

Circulant type matrices with heavy tailed entries

ARUP BOSE * SUMAN GUHA RAJAT SUBHRA HAZRA KOUSHIK SAHA

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Abstract

We study the limiting spectral distribution for a class of circulant type random matrices with heavy tailed input sequence. Unlike the light tailed case where the limit is nonrandom, here the limit is a random probability distribution. We provide an explicit representation of the limit.

Keywords Circulant matrix, eigenvalues, empirical spectral distribution, k circulant matrix, large dimensional random matrix, limiting spectral distribution, reverse circulant matrix, stable law, symmetric circulant matrix.

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1 Introduction

The study of the limit of the empirical spectral distribution (ESD) of random matrices when the dimension tends to infinity has a long history, specially where the input sequence has light tails. There have been very few articles where the input sequence has heavy tails.

For the Wigner matrix, when the input sequence belongs to the domain of attraction of an α stable law with $\alpha \in (0, 2)$, Ben Arous and Guionnet (2008) showed that the ESD converges weakly in probability to a measure (called the limiting spectral distribution (LSD)) that has unbounded support and has heavy tails. Later Belinschi et al. (2009) studied some symmetric band matrices and the sample variance covariance matrices with heavy tailed inputs. In both these articles the LSD was shown to be nonrandom.

The Wigner matrix is a particular pattern matrix. To the best of our knowledge, there has been no studies on the LSD for general patterned matrices with heavy tailed inputs. This appears to be a difficult problem for matrices such as Toeplitz and Hankel matrices. We investigate the LSD for a certain k -circulant matrices. For positive integers k and n , the k -circulant matrix is defined as

$$A_{k,n} = \begin{bmatrix} X_1 & X_2 & X_3 & \dots & X_{n-1} & X_n \\ X_{n-k+1} & X_{n-k+2} & X_{n-k+3} & \dots & X_{n-k-1} & X_{n-k} \\ X_{n-2k+1} & X_{n-2k+2} & X_{n-2k+3} & \dots & X_{n-2k-1} & X_{n-2k} \\ & & & \vdots & & \end{bmatrix}_{n \times n} .$$

The first row of $A_{k,n}$ is (X_1, X_2, \dots, X_n) and for $1 \leq j < n - 1$, its $(j + 1)$ -th row is obtained by giving its j -th row a right circular shift by k positions (equivalently, $k \bmod n$ positions). Note that $k = 1$ or equivalently $k = n + 1$ yields the usual “circulant matrix”. If $k = n - 1$ then we obtain the so called “reverse circulant” matrix.

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It is well known that the eigenvalues of the circulant are the discrete Fourier transform of the input sequence. The eigenvalues of the reverse circulant are square roots of the periodogram (see Bose and Mitra (2002)). Indeed, one important feature of the k -circulant matrices is that their eigenvalues can be explicitly expressed in terms of their entries (see Zhou (1996)). In general these are also (more complicated) functions of the discrete Fourier transform. The study of the LSD of these matrices can thus be in principle based on the study of the discrete Fourier transform. However, the number theoretic relation between k and n could be vastly varying and the formula for eigenvalues is not really amenable to a general treatment. It is not too difficult to see that if k is held fixed, then the only nontrivial case as far as the LSD is concerned is when $k = 1$. This is the reason why in addition to the case $k = 1$, one considers the case where $k \rightarrow \infty$ as $n \rightarrow \infty$. We are not aware of the study of these matrices for completely general combinations of k and n .

This crucial eigenvalue formula has been exploited by several articles to find the LSD, the limit distributions of the spacings of the ordered eigenvalues and the spectral norm, in the light tailed case for a subclass of k -circulants. See for example Bose et al. (2009a,b, 2010a,b,c), concentrating only on the cases $k^g = n \pm 1$ where g is a positive integer. This implies that k and n are relatively prime. Note that these include the usual circulant and the reverse circulant ($g = 1$).

We shall consider the k -circulant matrix for $k^g = n \pm 1$ and the input sequence belongs to the domain of attraction of an α stable law with $\alpha \in (0, 2)$. The LSD of the reverse circulant matrix with heavy tailed inputs was shown to exist in Bose et al. (2003) using the methods of Freedman and Lane (1981). Knight (1991) has been able to obtain some very nice representation of the empirical distribution of periodogram entries of $\{X_i\}$ and provide its limiting distribution including a representation for the limit. The LSD of the reverse circulant immediately follows from Theorem 5 of Knight (1991).

