PATTERNED MATRICES AND NOTIONS OF INDEPENDENCE

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Abstract

It is known that the joint limit distribution of independent Wigner matrices satisfies a very special asymptotic independence, called freeness. We study the joint convergence of a few other patterned matrices, providing a framework to accommodate other joint laws. In particular, the matricial limits of symmetric circulants and reverse circulants satisfy respectively the classical independence and the half independence. The matricial limits of Toeplitz and Hankel matrices do not seem to submit to any easy or explicit independence/dependence notions. Their limits are not independent, free or half independent. Finally, we prove a central limit theorem and a Cramér type characterization for identically distributed half independent random variables.

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1 Introduction

Wigner [14] showed how the semicircular law arises as the limit of the empirical spectral distribution of a sequence of Wigner matrices. See for example [2] and [1] for such results and their variations. Then researchers studied the arbitrary product of Wigner matrices formed from a class of independent Wigner matrices. The trace of any such product converges and this is tied to the idea of free independence developed by Voiculescu [15]. This freeness in the limit is very special to the Wigner type matrices.

It is natural to study the product formed from other symmetric independent ensembles. However, it appears that the study of joint distribution of random matrices has been mostly concentrated on Wigner like matrices. As one of the Referees pointed out, [12] considers the joint distribution of Vandermonde matrices and diagonal matrices and emphasises the importance of studying joint distribution of other pattern matrices.

We study the joint convergence of symmetric patterned matrices and in particular, show that for independent copies of the Toeplitz, Hankel, symmetric circulant and reverse circulant matrices,

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the tracial limits exist for any monomial formed with these independent copies. The Wigner, symmetric circulant and the reverse circulant yield respectively the free semicircular, the classical independent normal and the half independent Rayleigh limits. The Toeplitz and Hankel limits do not seem to submit to any easy or explicit independence/dependence notions. These limits are not free, independent or half independent and are worth further investigation. In what follows we show in Proposition 1 that under some conditions if the marginal distribution exists then the joint limit exists and can we expressed in terms of the marginals with the help of words and colored words.

In Section 2 we discuss some preliminaries on noncommutative probability spaces and recall the notions of independence, freeness and half independence. In Section 3 we introduce the notion of colored words in our setup and state our main result, Proposition 1. In Subsection 3.2 we discuss some examples and in Subsection 3.3 we explain how a restricted pyramidal multiplicativity appears for Hankel and Toeplitz limits. In Subsection 3.4 we give a proof of Proposition 1. Finally in Section 4 we prove a central limit theorem and a Cramér type characterization for identically distributed half independent random variables.

2 Noncommutative probability spaces and independence

A *noncommutative probability space* is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital complex algebra (with unity 1) and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\phi(1) = 1$.

Two important examples of such spaces are the following.

(1) Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $L(\mu) = \bigcap_{1 \le p < \infty} L^p(\mu)$ be the algebra of random variables with finite moments of all orders. Then $(L(\mu), \phi)$ becomes a (commutative) probability space where ϕ is the expectation operator, that is, integration with respect to μ.

(2) Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{A} = Mat_n(L(\mu))$ be the space of $n \times n$ complex random matrices with elements from $L(\mu)$. Then ϕ equal to $\frac{1}{n} \operatorname{E}_\mu[\operatorname{Tr}(\cdot)]$ or $\frac{1}{n}[\operatorname{Tr}(\cdot)]$ both yield noncommutative probabilities.

For any non commuting variables $x_1, \ldots, x_n$, let $\mathbb{C}\langle x_1, x_2, ..., x_n \rangle$ be the unital algebra of all complex polynomials in these variables. If $a_1, a_2, \ldots, a_n \in \mathcal{A}$ then their joint distribution $\mu_{\{a_i\}}$ is defined canonically by their mixed moments $\mu_{\{a_i\}}(x_{i_1} \ldots x_{i_m}) = \phi(a_{i_1} \ldots a_{i_m})$. That is,

$$\mu_{\{a_i\}}(P) = \phi(P(\{a_i\})) \text{ for } P \in \mathbb{C}\langle x_1, x_2, ..., x_n \rangle.$$

Definition 1. Convergence in Law. *Let $(\mathcal{A}_n, \phi_n)$, $n \ge 1$ and $(\mathcal{A}, \phi)$ be noncommutative probability spaces and let $\{a_i^n\}_{i \in J}$ be a sequence of subsets of $\mathcal{A}_n$ where J is any finite subset of $\mathbb{N}$. Then we say that $\{a_i^n\}_{i \in J}$ converges in law to $\{a_i\}_{i \in J} \subset \mathcal{A}$ if and only if for all complex polynomials P,*

$$\lim_{n \to \infty} \mu_{\{a_i^n\}_{i \in J}}(P) = \mu_{\{a_i\}_{i \in J}}(P).$$

To verify convergence in the above definition, it is enough to verify the convergence for all monomials $q = x_{i_1} \ldots x_{i_k}$, $k \ge 1$.

Independence and free independence of algebras: Suppose $\{\mathcal{A}_i\}_{i \in J} \subset \mathcal{A}$ are unital subalgebras. They are said to *independent* if they commute and $\phi(a_1 \ldots a_n) = \phi(a_1) \ldots \phi(a_n)$ for all $a_i \in \mathcal{A}_{k(i)}$ where $i \neq j \implies k(i) \neq k(j)$.

These subalgebras are called *freely independent* or simply *free* if $\phi(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for all j implies $\phi(a_1 \ldots a_n) = 0$. The random variables (or elements of an algebra)

$(a_1, a_2, \ldots)$ will be called independent (respectively free) if the subalgebras generated by them are independent (respectively free).

Half independence of elements of an algebra: Half independence is different from the above two notions and arises in classification results for easy quantum groups and some quantum analogue of de Finetti's theorem. We closely follow the developments in [4]. We shall see later how this half independence arises in the context of convergence of random matrices.

Half commuting elements. Let $\{a_i\}_{i \in J} \subset \mathcal{A}$. We say that they *half commute* if $a_i a_j a_k = a_k a_j a_i$, for all $i, j, k \in J$. Observe that if $\{a_i\}_{i \in J}$ half commute then a_i^2 commutes with a_j and a_j^2 for all $i, j \in J$.

