$

As $v_0 \leq v_2$ and $A_{w_1}(v_2)$ together imply $A_{w_1}(v_0)$, $B^1_w \supseteq B^1_{w'}$. Similarly it follows that $B^2_{w'} \supseteq B^2_w$.

Now,

$$\begin{aligned} B^1_w - B^1_{w'} &= N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_0)\{v_0 \leq v_2\} \cup (A_{w_1}(v_2))'; \\ B^2_{w'} - B^2_w &= N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_2) \cup A_{w_2}(v_0)\{v_2 \leq v_0\} \cup (A_{w_1}(v_0))'. \end{aligned}$$

Once again by interchange of two variables v_0 and v_2 we see that $Vol(B^2_{w'} - B^2_w) = Vol(C_w, w')$ where

$$C_{w,w'} = N_1 \cup \{v_1 + v_2 \leq 1\} \cup A_{w_1}(v_0) \cup A_{w_2}(v_2)\{v_0 \leq v_2\} \cup (A_{w_1}(v_2)'.$$

Again, $v_0 \leq v_2$ and $A_{w_2}(v_2)$ together imply $A_{w_2}(v_0)$ and hence $Cw, w' \subseteq B^1_w - B^1_{w'}$. It follows that,

$$\begin{aligned} Vol(Cw, w') &\leq Vol(B^1_w - B^1_{w'}) \\ \Rightarrow Vol(B^1_{w'} - B^1_w) &\geq Vol(B^2_w - B^2_{w'}) \end{aligned}$$

and the proof is completed as in the previous Lemma. □

The next proposition gives an upper bound on $p_u(w)$'s.

Proposition 5.1. *Let w be a Catalan word of length $2k$. Then $p_u(w) \leq \frac{1}{k+1}$.*

Proof. If $w = a_1a_1a_2a_2...a_ka_k$, then the result follows from Lemma 5.1. If not, by left rotation, we can obtain, from w, a word w' such that w' does not start with a double letter and w' is not of the form aw_1a either. Since $p_u(w)$ is invariant under rotation, $p_u(w) = p_u(w')$. By hypothesis, w' must be of the form $w' = aw_1aw_2$ where w_1 and w_2 are Catalan words. By Lemma 5.3, $p_u(w) = p_u(w') \leq p_u(aaw_1w_2) = p_u(w_1w_2aa)$ and the number of consecutive double letters at the end of w_1w_2aa is strictly greater than that of w. The proof can now be completed by induction on the number of consecutive double letters at the end of w. □

We can see from the above results that $p_u(w)$ depends upon the Noncrossing structure of the word w. It is an interesting combinatorial problem to obtain a formula for $p_u(w)$ as a function of the Catalan structure of w or to investigate what Catalan structures gives rise to the same $p_u(w)$'s.

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Department of Statistics University of California, Berkeley
E-mail address: `riddhipratim@stat.berkeley.edu`

Statistics and Mathematics Unit Indian Statistical Institute 203 B.T. Road Kolkata 700108 India
E-mail address: `bosearu@gmail.com`

Department of Mathematics, University of Washington, Seattle
E-mail address: `shirshenduganguly2006@gmail.com`

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057, Zürich
E-mail address: `rajatmaths@gmail.com`