We show that the LSD exists in the weak convergence sense for the above mentioned k -circulant matrices. We also determine explicit representations of the limit. Our method of proof heavily relies on the extension of the results of Freedman and Lane (1981) accomplished in Knight (1991) together with some intricate study of the eigenvalue structure of the k -circulant.

In Section 3 we describe the eigenvalue structure and state the main results. In Section 4 we provide short proofs of our results.

2 Preliminaries

2.1 Input sequence and ESD

The sequence of random variables used to build the relevant matrices will be called the input sequence. We initially assume that this sequence $\{X_i\}$ is i.i.d., defined on a probability space (Ω, \mathcal{A}, P) . Let $\{M_n\}$ be a sequence of random matrix with this input sequence.

Let $\{\lambda_j\}$ denote the eigenvalues of the random matrix $\frac{1}{a_n}M_n$, where $\{a_n\}$ is a positive sequence of constants. Then the ESD of $\frac{1}{a_n}M_n$ is defined as

$$L_{M_n}(A) = \frac{1}{n} \sum_{j=1}^n I(\lambda_j \in A).$$

If the input sequence for M_n is $\{X_i - c_n\}$ with $c_n = E[X_1 I(|X_1| \leq a_n)]$ and the eigenvalues of $\frac{1}{a_n}M_n$ are denoted by $\{\tilde{\lambda}_j\}$, then the corresponding ESD will be

$$\tilde{L}_{M_n}(A) = \frac{1}{n} \sum_{j=1}^n I(\tilde{\lambda}_j \in A).$$

2.2 Eigenvalues of the k -circulant

We first describe the eigenvalues of the k -circulant matrix. This is based on Zhou (1996). Let

$$\nu = \nu_n := \cos(2\pi/n) + i \sin(2\pi/n), \quad i^2 = -1 \quad \text{and} \quad \lambda_k = \sum_{l=1}^n X_l \nu^{k(l-1)}, \quad 0 \leq k < n. \quad (2.1)$$

For any positive integers k, n , let $p_1 < p_2 < \dots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^c p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^c p_q^{\alpha_q}.$$

Here $\alpha_q, \beta_q \geq 1$ and n', k', p_q are pairwise relatively prime. For any positive integer s , let $\mathbb{Z}_s = \{0, 1, 2, \dots, s-1\}$. Define

$$S(x) = \left\{ xk^b \bmod n' : b \geq 0 \right\}, \quad 0 \leq x < n' \quad \text{and} \quad g_x = |S(x)|. \quad (2.2)$$

Define

$$v_{k,n'} := |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}|. \quad (2.3)$$

We observe the following about the sets $S(x)$.

- (i) $S(x) = \{xk^b \bmod n' : 0 \leq b < |S(x)|\}$.
- (ii) For $x \neq u$, either $S(x) = S(u)$ or $S(x) \cap S(u) = \emptyset$.

As a consequence, the distinct sets from the collection $\{S(x) : 0 \leq x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$. We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \dots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{l-1}\}, \quad \text{and} \quad n_i = |\mathcal{P}_i|, \quad 0 \leq i < l. \quad (2.4)$$

Define

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \dots, l-1 \quad \text{where} \quad y = n/n'.$$

Then the characteristic polynomial of $A_{k,n}$ (whence its eigenvalues follow) is given by

$$\chi(A_{k,n}) = \lambda^{n-n'} \prod_{j=0}^{l-1} (\lambda^{n_j} - y_j). \quad (2.5)$$

3 Main results

Assumption on the input sequence: Let $\{X_i\}$ be i.i.d. random variables in the domain of attraction of a stable law with index $\alpha \in (0, 2)$, that is, there exists $a_n \rightarrow \infty$ such that

$$a_n^{-1} \sum_{k=1}^n (X_k - c_n) \xrightarrow{d} S_\alpha,$$

where S_α is a stable random variable and $c_n = \mathbb{E}[X_1 I(|X_1| \leq a_n)]$.

It is well known that a random variable X is in the domain of attraction of a (nonnormal) stable law with index $\alpha \in (0, 2)$ if and only if $\mathbb{P}[|X| > t] = t^{-\alpha}l(t)$, for some slowly varying l and

$$\lim_{t \rightarrow \infty} \frac{\mathbb{P}[X > t]}{\mathbb{P}[|X| > t]} = p \in [0, 1]. \quad (3.1)$$

Also the normalizing constants a_n are defined so that

$$n \mathbb{P}[|X| > a_n x] \rightarrow x^{-\alpha}.$$

3.1 k -circulant with $n = k^g + 1$

We now analyze the eigenvalues for this particular case in more detail. First suppose $n = k^2 + 1$. Then from Bose et al. (2010a), if k is even then there is one singleton partition set $\{0\}$ and if k is odd then there are two singleton partition sets $\{0\}$ and $\{n/2\}$ respectively; all the remaining partitions have four elements each. Thus apart from these finitely many (hence negligible) singleton partitions all others are of equal size of four.