Symmetric monomials. Suppose $\{a_i\}_{i \in J} \subset \mathcal{A}$. For any $k \geq 1$, and any $\{i_j\} \subset J$, let $a = a_{i_1} a_{i_2} \ldots a_{i_k}$ be an element of $\mathcal{A}$. For any $i \in J$, let $E_i(a)$ and $O_i(a)$ be respectively, the number of times a_i has occurred in the even positions and in the odd positions in a. The monomial a is said to be **symmetric** (with respect to $\{a_i\}_{i \in J}$) if $E_i(a) = O_i(a)$ for all $i \in J$. Else it is said to be nonsymmetric.

Definition 2. ***Half independent elements***. *Let $\{a_i\}_{i \in J}$ be a family of half commuting variables in $(\mathcal{A}, \phi)$. We say $\{a_i\}$ are half independent if (i) $\{a_i^2\}_{i \in J}$ are independent and (ii) whenever a is nonsymmetric with respect to $\{a_i\}_{i \in J}$, then $\phi(a) = 0$.*

Remark 1. *The above definition is equivalent to that given in [4], although there is no notion of symmetric monomials there. As pointed out in [13], the concept of half independence does not extend to subalgebras.*

Example 1. *This is Example 2.4 of [4]. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $\{\eta_i\}$ be a family of independent complex Gaussian random variables. Define $a_i \in (M_2(L(\mu)), \mathrm{E}[\mathrm{Tr}(\cdot)])$ by*

$$a_i = \begin{bmatrix} 0 & \eta_i \\ \bar{\eta}_i & 0 \end{bmatrix}.$$

These $\{a_i\}$ are half independent.

Remark 2. *Let X, Y and Z be three elements of $(\mathcal{A}, \phi)$ such that $\phi(X) = \phi(Y) = \phi(Z) = 0$ but $\phi(X^2), \phi(Y^2), \phi(Z^2) \neq 0$. Following Remark 5.3.2 of [1],*

(i) If X, Y commute and are independent then $\phi(XY) = 0$ and $\phi(XYXY) = \phi(X^2)\phi(Y^2) \neq 0$.

(ii) If X, Y and Z are half independent, then $\phi(XYZZXY) = 0$. This happens since X appears two times in odd positions but zero times in even positions.

(iii) If X, Y, Z are free then $\phi(XYZXYZ) = 0$. If they are half independent then $\phi(XYZXYZ) = \phi(X^2)\phi(Y^2)\phi(Z^2) \neq 0$.

3 Joint convergence of patterned matrices

3.1 Some preliminaries and the main result

It is well known that, the joint limit of independent Wigner matrices yields the freely independent semicircular law. At the same time, there are a host of results on the limiting spectrum of other matrices. Important examples include the sample variance covariance, the Toeplitz and the Hankel matrices. The joint convergence of several sample variance covariance matrices has also been investigated in the literature. However, this convergence does not seem to have been addressed in any generality. In particular, it is not clear what other notions of independence are possible when we consider the joint limit of (independent) matrices.

It appears that patterned matrices offer a general framework for which this question may be worth investigating. A general pattern matrix may be defined through a *link function*. Let d be a positive integer. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_+^d$ denote the d-dimensional lattice of nonnegative integers. Let $L_n : \{1, 2, \dots n\}^2 \to \mathbb{Z}_+^d,\ n \geq 1$ be a sequence of functions such that $L_{n+1}(i, j) = L_n(i, j)$ whenever $1 \leq i, j \leq n$. We shall write $L_n = L$ and call it the **link** function and by abuse of notation we write $\mathbb{Z}^2$ as the common domain of $\{L_n\}$. For our examples later, the value of d is either 1 or 2.

A typical patterned matrix is then of the form $A_n = ((x(L(i, j)))$ where $\{x(i); i \geq 0\}$ or $\{x(i\ j); i, j \geq 1\}$ is a sequence of variables. In what follows we only consider real symmetric matrices. Here are some well known matrices and their link functions:

(i) Wigner matrix W_n. $L : \mathbb{Z}_+^2 \to \mathbb{Z}_+^2$ where $L(i, j) = (\min(i, j), \max(i, j))$.
(ii) Symmetric Toeplitz matrix T_n. $L : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ where $L(i, j) = |i - j|$.
(iii) Symmetric Hankel matrix H_n. $L : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ where $L(i, j) = i + j$.
(iv) Reverse circulant matrix RC_n. $L : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ where $L(i, j) = (i + j) \mod n$.
(v) Symmetric circulant matrix SC_n. $L : \mathbb{Z}_+^2 \to \mathbb{Z}_+$ where $L(i, j) = n/2 - |n/2 - |i - j||$.

In general, we assume that the link function L satisfies Property B:

Property B: $\Delta(L) = \sup_n \sup_{t\in\mathbb{Z}_+^d} \sup_{1\leq k\leq n} \#\{l : 1 \leq l \leq n,\ L(k, l) = t\} < \infty$.

In particular, $\Delta(L) = 2$ for T_n, SC_n, and $\Delta(L) = 1$ for W_n, H_n and RC_n.

Now let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $X_{i,n} : \Omega \to M_n$ for $1 \leq i \leq p$ be symmetric patterned random matrices of order n each with the same link function L. We shall refer to the p indices as p distinct **colors**. The (j, k)-th entry of the matrix $X_{i,n}$ will be denoted by $X_{i,n}(L(j, k))$.

Assumption I Let $\{X_{i,n}\}_{1\leq i\leq p}$ be independent across i with mean zero, variance 1 entries and a common link function L which satisfies Property B. Suppose

$$\sup_{n\in\mathbb{N}} \sup_{1\leq i\leq p} \sup_{1\leq m\leq l\leq N} \mathrm{E}[|X_{i,n}(L(m, l))|^k] \leq c_k < \infty.$$

Suppressing the dependence on n, we shall simply write X_i for $X_{i,n}$. We view $\{\frac{1}{\sqrt{n}}X_i\}_{1\leq i\leq p}$ as elements of $(\mathcal{A}_n = Mat_n(L(\mu)), \phi_n)$ where $\phi_n = n^{-1}\,\mathrm{E}[\mathrm{Tr}]$. Denote the joint distribution of $\{\frac{1}{\sqrt{n}}X_i\}_{1\leq i\leq p}$ by $\widehat{\mu_n}$. Then $\{\frac{1}{\sqrt{n}}X_i\}_{1\leq i\leq p}$ converges in law if and only if

$$\begin{aligned} \widehat{\mu_n}(q) = \phi_n(q) &= \frac{1}{n^{1+k/2}}\,\mathrm{E}[\mathrm{Tr}(X_{i_1}\dots X_{i_k})] && (1)\\ &= \frac{1}{n^{k/2+1}} \sum_{j_1,\dots,j_k} \mathrm{E}[X_{i_1}(L(j_1, j_2))X_{i_2}(L(j_2, j_3))\dots X_{i_k}(L(j_k, j_1))]. && (2) \end{aligned}$$

converges for all monomials q of the form $q(\{X_i\}_{1\leq i\leq p}) = X_{i_1}\dots X_{i_k}$.

To upgrade to almost sure convergence, we also define,

$$\widetilde{\mu_n}(q) = \frac{1}{n^{k/2+1}}\,\mathrm{Tr}[X_{i_1}\dots X_{i_k}] = \frac{1}{n^{k/2+1}} \sum_{j_1,\dots,j_k} X_{i_1}(L(j_1, j_2))X_{i_2}(L(j_2, j_3))\dots X_{i_k}(L(j_k, j_1)).$$

All developments below are with respect to one fixed monomial q at a time.

Circuit: Any $\pi : \{0, 1, 2, \ldots, h\} \to \{1, 2, \ldots, n\}$ with $\pi(0) = \pi(h)$ is a **circuit** of **length** $l(\pi) := h$. The dependence of a circuit on h and n will be suppressed. A typical element in (2) can be now written as

$$\mathrm{E}[\prod_{j=1}^{k} X_{i_j}(L(\pi(j-1), \pi(j)))] \tag{3}$$

If all L-values $L(\pi(j-1), \pi(j))$ are repeated more than once in (3), then the circuit is **matched**. If the L values are repeated within the same color then it is **color matched.**

For $q = X_{i_1}X_{i_2}\ldots X_{i_k}$, let for convenience, the corresponding sequence of colors be denoted by $\{c_1, c_2, \ldots, c_k\}$. Also let $H = \{\pi : \pi$ is a color matched circuit$\}$. Define an equivalence relation on H by defining $\pi_1 \sim_C \pi_2$ if and only if, $c_i = c_j$ *and*

$$X_{c_i}(L(\pi_1(i-1), \pi_1(i))) = \mathrm{X}_{c_j}(L(\pi_1(j-1), \pi_1(j))) \Longleftrightarrow X_{c_i}(L(\pi_2(i-1), \pi_2(i))) = \mathrm{X}_{c_j}(L(\pi_2(j-1), \pi_2(j))).$$

Colored Words: An equivalence class induces a partition of $\{1, 2, \ldots, k\}$ and each block of the partition is associated with a color. Any such class can be expressed as a (colored) word w where letters appear in alphabetic order of their first occurrence and with a subscript to distinguish the color. For example the partition $(\{\{1, 3\}, 1\}, \{\{2, 4\}, 2\}, \{\{5, 7\}, 1\}, \{\{6, 8\}, 3\})$ is identified with the word $a_1b_2a_1b_2c_1d_3c_1d_3$. A typical position in a colored word would be referred to as $w_{c_i}[i]$.

Let the class of all (colored) circuits corresponding to a color matched word w and the class of all pair matched colored words be denoted respectively by

$$\begin{aligned}\Pi_{C_q}(w) &= \{\pi : w_{c_i}[i] = w_{c_j}[j] \Longleftrightarrow X_{c_i}(L(\pi(i-1), \pi(i))) = X_{c_j}(L(\pi(j-1), \pi(j)))\} && (4)\\ CW_k(2) &= \{\text{all paired matched (within same color) words } w \text{ of length } k\} \ (k \text{ is even}). && (5)\end{aligned}$$

All the above notions have the corresponding non-colored versions. For instance, if we drop the colors from a colored word, then we obtain a **non-colored** word. For any monomial q, dropping the color amounts to dealing with only one matrix or in other words with the marginal distribution.

Let $w[i]$ denote the i-th entry of a noncolored word w. The equivalence class corresponding to w and the set of pair matched non-colored words will be denoted respectively by

$$\begin{aligned}\Pi(w) &= \{\pi : w[i] = w[j] \Leftrightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\} && (6)\\ W_k(2) &= \{\text{all paired matched words } w \text{ of length } k\} \ (k \text{ is even}). && (7)\end{aligned}$$

For any word $w \in CW_k(2)$, consider the noncolored word w' obtained by dropping the color. Then $w' \in W_k(2)$. Since we are dealing with one fixed monomial at a time, this yields a bijective mapping say

$$\psi_q : CW_k(2) \to W_k(2). \tag{8}$$

For any $w \in CW_k(2)$ define $p_{C_q}(w) = \lim_{n\to\infty} \frac{1}{n^{k/2+1}}|\Pi_{C_q}(w)|$ if the limit exists.

Proposition 1. *Let $\{X_i\}_{1\leq i\leq p}$ be independent patterned matrices satisfying Assumption I. Fix any monomial $q = X_{i_1}X_{i_2}\ldots X_{i_k}$. Assume that, whenever k is even,*

$$p(w) = \lim_{n\to\infty} \frac{1}{n^{k/2+1}}|\Pi(w)| \text{ exists for all } w \in W_k(2). \tag{9}$$

(a) Then $p_{C_q}(w) = p(\psi_q(w))$ and for any k,

$$\lim_{n\to\infty} \widehat{\mu_n}(q) = \sum_{w\in CW_k(2)} p_{C_q}(w) = \alpha(x_{i_1}\ldots x_{i_k}) \ (say)$$

with

$$\begin{aligned}|\alpha(x_{i_1}\dots x_{i_k})| &\le \frac{k!\Delta(L)^{k/2}}{(k/2)!2^{k/2}} && \textit{if } k \textit{ is even and each color appear even number of times}\\ &= 0 && \textit{if } k \textit{ is odd or a color appears odd number of times.}\end{aligned}$$

(b) $\mathrm{E}\,[|\widetilde{\mu_n}(q)-\widehat{\mu_n}(q)|^4]=O(n^{-2})$ *and hence* $\lim_{n\to\infty}\widetilde{\mu_n}(q)=\alpha(x_{i_1}\dots x_{i_k})$ *almost surely.*

Remark 3. *It is known from [8] that (9) holds true for Wigner, Toeplitz, Hankel, reverse circulant and symmetric circulant matrices. The quantity* $\frac{k!}{(k/2)!2^{k/2}}$ *above is the total number of non-clored pair matched words of length k (k even). Often not all pair matched words contribute to the limit and in such cases, this bound can be improved.*

Consider the polynomial algebra $\mathbb{C}\langle a_1,a_2,\dots,a_p\rangle$ *in non commutative indeterminates* $\{a_i\}_{1\le i\le p}$ *and define a linear functional* ϕ *on it by*

$$\phi(a_{i_1}\dots a_{i_k})=\lim_{n\to\infty}\widehat{\mu_n}(X_{i_1}\dots X_{i_k}).$$