In general for $n = k^g + 1$, $g \geq 1$, the eigenvalue partition (see Section 4.1 of Bose et al. (2010a)) of $\{0, 1, 2, \dots, n-1\}$ contains approximately $q = \lfloor \frac{n}{2g} \rfloor$ sets each of size $(2g)$ and each set is self-conjugate; in addition, the remaining sets do not contribute to the LSD. We shall call the partition sets of size $(2g)$ as *major partition sets*.

We will benefit from expressing the eigenvalues in a convenient form. This is given in the following Lemma for easy reference. To do this, observe that a typical $S(x)$ may be written as

$$S(b_1 k^{g-1} + b_2 k^{g-2} + \dots + b_g)$$

which in turn is the union of the following two sets

$$\{b_1 k^{g-1} + b_2 k^{g-2} + \dots + b_g, b_2 k^{g-1} + b_3 k^{g-2} + \dots + b_g k - b_1, \dots, b_g k^{g-1} - b_1 k^{g-2} - \dots - b_{g-1}\}$$

and its conjugate i.e.

$$\{n - (b_1 k^{g-1} + b_2 k^{g-2} + \dots + b_g), \dots, n - (b_g k^{g-1} - b_1 k^{g-2} - \dots - b_{g-1})\}$$

where

$$0 \leq b_1 \leq k-1, \dots, 0 \leq b_{g-1} \leq k-1 \text{ and } 1 \leq b_g \leq k.$$

Define

$$T_n = \{(b_1, b_2, \dots, b_g) : 0 \leq b_1 \leq k-1, \dots, 1 \leq b_g \leq k\},$$

$$C_t = \sum_{j=1}^n X_j \cos\left(\frac{2\pi jt}{n}\right) \quad \text{and} \quad S_t = \sum_{j=1}^n X_j \sin\left(\frac{2\pi jt}{n}\right) \quad \text{for } t \in \mathbb{N}.$$

Lemma 3.1. *The eigenvalues of the k -circulant with $n = k^g + 1$ corresponding to the major partition sets may be written as*

$$\left\{ \lambda_{(b_1, b_2, \dots, b_g)}, \lambda_{(b_1, b_2, \dots, b_g)} \omega_{2g}, \dots, \lambda_{(b_1, b_2, \dots, b_g)} \omega_{2g}^{2g-1} : (b_1, b_2, \dots, b_g) \in T_n \right\}$$

where ω_{2g} is the primitive $(2g)$ -th root of unity and

$$\lambda_{(b_1, b_2, \dots, b_g)} = a_n^{-1} \left(C_{b_1 k^{g-1} + \dots + b_g}^2 + S_{b_1 k^{g-1} + \dots + b_g}^2 \right)^{1/2g} \cdots \left(C_{b_g k^{g-1} - \dots - b_{g-1}}^2 + S_{b_g k^{g-1} - \dots - b_{g-1}}^2 \right)^{1/2g}.$$

In view of Lemma 3.1, to find the LSD of $A_{k,n}$ where $n = k^g + 1$, it suffices to consider the ESD of $\{\lambda_{(b_1, b_2, \dots, b_g)} : (b_1, \dots, b_g) \in T_n\}$: if these have an LSD F , then the LSD of $A_{k,n}$ will be (r, θ) in polar coordinates where r is distributed according to F , and θ is distributed uniformly across all the $(2g)$ -th roots of unity and r and θ are independent. With this in mind, define

$$\begin{aligned}\tilde{L}_{A_{k,n}}(A, \omega) &= \frac{1}{|T_n|} \sum_{(b_1, \dots, b_g) \in T_n} I(\tilde{\lambda}_{(b_1, \dots, b_g)} \in A), \\ L_{A_{k,n}}(A, \omega) &= \frac{1}{|T_n|} \sum_{(b_1, \dots, b_g) \in T_n} I(\lambda_{(b_1, \dots, b_g)} \in A).\end{aligned}$$