Then Proposition 1 implies we have convergence in law of $(\{\frac{1}{\sqrt{n}}X_i\}_{1\le i\le p},\ \phi_n)$ *to* $(\{a_i\}_{1\le i\le p},\ \phi)$ *in sense of Definition 1. In the next section we explore further properties of the limit* $(\{a_i\}_{1\le i\le p},\phi)$ *for particular matrices.*

Remark 4. *Proposition 1 shows that the joint moments of pattern matrices can be expressed as functions of pair matched words or, in other words, pair partitions. [5] consider bosonic, fermionic and q-Brownian motions and show that the joint distribution of certain operators (in some appropriate sense) can be expressed as functions on the set of pair partitions. It would be interesting to investigate if there are connections between the two types of models.*

3.2 Some examples

From the above result, $\lim\widehat{\mu_n}(q)=0$ when k is odd or when there is a color which appears an odd number of times in the monomial q. Henceforth we thus assume that the order of the monomial is even and each color appears an even number of times.

Example 2 (Wigner matrices) Joint convergence of the Wigner matrices was first studied in [15] and later many authors extended it. For details of the classical proof and further extensions we refer the readers to [1]. Here we give a quick partial proof essentially translating the concept of non-crossing partitions that is used in the standard proof, into the language of words.

Colored Catalan words: Fix $k\ge 2$. If for a $w\in CW_k(2)$, sequentially deleting all double letters of the same color each time leads to the empty word then we call w a *colored Catalan word.* For example, the monomial $X_1X_2X_2X_1X_1X_1$ has exactly two colored Catalan words $a_1b_2b_2a_1c_1c_1$ and $a_1b_2b_2c_1c_1a_1$. A colored Catalan word associated with $X_1X_2X_2X_1X_1X_1X_2X_2$ is $a_1b_2b_2a_1c_1c_1d_2d_2$ which is not even a valid colored word for the monomial $X_1X_1X_1X_1X_2X_2X_2X_2$.

Let $\{W_i\}_{1\le i\le p}$ be an independent sequence of $n\times n$ Wigner matrices satisfying Assumption I. Then from Proposition 1 and Remark 3, $\{n^{-1/2}W_i\}_{1\le i\le p}$ converges in law to $\{a_i\}_{1\le i\le p}$. We show that $\{a_i\}_{1\le i\le p}$ are free and the marginals are distributed according to the semicircular law.

From Table 1 of [8], for non-colored words, $p(w)$ equals 1 if w is a Catalan word and otherwise $p(w)=0$. As a consequence, the marginals are semicircular. Now fix any monomial $q=x_{i_1}x_{i_2}\dots x_{i_{2k}}$ where each color appears an even number of times in the monomial.

Let w be a colored Catalan word. It remains Catalan when we ignore the colors. Hence from above, $p_{C_q}(w) = p(\psi_q(w)) = 1$. Likewise, if w is not colored Catalan then the word $\psi_q(w)$ cannot be Catalan and hence $p_{C_q}(w) = p(\psi_q(w)) = 0$. Hence if CAT_q denotes the set of colored Catalan words corresponding to a monomial q then from the above discussion,

$$\lim_{n\to\infty} \widehat{\mu_n}(q) = |CAT_q|.$$

Any double letter corresponds to a pair partition (within the same color) by the equivalence relation $\sim_C$. It is known that the number of Catalan words of length $2k$ is same as the number of non crossing pair partitions (denoted by $NC_{2k}(2)$) of length $2k$. See [8] and Chapter 1 of [1] for proofs.

Since the elements of the same pair partition must belong to the same color, we have

$$|CAT_q| = \sum_{\pi\in NC_{2k}(2)} \prod_{(j,j')\in\pi} \mathbb{I}_{i_j = i_{j'}}.$$

This is precisely the free joint semicircular law corresponding to q (see Theorem 5.4.2 of [1]).

Incidentally, since the number of noncolored Catalan words of lengthe $2k$ is $\frac{2k!}{k!(k+1)!}$, in this case, the bound in Proposition 1 can be sharpened to $|\alpha(x_{i_1}\dots x_{i_{2k}})| \le \frac{2k!}{k!(k+1)!}$. Corollary 5.2.16 of [1] can hence be applied to claim the existence of a C^*-probability space and with a state ϕ and containing free semicircular random variables $\{a_i\}$ in it.

Example 3 (Symmetric circulants) The case of symmetric circulant is rather easy. These matrices are commutative and so the limit is also commutative.

Let $\{SC_i\}_{1\le i\le p}$ be an independent sequence of $n\times n$ symmetric circulant matrices satisfying Assumption I. Then $\{n^{-1/2}SC_i\}_{1\le i\le p}$ converges in law to $\{a_i\}_{1\le i\le p}$ which are independent and the marginals are distributed according to the standard Gaussian law. To see this, first recall that the total number of (non-colored) pair matched words of length $2k$ equals $\frac{2k!}{k!2^k} = C_k$, (say). Further, in this case (see [8]), for *any* pair matched word $w \in W_{2k}(2)$,

$$p(w) = \lim_n \frac{1}{n^{1+k}}|\Pi(w)| = 1.$$

Now consider an order $2k$ monomial where each color appears an even number of times. Hence, from Proposition 1, for any fixed monomial q,

$$\lim_{n\to\infty} \widehat{\mu_n}(q) = \sum_{w\in CW_{2k}(2)} p_{C_q}(w) = |CW_{2k}(2)|.$$

Let l be the total number of distinct colors (distinct matrices) in the monomial $q = x_{i_1}x_{i_2}\dots x_{i_{2k}}$. Let $2\times n_i$ be the number of matrices of the i^{th} color. Then the set of all pair matched colored words of length $2k$ is obtained by forming pair matched subwords of color i of lengths $2n_i$, $1\le i\le l$. Hence

$$\phi(a_{i_1}\dots a_{i_{2k}}) = \prod_{i=1}^{l} C_{n_i}. \tag{10}$$

Thus if $(a_1,\dots,a_p)$ denotes i.i.d. standard normal random variables then the above is the mixed moment $\mathrm{E}[\prod_{i=1}^{l} a_i^{2n_i}]$.

Example 4 (Reverse circulant) It can be easily observed using the link function that the reverse circulant matrices are half commuting. This motivates the next theorem.