Further, let $\{\Gamma_j\}$, $\{B_j\}$, $\{U_j\}$, $\{U_j^*\}$ and $\{U_{t,j}^*\}$, be independent random sequences defined on the same probability space where $\Gamma_j = \sum_{i=1}^j E_i$ and $\{E_i\}$ is a sequence of i.i.d. exponential with mean 1, and B_j are i.i.d. satisfying $P[B_1 = 1] = p = 1 - P[B_1 = -1]$ where p is defined by equation (3.1) and the rest of the variables are i.i.d $U(0, 1)$. Finally, let

$$Z_j = \Gamma_j^{-1/\alpha} = \left(\sum_{t=1}^j E_t \right)^{-1/\alpha} \quad \text{and} \quad \mu_t = E[B_t Z_t I(Z_t \leq 1)].$$

We now state the following theorem. A typical element of Ω will be denoted by ω .

Theorem 3.1. *Let $n = k^g + 1$.*

(a) *Then $\tilde{L}_{A_{k,n}} \xrightarrow{d} \tilde{L}_{A_k}$, $\tilde{L}_{A_k}(\cdot, \omega)$ being the random distribution induced by $\tilde{L}_1(\omega)^{1/2g} \tilde{L}_2(\omega)^{1/2g} \dots \tilde{L}_g(\omega)^{1/2g}$,*

$$\tilde{L}_j(\omega) = \left(\sum_{t=1}^{\infty} \sin(2\pi U_{t,j}^*) (B_t(\omega) Z_t(\omega) - \mu_t) \right)^2 + \left(\sum_{t=1}^{\infty} \cos(2\pi U_{t,j}^*) (B_t(\omega) Z_t(\omega) - \mu_t) \right)^2, \quad 1 \leq j \leq g.$$

(b) *Then $L_{A_{k,n}} \xrightarrow{d} L_{A_k}$, $L_{A_k}(\cdot, \omega)$ being the random distribution induced by $L_1(\omega)^{\frac{1}{2g}} \dots L_g(\omega)^{\frac{1}{2g}}$,*

$$L_j(\omega) = \left(\sum_{t=1}^{\infty} \sin(2\pi U_{t,j}^*) B_t(\omega) Z_t(\omega) \right)^2 + \left(\sum_{t=1}^{\infty} \cos(2\pi U_{t,j}^*) B_t(\omega) Z_t(\omega) \right)^2, \quad 1 \leq j \leq g.$$

3.2 k -circulant with $n = k^g - 1$

As before, $n' = n$ and $k' = k$. Now the eigenvalue partition of $\{0, 1, 2, \dots, n-1\}$ contains approximately $q = \lfloor \frac{n}{g} \rfloor$ sets of size g which are the major partition sets. The remaining sets do not contribute to the LSD. For detail illustration see Bose et al. (2010a). Similar to the developments in the previous section, now the major partition sets $\{S(x)\}$ may be listed as

$$\{b_1 k^{g-1} + b_2 k^{g-2} + \dots + b_g, b_2 k^{g-1} + b_3 k^{g-2} + \dots + b_g k + b_1, \dots, b_g k^{g-1} + b_1 k^{g-2} + \dots + b_{g-1}\}$$

where $0 \leq b_1 \leq k-1, \dots, 0 \leq b_{g-1} \leq k-1, 1 \leq b_g \leq k$, with $(b_1, b_2, \dots, b_g) \neq (k-1, k-1, \dots, k-1)$ and $(b_1, b_2, \dots, b_g) \neq (k-1, k-1, \dots, k-1, k)$. Now define

$$T'_n = \left\{ (b_1, b_2, \dots, b_g) : \begin{aligned} &0 \leq b_1 \leq k-1, \dots, 1 \leq b_g \leq k, (b_1, b_2, \dots, b_g) \neq (k-1, k-1, \dots, k-1) \\ &\text{and } (b_1, b_2, \dots, b_g) \neq (k-1, k-1, \dots, k-1, k) \end{aligned} \right\},$$

$$\gamma_{(b_1, b_2, \dots, b_g)} = a_n^{-1} \left(C_{b_1 k^{g-1} + \dots + b_g} + i S_{b_1 k^{g-1} + \dots + b_g} \right) \dots \left(C_{b_g k^{g-1} + \dots + b_{g-1}} + i S_{b_g k^{g-1} + \dots + b_{g-1}} \right),$$

$$\eta_{(b_1, b_2, \dots, b_g)} = |\gamma_{(b_1, b_2, \dots, b_g)}|^{1/g} \exp\left\{\frac{i \arg(\gamma_{(b_1, b_2, \dots, b_g)})}{g}\right\}.$$

Then the eigenvalues of the k -circulant with $n = k^g - 1$ corresponding to the partition set $S(b_1 k^{g-1} + b_2 k^{g-2} + \dots + b_g)$ are

$$\eta_{(b_1, b_2, \dots, b_g)}, \eta_{(b_1, b_2, \dots, b_g)} \omega_g, \eta_{(b_1, b_2, \dots, b_g)} \omega_g^2, \dots, \eta_{(b_1, b_2, \dots, b_g)} \omega_g^{g-1}$$

where ω_g is g -th root of unity. So, to find the LSD, it suffices to consider the ESD of $\{\gamma_{(b_1, b_2, \dots, b_g)} : (b_1, \dots, b_g) \in T'_n\}$: if these have an LSD F , then the LSD of $A_{k,n}$ will be (r', θ) where r' is distributed according to $h(F)$ where $h(z) = |z|^{1/g} e^{\frac{i \arg(z)}{g}}$ and θ is distributed uniformly across all the g -th roots of unity, and r' and θ are independent. Hence define

$$\tilde{L}_{A_{k,n}}(A, \omega) = \frac{1}{|T'_n|} \sum_{(b_1, \dots, b_g) \in T'_n} I(\tilde{\gamma}_{(b_1, \dots, b_g)} \in A), \quad L_{A_{k,n}}(A, \omega) = \frac{1}{|T'_n|} \sum_{(b_1, \dots, b_g) \in T'_n} I(\gamma_{(b_1, \dots, b_g)} \in A).$$

Theorem 3.2. *Let $n = k^g - 1$.*

(a) *Let $\tilde{L}_{A_{k,n}}(\cdot, \omega)$ be the ESD of $\{\tilde{\gamma}_{(b_1, \dots, b_g)} : (b_1, \dots, b_g) \in T'_n\}$. Then $\tilde{L}_{A_{k,n}} \xrightarrow{d} \tilde{L}_{A_k}$, $\tilde{L}_{A_k}(\cdot, \omega)$ being the random distribution induced by $\tilde{L}_1(\omega) \tilde{L}_2(\omega) \dots \tilde{L}_g(\omega)$,*

$$\tilde{L}_j(\omega) = \left(\sum_{t=1}^{\infty} \cos(2\pi U_{t,j}^*) (B_t(\omega) Z_t(\omega) - \mu_t) \right) + i \left(\sum_{t=1}^{\infty} \sin(2\pi U_{t,j}^*) (B_t(\omega) Z_t(\omega) - \mu_t) \right), \quad 1 \leq j \leq g.$$

(b) *Then $L_{A_{k,n}} \xrightarrow{d} L_{A_k}$, $L_{A_k}(\cdot, \omega)$ being the random distribution induced by $L_1(\omega) \dots L_g(\omega)$,*

$$L_j(\omega) = \left(\sum_{t=1}^{\infty} \cos(2\pi U_{t,j}^*) B_t(\omega) Z_t(\omega) \right) + i \left(\sum_{t=1}^{\infty} \sin(2\pi U_{t,j}^*) B_t(\omega) Z_t(\omega) \right), \quad 1 \leq j \leq g.$$

3.3 Symmetric Circulant Matrices:

The (i, j) -th element of the symmetric circulant, SC_n is given by $X_{n/2+1-|n/2-|i-j||}$. Let $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ be the eigenvalues of $a_n^{-1} SC_n$. These are given by:

(i) for n odd:

$$\begin{cases} \lambda_0 &= a_n^{-1} (X_1 + 2 \sum_{j=1}^{\lfloor n/2 \rfloor} X_{j+1}) \\ \lambda_k &= a_n^{-1} (X_1 + 2 \sum_{j=1}^{\lfloor n/2 \rfloor} X_{j+1} \cos(\frac{2\pi jk}{n})), \quad 1 \leq k \leq \lfloor n/2 \rfloor \end{cases} \quad (3.2)$$

(ii) for n even:

$$\begin{cases} \lambda_0 &= a_n^{-1} (X_1 + 2 \sum_{j=1}^{\frac{n}{2}-1} X_{j+1} + X_{n/2}) \\ \lambda_k &= a_n^{-1} (X_1 + 2 \sum_{j=1}^{\frac{n}{2}-1} X_{j+1} \cos(\frac{2\pi jk}{n}) + (-1)^k X_{\frac{n}{2}+1}), \quad 1 \leq k \leq \frac{n}{2} \end{cases} \quad (3.3)$$

with $\lambda_{n-k} = \lambda_k$ in both cases.