Theorem 1. *Let* $\{RC_i\}_{1\leq i\leq p}$ *be an independent sequence of* $n\times n$ *reverse circulant matrices satisfying Assumption I. Then* $\{n^{-1/2}RC_i\}_{1\leq i\leq p}$ *converges in law to half independent* $\{a_i\}_{1\leq i\leq p} \in (M_2(L(\mu)), E[Tr(\cdot)])$ *where* $a_i = \begin{bmatrix} 0 & \eta_i \\ \bar{\eta}_i & 0 \end{bmatrix}$ *and* η_i *are i.i.d. complex Gaussian.*

To prove the result, we need the following notion:

Colored symmetric words: Fix $k \geq 2$. A word $w \in CW_k(2)$ is called *colored symmetric* if each letter occurs once each in an odd and an even position *within the same color*. Clearly, every colored Catalan word is a colored symmetric word.

Proof of Theorem 1. Consider a monomial q of length $2k$ where each color appears an even number of times. From the single matrix case, it follows that $p(w) = 0$ if w is not a symmetric word (see Table 1 of [8]). If w is not a colored symmetric word then $\psi_q(w)$ is not a symmetric word and hence for such w, $p_{C_q}(w) = p(\psi_q(w)) = 0$. Hence we may restrict to colored symmetric words and then we have by Proposition 1 (a) that

$$\lim_{n\to\infty} \widehat{\mu_n}(q) = |CS_q(w)|,$$

where $CS_q(w)$ is the collection of all colored symmetric words of length $2k$.

The number of symmetric words of length $2k$ is $k!$. Let, as before, l be the number of distinct colors in the monomial and $2n_i$ be the number of matrices of the i^{th} color. All symmetric words are obtained by arranging the $2n_i$ letters of the i^{th} color in a symmetric way for $i = 1, 2, \ldots l$.

It is then easy to see that these arguments imply that

$$|CS_q(w)| = n_1! \times n_2! \times \cdots \times n_l!. \tag{11}$$

First observe that if the monomial $a_{i_1}a_{i_2}\ldots a_{i_k} \in (M_2(L(\mu)), \mathrm{E}[Tr(\cdot)])$ is non-symmetric, then

$$\mathrm{E}(Tr[a_{i_1}a_{i_2}\ldots a_{i_k}]) = 0.$$

If instead $q(\{a_i\}) = a_{i_1}a_{i_2}\ldots a_{i_{2k}}$ is symmetric then we have by half independence (Example 1),

$$E(Tr[a_{i_1}\ldots a_{i_{2k}}]) = n_1! \times n_2! \times \cdots \times n_l! = \lim_{n\to\infty} \widehat{\mu_n}(q).$$

So it follows from (11) that the joint limit is asymptotically half independent. Incidentally the moments $\{k!,\ k \geq 1\}$ is identified with the symmetrized Rayleigh distribution. □

Example 5 (Toeplitz and Hankel) Since $p(w)$ exists for Toeplitz and Hankel matrices, we have the joint convergence for these two matrices. For any fixed monomial q let SNC_q be the colored symmetric words which are not Catalan. Then for the Toeplitz matrix, we obtain the following:

$$\begin{aligned}\phi(a_{i_1}\ldots a_{i_k}) = \sum_{w\in CW_k(2)} p_{C_q}(w) &= \sum_{w\in CAT_q} p_{C_q}(w) + \sum_{w\in SNC_q} p_{C_q}(w) + \sum_{\text{other pair matched colored words}} p_{C_q}(w)\\ &= |CAT_q| + \sum_{w\in SCN_q} p_{C_q}(w) + \sum_{\text{other pair matched colored words}} p_{C_q}(w).\end{aligned}$$

Consider $q(X_1, X_2, X_3) = X_1X_2X_3X_1X_2X_3$ where X_1, X_2 and X_3 are scaled independent Toeplitz matrices. From Table 4 of [8], $p_{C_q}(a_1b_2c_3a_1b_2c_3) = p(\psi_q(a_1b_2c_3a_1b_2c_3)) = p(abcabc) = \dfrac{1}{2}$. Since for this monomial only pair matched colored word possible is $a_1b_2c_3a_1b_2c_3$, so we have $\phi(a_1a_2a_3a_1a_2a_3) = \frac{1}{2} \neq 0$. Thus the limit is not free.

Now let $q(X_1, X_2, X_3) = X_1X_2X_3X_2X_3X_1$. Then the only pair matched colored word is $a_1b_2c_3b_2c_3a_1$ and $\phi(a_1a_2a_3a_2a_3a_1) = p_{C_q}(a_1b_2c_3b_2c_3a_1) = p(abcbca) = \dfrac{2}{3}$. On the other hand we have already seen that $\phi(a_1a_2a_3a_1a_2a_3) = \frac{1}{2}$. Since the two contributions are not equal, the Toeplitz limit is not independent.

If they had been half independent then $\phi(a_1a_2a_3a_1a_2a_3) = \phi(a_1^2)\phi(a_2^2)\phi(a_3^2) = 1$, but that is not the case. Thus, the Toeplitz limit is not free, independent or half independent.

For Hankel matrices, the colored non symmetric words *do not* contribute to the limit. So for any fixed monomial q we have

$$\phi(a_{i_1} \dots a_{i_k}) = |CAT_q| + \sum_{w \in SCN_q} p_{C_q}(w).$$

That the Hankel limit is also not free, half independent or independent can be checked along the above lines by considering appropriate monomials and their contributions. It is interesting to note that Hankel matrices do not half commute in the limit and that is why even though the limits vanish on non symmetric words they are not half independent.

3.3 Pyramidal multiplicativity

Not much is known even about the limiting marginal distributions for the Toeplitz and Hankel matrices. Several researchers have attempted to clarify the nature of the (marginal) limit. For instance it is known that both the Toeplitz and the Hankel marginals have unbounded support and that the Hankel limit is not unimodal. Some asymptotic bounds on the moments have been derived in [10]. It is interesting to note that, from the results of [8] the limit for Catalan words equals one for all the four matrices discussed above. [3] discusses this issue for general pattern matrices. Attempts by different researchers to discover patterns in the sequence of moments and to compute them for Toeplitz and Hankel limits, have largely failed.

However, the following is worth mentioning. We need a definition from [5] which was later used by [9] to compute the marginal limit of Markov matrices. A function $f : \mathcal{D} \subset \cup_k W_k(2) \to \mathbb{R}$ is called pyramidally multiplicative, if for every $w \in \mathcal{D}$ of the form $w = xw_1y$ where w_1 is pair matched, we have

$$f(w) = f(w_1)f(xy).$$

An easy example of this is the Wigner limit $p(w)$ which is pyramidally multiplicative on entire class of pair matched words. This is because $p(w) = 1$ if w is Catalan and $p(w)$ is zero if it is non Catalan.