Theorem 3.3. (a) $\tilde{L}_{SC_n} \xrightarrow{d} \tilde{L}_{SC}$, $\tilde{L}_{SC}(\cdot, \omega)$ being the distribution of symmetrized

$$2 \sum_{t=1}^{\infty} \cos(2\pi U_t^*) (B_t(\omega) Z_t(\omega) - \mu_t).$$

(b) $L_{SC_n} \xrightarrow{d} L_{SC}$, $L_{SC}(\cdot, \omega)$ being the distribution of symmetrized $2 \sum_{t=1}^{\infty} \cos(2\pi U_t^*) B_t(\omega) Z_t(\omega)$.

Remark 3.1. Even though the input sequence is heavy tailed, the LSD (in a.s. sense) is light tailed. Indeed for $\alpha \in (0, 1)$ the LSD (in a.s. sense) has bounded support. We briefly sketch the proof.

Recall that $Z_j = \Gamma_j^{-1/\alpha} = \left(\sum_{t=1}^j E_t \right)^{-1/\alpha}$, where $\{E_i\}$ is a sequence of i.i.d. exponential with mean 1. Hence, by LIL for almost all ω and for arbitrary $\epsilon > 0$ there exist $j_0(\omega)$ so that for $j \geq j_0(\omega)$,

$$\left(\frac{1}{j + (\sqrt{2} + \epsilon)\sqrt{j \log \log j}} \right)^{\frac{1}{\alpha}} < Z_j(\omega) < \left(\frac{1}{j - (\sqrt{2} + \epsilon)\sqrt{j \log \log j}} \right)^{\frac{1}{\alpha}}.$$

Hence

$$\begin{aligned} - \sum_{t=1}^{j_0(\omega)} Z_t(\omega) - \sum_{t=j_0(\omega)}^{\infty} \left(\frac{1}{t - (\sqrt{2} + \epsilon)\sqrt{t \log \log t}} \right)^{\frac{1}{\alpha}} \\ \leq \sum_{t=1}^{\infty} \cos(2\pi U_t^*) B_t(\omega) Z_t(\omega) \leq \sum_{t=1}^{j_0(\omega)} Z_t(\omega) + \sum_{t=j_0(\omega)}^{\infty} \left(\frac{1}{t - (\sqrt{2} + \epsilon)\sqrt{t \log \log t}} \right)^{\frac{1}{\alpha}}. \end{aligned}$$

Now it is clear from the last expression that the LSD obtained in Theorems 3.1(b), 3.2(b) and 3.3(b) will have bounded support for $\alpha \in (0, 1)$. For $\alpha \in [1, 2)$ and for fixed ω ,

$$\begin{aligned} \text{Var} \left(\sum_{t=1}^{\infty} \cos(2\pi U_t^*) B_t(\omega) Z_t(\omega) \right) &= \sum_{t=1}^{\infty} B_t^2(\omega) Z_t^2(\omega) \text{Var}(\cos(2\pi U_t^*)) \\ &= \text{Var}(\cos(2\pi U_1^*)) \sum_{t=1}^{\infty} B_t^2(\omega) Z_t^2(\omega) \\ &\leq \sum_{t=1}^{j_0(\omega)} Z_t^2(\omega) + \sum_{t=j_0(\omega)}^{\infty} \left(\frac{1}{t - (\sqrt{2} + \epsilon)\sqrt{t \log \log t}} \right)^{\frac{2}{\alpha}} < \infty \end{aligned}$$

Hence, the LSDs in Theorems 3.1(b), 3.2(b) and 3.3(b) has light tail (in a.s. sense) for $\alpha \in [1, 2)$.

Similar conclusions hold about the LSD in Theorems 3.1(a) and 3.2(a).

4 Proof of the results

Following Knight (1991), order the X_k 's, $1 \leq k \leq n$ by their magnitudes $|X_{n1}| \geq |X_{n2}| \geq \dots |X_{nn}|$. Let $Y_{nk} = |X_{nk}|$ and B_{nk} be the random signs so that $X_{nk} = B_{nk} Y_{nk}$. Let $\pi_{n1}, \pi_{n2}, \dots, \pi_{nn}$ be the anti-ranks of Y_{n1}, \dots, Y_{nn} and let $U_{nk} = \frac{\pi_{nk}}{n}$. and $Z_{nk} = a_n^{-1} Y_{nk}$.

It follows from Lemma 1 of Knight (1991) or Lemma 1 and 2 of LePage et al. (1981) that $\mathbf{U}^n = (U_{n1}, \dots, U_{nn}, 0, 0, \dots)$, $\mathbf{B}^n = (B_{n1}, \dots, B_{nn}, 0, 0, \dots)$ and $\mathbf{Z}^n = (Z_{n1}, \dots, Z_{nn}, 0, 0, \dots)$ converge in distribution to $\mathbf{U} = (U_1, U_2, \dots)$, $\mathbf{B} = (B_1, B_2, \dots)$ and $\mathbf{Z} = (Z_1, Z_2, \dots)$ respectively. Moreover the limiting sequence are mutually independent and they converge jointly. Also the limits can and will be assumed to be defined on the same probability space with the convergence in distribution substituted by almost sure convergence.

Now we state a key Lemma which shall be used in the proof of the results. The first part of the result was shown in Lemma 3 of Knight (1991) and the second part is a modification of the first part, and shall be used in the proof of Theorem 3.1 We need the notion of rational independence. A sequence of real numbers $\{b_j\}_{1 \leq j \leq m} \in (0, 1)$ is *rationally independent* if for integers $\{a_j\}_{1 \leq j \leq m}$,

$$a_1 b_1 + \dots + a_m b_m = 0 \pmod{1} \Leftrightarrow a_1 = \dots = a_m = 0.$$

Lemma 4.1. (a) Let $\{\pi_{nj}\}_{j=1,\dots,m}$, $n \geq 1$ be positive real numbers such that

$$\frac{\pi_{nj}}{n} \rightarrow u_j \in (0, 1) \quad (4.1)$$

where $\{u_j\}$ is rationally independent. Let \mathbf{R}_n be a random vector taking values from

$$\left(t \frac{\pi_{n1}}{n} (\text{mod } 1), \dots, t \frac{\pi_{nm}}{n} (\text{mod } 1) \right)$$

with probability $1/n$ for each $t = 1, \dots, n$. Then the distribution of \mathbf{R}_n converges weakly to the uniform distribution on the unit cube $[0, 1]^m$.

(b) Let $n = k^g + 1$ and $\{\pi_{nj}\}_{j=1,\dots,m}$, $n \geq 1$ satisfy (4.1) and

$$\frac{k\pi_{nj}}{n} (\text{mod } 1) \rightarrow v_{1,j} \in (0, 1), \dots, \frac{k^{g-1}\pi_{nj}}{n} (\text{mod } 1) \rightarrow v_{g-1,j} \in (0, 1). \quad (4.2)$$

where $\{v_{1,j}, \dots, v_{g-1,j}\}$ is a rationally independent. Let

$$\tilde{\pi}_{nj}(b_1, b_2, \dots, b_g) = \left(\frac{\pi_{nj}}{n} (b_1 k^{g-1} + \dots + b_g) (\text{mod } 1), \dots, \frac{\pi_{nj}}{n} (b_g k^{g-1} - \dots - b_{g-1}) (\text{mod } 1) \right)$$

and $\tilde{\mathbf{R}}_n$ be a random vector which takes values from the set

$$(\tilde{\pi}_{n1}(b_1, b_2, \dots, b_g), \dots, \tilde{\pi}_{nm}(b_1, b_2, \dots, b_g))$$

with probability $1/|T_n|$ for each $(b_1, b_2, \dots, b_g) \in T_n$. Then $\tilde{\mathbf{R}}_n$ converges weakly to the uniform distribution on the unit cube $[0, 1]^{gm}$.

Proof. We will give a proof only for the case $g = 2$. The proof for the general case is along similar lines. A proof for the first part appears in Knight (1991) and hence we provide a proof for the second part. It is enough to show

$$\int \exp(2\pi i \mathbf{w} \cdot \mathbf{x}) d\tilde{\mathbf{R}}_n(\mathbf{x}) \rightarrow 0, \quad (4.3)$$

where $\mathbf{w} = (w_1, w_2, \dots, w_{2m})$ is a vector in \mathbb{R}^{2m} with integer coordinates. Note that we can write the left side of ((4.3)) as

$$\frac{1}{|T_n|} \sum_{(a,b) \in T_n} \exp \left(2\pi i (ak + b) \sum_{j=1}^m w_{2j-1} \frac{\pi_{nj}}{n} + 2\pi i (bk - a) \sum_{j=1}^m w_{2j} \frac{\pi_{nj}}{n} \right).$$