$p(w)$ is not pyramidally multiplicative for Toeplitz and Hankel limit on the full set. For example, consider the Toeplitz matrix and the word $w = ababcdcd = w_1w_2$ where $abab = w_1$ and $cdcd = w_2$. It is known that $p(w_1) = 2/3 = p(w_2)$ but $p(w) = 9/20 \neq 4/9$ and hence pryramidal multiplicativity is absent.

However, this multiplicativity appears in a restricted way. First observe that for any Catalan word of any order, $p(w) = 1$ and hence $p(w)$ is trivially pyramidally multiplicative on the subclass of Catalan words. Now we claim that the multiplicativity holds whenever w_1 is Catalan. To see this, first consider the Toeplitz limit and consider a word of length $2k$ of the form xw_1y where w_1 is Catalan of reduced length. Consider a typical double letter in $w[i] = w[i+1]$ in w_1. Now [9] has shown that the value $p(w)$ is identified by the system of k equations in $2k+1$ unknowns $x_{i0 \leq i \leq 2k}$, one for each pair match. In particular, the pair match from the double letter gives rise to the equation $x_{i-1} - x_i + x_{i+1} - x_i = 0$, implying $x_{i-1} = x_{i+1}$. Now consider the reduced word $w' = a \cdots cd \cdots z$

after dropping w_1 and relabel the unknowns as $y_0 = x_0, \cdots y_{i-1} = x_{i-1}, y_i = x_{i+2}, \cdots y_{2k-1} = x_{2k+1}$ and consider the $(k-1)$ equations that still exist (one equation $x_{i-1} = x_{i+1}$ is dropped). But this is the same as the equations for the reduced pair matched word of length $2(k-1)$. Inductively, one can again drop a double letter and eventually entire w_1 is dropped So we have $p(w) = p(w') = p(w_1)p(w')$ since $p(w_1) = 1$. The same argument holds for Hankel matrices too.

Since ψ_q is a bijection and $p_{C_q}(w) = 1$ for any colored Catalan word w, this restricted multiplicativity continues to hold when we have more than one color.

3.4 Proof of Proposition 1

Proof. (a) Fix a monomial $q = q(\{X_i\}_{1\le i\le p}) = X_{i_1} \cdots X_{i_k}$. Now due to bijection of ψ_q observe that

$$\Pi_{C_q}(w) = \Pi(\psi_q(w)) \text{ for } w \in CW_k(2).$$

Hence using (9),

$$\lim_{n\to\infty} \frac{1}{n^{k/2+1}} |\Pi_{C_q}(w)| = \lim_{n\to\infty} \frac{1}{n^{k/2+1}} |\Pi(\psi_q(w))| = p(\psi_q(w)) = p_{C_q}(w).$$

For simplicity denote

$$\mathbb{T}_{\boldsymbol{j}} = \mathrm{E}[X_{i_1}(L(j_1, j_2))X_{i_2}(L(j_2, j_3)) \cdots X_{i_k}(L(j_k, j_1))] \text{ for } \boldsymbol{j} = (j_1, \ldots, j_k).$$

Then

$$\widehat{\mu_n}(q) = \frac{1}{n^{k/2+1}} \sum_{j_1,\ldots,j_k} \mathbb{T}_{\boldsymbol{j}}. \tag{12}$$

In the monomial, if any color appears once, then by independence and mean zero condition, $\mathbb{T}_{\boldsymbol{j}} = 0$ for every $\boldsymbol{j}$. Hence $\widehat{\mu_n}(q) = 0$.

So henceforth assume that each color appearing in the monomial, appears at least twice. Now again, if $\boldsymbol{j}$ belongs to a circuit which is not color matched, then $\mathbb{T}_{\boldsymbol{j}} = 0$.

Now form the following matrix M:

$$M(L(i, j)) = |X_{i_1}(L(i, j))| + |X_{i_2}(L(i, j))| + \cdots + |X_{i_k}(L(i, j))|.$$

Observe that,

$$|\mathbb{T}_{\boldsymbol{j}}| \le \mathrm{E}[M(L(j_1, j_2) \cdots M(L(j_k, j_1)].$$

From Lemma 1 of [8] it is known that the total contribution of all circuits which have at least one three match, is zero in the limit.

As a consequence of the above discussion, if k is odd, then $\widehat{\mu_n}(q) \to 0$. So assume k is even. In that case, we need to consider only circuits which are pair matched. Further this pair matching must occur within the same color. If $\boldsymbol{j}$ belongs to any such circuit, then by independence, mean and zero variance one condition, $\mathbb{T}_{\boldsymbol{j}} = 1$.

Then using all the facts established so far,

$$\begin{aligned} \lim_{n\to\infty} \widehat{\mu_n}(q) &= \lim_{n\to\infty} \frac{1}{n^{k/2+1}} \sum_{\substack{\pi:\pi \text{ pair matched} \\ \text{within colors}}} \mathrm{E}[X_{i_1}(L(\pi(0), \pi(1))) \cdots X_{i_k}(L(\pi(k-1), \pi(k)))] \\ &= \lim_{n\to\infty} \frac{1}{n^{k/2+1}} \sum_{w\in CW_k(2)} \sum_{\pi\in\Pi_{C_q}(w)} \mathrm{E}[X_{i_1}(L(\pi(0), \pi(1))) \cdots X_{i_k}(L(\pi(k-1), \pi(k)))] \end{aligned}$$

$$= \sum_{w \in CW_k(2)} p_{C_q}(w) \ .$$

The last claim in part (a) follows since

$$\sum_{w \in CW_{2k}(2)} p_{C_q}(w) = \sum_{w \in CW_{2k}(2)} p(\psi_q(w)) \leq \sum_{w \in W_{2k}(2)} p(w) \leq \frac{(2k)!\Delta(L)^k}{k!2^k}.$$

The last inequality above is shown in [8].

(b) For part (b) the following notions will be useful: k circuits $\pi_1, \pi_2, \cdots, \pi_k$ are said to be **jointly matched** if each L-value occurs at least twice across all circuits. They are said to be **cross matched** if each circuit has at least one L-value which occurs in at least one of the other circuits. We have

$$E[|\widetilde{\mu_n}(q) - \mu_n(q)|^4] = \frac{1}{n^{2k+4}} \sum_{\pi_1,\pi_2,\pi_3,\pi_4} \mathrm{E}\,[\prod_{l=1}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})]. \tag{13}$$

If $(\pi_1, \pi_2, \pi_3, \pi_4)$ are not jointly matched, then one of the circuits, say π_j, has an L value which does not occur anywhere else. Also note that $\mathrm{E}\,\mathbb{X}_{\pi_j} = 0$. Hence, using independence

$$\mathrm{E}\,[\prod_{l=1}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})] = \mathrm{E}\,[\mathbb{X}_{\pi_j} \prod_{l=1, l\neq j}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})] = 0. \tag{14}$$

If $(\pi_1, \pi_2, \pi_3, \pi_4)$ is jointly matched but is not cross matched then one of the circuits, say π_j is only self matched, that is, none of the L-values is shared by the other circuits. Then by independence,

$$\mathrm{E}\,[\prod_{l=1}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})] = \mathrm{E}\,[(\mathbb{X}_{\pi_j} - \mathrm{E}\,\mathbb{X}_{\pi_j}) \prod_{l=1, l\neq j}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})] = 0. \tag{15}$$

Since $\{X_{i,n}\}_{1 \leq i \leq n}$ satisfy Assumption I, $\mathrm{E}\,[\prod_{l=1}^{4} (\mathbb{X}_{\pi_l} - \mathrm{E}\,\mathbb{X}_{\pi_l})]$ uniformly bounded over all $(\pi_1, \pi_2, \pi_3, \pi_4)$.

The arguments given in [9] for Toeplitz and Hankel matrices can be extended to our set up easily to yield the following: Let $Q_{k,4}$ be the number of quadruples of circuits $(\pi_1, \pi_2, \pi_3, \pi_4)$ of length k such that they are jointly matched and cross matched with respect to L. If L satisfy Property B, then there exists a constant K such that $Q_{k,4} \leq K n^{2k+2}$. Using this, and (13)–(15),

$$\mathrm{E}\,[|\widetilde{\mu_n}(q) - \widehat{\mu_n}(q)|^4] \leq K \frac{n^{2k+2}}{n^{2k+4}} = O(n^{-2}).$$

Now by an easy application of Borel Cantelli lemma $\widetilde{\mu_n}(q)$ converges almost surely. □

4 Half independence CLT and Cramér type characterisations

4.1 Half independence CLT with Rayleigh limit

Suppose $\{x_i\}$ are independent and identically distributed random variables with mean zero and variance one and all moments finite. Let $Y_n = n^{-1/2}(x_1 + x_2 + \ldots + x_n)$. By using binomial expansion and taking term by term expectation and then using elementary order calculations, $\mathrm{E}[Y_n^{2k+1}] \to 0$ and $\mathrm{E}[Y_n^{2k}] \to \frac{2k!}{2^k k!}$. Using Stirling's approximation, it can be easily checked that $\{\beta_{2k} = \frac{2k!}{2^k k!}\}$ satisfies Carleman's condition. Since β_{2k} are the $2k$-th moments of the standard normal distribution,

Y_n converges in law to the standard normal distribution. Note that $\frac{2k!}{2^k k!}$ is the total number of pair matched words of length $2k$.

Now suppose that $\{x_i\}$ (noncommutative) are free and identically distributed random variables with mean zero and variance one and all moments finite. Then a similar order calculation may be done on $Y_n = n^{-1/2}(x_1 + x_2 + \ldots + x_n)$. However, this time because of freeness, not all pair matched words contribute in the term by term expectation. Instead, in this case only the Catalan words contribute and hence $\mathrm{E}[Y_n^{2k}] \to \frac{2k!}{k!(k+1)!}$ while the odd moments converge to 0. Arguing as before, Y_n converges in law to the semicircular distribution. In Theorem 5 of [11] a central limit theorem was proved in a more general setup. The free random variables fall into this setup but it can be easily checked that half independent random variables do not satisfy the condition (Condition A(3)) of the theorem. The following result gives a central limit theorem for half independent random variables.

Theorem 2. *Let $\{x_i\}$ be sequence of half commuting self adjoint (non commutative) random variables in a non commutative probability space $(\mathcal{A}, \phi)$ which are half independent and identically distributed with $\phi(x_i) = 0$ and $\phi(x_i^2) = 1$ and $\phi(x_i^k) < \infty$ for all $k \geq 1$. If $S_n = x_1 + x_2 + \cdots + x_n$, then $S_n/\sqrt{n}$ converges in law to a symmetrized Rayleigh distribution.*

Proof. To show the required convergence it is enough to show that,

$$\frac{\phi(S_n^{2k})}{n^k} \to k! \text{ and } \frac{\phi(S_n^{2k+1})}{n^{k+1/2}} \to 0.$$

For the latter case, observe that in the expansion of $(x_1 + \cdots + x_n)^{2k+1}$, every monomial is of the form $x_{i_1} \cdots x_{i_{2k+1}}$ (with $i_j \in \{1, 2, \cdots, n\}$) and there will exists an l such that

$$E_l(x_{i_1} \cdots x_{i_{2k+1}}) \neq O_l(x_{i_1} \cdots x_{i_{2k+1}})$$

and hence $\phi((x_1 + \cdots + x_n)^{2k+1}) = 0$.

We now consider the even moments. Observe the following:

(i) If a monomial $q = x_{i_1} \cdots x_{i_{2k}}$ in the expansion of $(x_1 + \cdots + x_n)^{2k}$ contains any x_j an odd number of times for some j, then $E_j(x_{i_1} \cdots x_{i_{2k}}) \neq O_j(x_{i_1} \cdots x_{i_{2k}})$. Hence $\phi(q) = 0$.

(ii) Now we are left with monomials q which contain every random variable an even number of times. Again if q forms an asymmetric word, that is, $E_j(q) \neq O_j(q)$ for some j then also $\phi(q) = 0$.

So we deal only with symmetric monomials q. We divide such monomials into two sets:

$M_1 = \{x_{i_1} \cdots x_{i_{2k}}; i_j \in \{1, 2, \cdots, n\}$ symmetric and every random variable appears exactly twice$\}$,

$M_2 = \{x_{i_1} \cdots x_{i_{2k}}; i_j \in \{1, 2, \cdots, n\}$ symmetric and at least one random variable appears more than twice$\}$.

Observe that M_2 has at most $(k-1)$ many distinct random variables and hence

$$\#M_2 \leq C_k n(n-1) \cdots (n-k+2),$$

for some constant C_k which depends only on k. Now since all moments are finite, we have as $n \to \infty$

$$\frac{1}{n^k} \sum_{M_2} \phi(x_{i_1} \cdots x_{i_{2k}}) \leq C'_k \frac{n(n-1) \cdots (n-k+2)}{n^k} \to 0.$$

Now consider M_1. Pick a fixed set of k variables from $\{x_1, \ldots, x_n\}$. Then there are exactly $k!$ symmetric monomials in M_1 of these k variables. Hence

$$\#M_1 = k! \times n(n-1)\cdots(n-k+1).$$

Therefore as $n \to \infty$

$$\frac{1}{n^k} \sum_{M_1} \phi(x_{i_1} \cdots x_{i_{2k}}) = \frac{k! \times \#M_1}{n^k} \phi(x_1^2 \cdots x_k^2) = \frac{k! \times \#M_1}{n^k} \to k!.$$

Hence the result follows. □

4.1.1 Half independent Cramér type theorem for Rayleigh distribution

If $x = \frac{x_1+x_2}{\sqrt{2}}$ has the standard normal law where x_1 and x_2 are independent then each x_i also has the normal distribution. This is the famous Cramér's theorem. The corresponding free version is not true. That is, there exists two free random variables X_1 and X_2 which are not semicircular but their sum is semicircular (see [6] for details).

Under some moment conditions and assumption of identically distributed we prove a Cramér type theorem for half independence with symmetrized Rayleigh distribution. A version of the following Theorem can be proved also for free and identically distributed random variables.

Theorem 3. *Let x_1 and x_2 be noncommutative half commuting half independent self adjoint identically distributed random variables.*
(a) If x_1 and x_2 are distributed as symmetrized Rayleigh, then $\frac{x_1+x_2}{\sqrt{2}}$ again has a symmetrized Rayleigh distribution.
(b) If $\frac{x_1+x_2}{\sqrt{2}}$ is distributed as symmetrized Rayleigh and x_1 and x_2 have all moments finite then they are also distributed as symmetrized Rayleigh.

Proof. (a) We show that

$$\frac{\phi((x_1 + x_2)^{2k})}{2^k} = k! \text{ and } \phi((x_1 + x_2)^{2k+1}) = 0.$$

Since the Rayleigh distribution is characterized by its moments so this will complete the proof.

For the latter case observe that in the expansion of $(x_1 + x_2)^{2k+1}$, for every monomial $x_{i_1} \cdots x_{i_{2k+1}}$ with $x_{i_j} \in \{x_1, x_2\}$ either $E_1(x_{i_1} \cdots x_{i_{2k+1}}) \neq O_1(x_{i_1} \cdots x_{i_{2k+1}})$ or $E_2(x_{i_1} \cdots x_{i_{2k+1}}) \neq O_2(x_{i_1} \cdots x_{i_{2k+1}})$. Hence $\phi(x_{i_1} \cdots x_{i_{2k+1}}) = 0$.

For the first part, following the line of argument given in Lemma 2, if we expand $(x_1 + x_2)^{2k}$, then we will be left only with symmetric monomials, that is monomials where x_1 and x_2 appear even number of times and $E_1(x_{i_1} \cdots x_{i_{2k}}) = O_1(x_{i_1} \cdots x_{i_{2k}})$, $E_2(x_{i_1} \cdots x_{i_{2k}}) = O_2(x_{i_1} \cdots x_{i_{2k}})$. So we will concentrate only on the following set

$$M = \{x_{i_1} \cdots x_{i_{2k}};\ i_j \in \{1, 2\} \text{ and } E_1(x_{i_1} \cdots x_{i_{2k}}) = O_1(x_{i_1} \cdots x_{i_{2k}}),\ E_2(x_{i_1} \cdots x_{i_{2k}}) = O_2(x_{i_1} \cdots x_{i_{2k}})\}.$$

Suppose $x_{i_1} \cdots x_{i_{2k}} \in M$ and x_1 appears $(2j)$ times. Since we can choose j many odd positions for x_1 from k many odd positions in $\binom{k}{j}$ ways, the number of such monomials in M is $\binom{k}{j}\binom{k}{j}$. Hence

$$\frac{\phi((x_1 + x_2)^{2k})}{2^k} = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j}\binom{k}{j} \phi(x_1^{2j})\phi(x_2^{2(k-j)}) = \frac{1}{2^k} \sum_{j=0}^{k} \binom{k}{j}\binom{k}{j} j!(k-j)! = \frac{k!2^k}{2^k} = k!.$$

(b) First note that due to half independence,

$$1 = \phi(\frac{(x_1 + x_2)^2}{2}) = \frac{1}{2}(\phi(x_1^2) + \phi(x_2^2)) = \phi(x_1^2).$$

Similarly for the fourth moment, noting that only symmetric monomials contribute, we get

$$\begin{aligned}
2 &= \frac{1}{4}\phi((x_1 + x_2)^4) \\
&= \frac{1}{4}\left[\phi(x_1^4 + x_2^4 + x_1x_1x_2x_2 + x_1x_2x_2x_1 + x_2x_2x_1x_1 + x_2x_1x_1x_2)\right] \\
&= \frac{1}{4}\left[2\phi(x_1^4) + 4\phi(x_1^2)\phi(x_2^2)\right].
\end{aligned}$$

This implies $\phi(x_1^4) = 2$. Now we use mathematical induction on the moments. Suppose it is true that $\phi(x_1^{2j}) = j!$ for all $j \leq k$. Then

$$\begin{aligned}
(k+1)! &= \frac{1}{2^{k+1}}\phi((x_1 + x_2)^{2(k+1)}) \Rightarrow \\
2^{k+1}(k+1)! &= \phi(x_1^{2(k+1)}) + \phi(x_2^{2(k+1)}) + \sum_{j=1}^{k} \binom{k+1}{j}^2 j!(k+1-j)! \\
&= 2\phi(x_1^{2(k+1)}) + \sum_{j=0}^{k+1} \binom{k+1}{j}^2 j!(k+1-j)! - 2(k+1)! \\
&= 2\phi(x_1^{2(k+1)}) + 2^{k+1}(k+1)! - 2(k+1)!
\end{aligned}$$

which implies that $\phi(x_1^{2(k+1)}) = (k+1)!$. This completes the proof of Part (b). □

Remark 5. *It should be noted that the assumption of identically distributed random variables plays a crucial role in the proof of the above Theorem. It would be interesting to see whether the Cramér's Theorem hold for half independent random variables. We are actively looking into this problem.*

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