As $|T_n| \sim n$ we can write the above expression as

$$\approx \frac{1}{n} \sum_{1 \leq a \leq k} \sum_{1 \leq b \leq k} \exp(2\pi i a x_{kn}) \exp(2\pi i b y_{kn}) \quad (4.4)$$

where

$$x_{kn} = \sum_{j=1}^m w_{2j-1} k \frac{\pi_{nj}}{n} - \sum_{j=1}^m w_{2j} \frac{\pi_{nj}}{n} \quad \text{and} \quad y_{kn} = \sum_{j=1}^m w_{2j} k \frac{\pi_{nj}}{n} + \sum_{j=1}^m w_{2j-1} \frac{\pi_{nj}}{n}.$$

Note that both x_{kn} and y_{kn} can be considered with $(\text{mod } 1)$ and

$$x_{kn}(\text{mod } 1) \rightarrow \sum_{j=1}^m w_{2j-1}v_j - \sum_{j=1}^m w_{2j}u_j := x^*.$$

Now for nonzero integers $\{w_j\}$, by rational independence of $\{u_j, v_j\}$ we have $x^* \neq 0 \pmod{1}$. Hence for all large n , x_{kn} is not an integer. Also note that for all large n , this is bounded away from zero and one. Similar conclusion holds for $\{y_{kn}\}$. Hence, summing the above geometric series (4.4) we get

$$\frac{1}{n} \exp(2\pi i x_{kn}) \frac{1 - \exp(2\pi i k x_{kn})}{1 - \exp(2\pi i x_{kn})} \exp(2\pi i y_{kn}) \frac{1 - \exp(2\pi i k y_{kn})}{1 - \exp(2\pi i y_{kn})}.$$

Now using the fact that the numerator is bounded and $\{x_{kn}\}$ and $\{y_{kn}\}$ stay bounded away from 0 and 1 for sufficiently large n , it easily follows that the above expression goes to zero. Hence this completes the proof. \square

4.1 Proof of Theorem 3.1

We shall prove the theorem only for $g = 2$. For $g > 2$, argument will be similar with more complicated algebraic calculations. Now define

$$\begin{aligned} h(x, y) &= (\cos(2\pi x), \sin(2\pi x), \cos(2\pi y), \sin(2\pi y)), \text{ for } (x, y) \in \mathbb{R}, \\ \tilde{X}_n(a, b) &= a_n^{-1} \left(\tilde{C}_{ak+b}, \tilde{S}_{ak+b}, \tilde{C}_{bk-a}, \tilde{S}_{bk-a} \right), \text{ for } (a, b) \in T_n, \end{aligned}$$

where

$$\tilde{C}_t = \sum_{j=1}^n (X_j - c_n) \cos\left(\frac{2\pi jt}{n}\right) \text{ and } \tilde{S}_t = \sum_{j=1}^n (X_j - c_n) \sin\left(\frac{2\pi jt}{n}\right).$$

Let $\tilde{P}_n(\cdot, \omega)$ be the empirical distribution function of $\tilde{X}_n(a, b)$ defined by

$$\frac{1}{|T_n|} \sum_{(a,b) \in T_n} I(\tilde{X}_n(a, b) \in \cdot).$$

Note that if we show that \tilde{P}_n converges in distribution then the result follows as the empirical measure of $f(\tilde{X}_n(a, b))$ will also converge in distribution, where

$$f(x_1, x_2, x_3, x_4) = (x_1^2 + x_2^2)^{\frac{1}{4}} (x_3^2 + x_4^2)^{\frac{1}{4}}.$$

As discussed, we ignore the eigenvalues coming from the partition sets \mathcal{P}_j with $|\mathcal{P}_j| < 4$. Now,

$$\begin{aligned} \tilde{X}_n(a, b) &= \sum_{j=1}^n h\left(\frac{(ak+b)j}{n}, \frac{(bk-a)j}{n}\right) \frac{X_j - \mu_n}{a_n} \\ &= \sum_{j=1}^n h((ak+b)U_{nj}, (bk-a)U_{nj})(B_{nj}Z_{nj} - \mathbb{E}[B_{nj}Z_{nj}I(Z_{nj} \leq 1)]). \end{aligned}$$

Let

$$W_{nj}(a, b) = ((ak+b)U_{nj}(\text{mod } 1), (bk-a)U_{nj}(\text{mod } 1)).$$

To study the behavior of \tilde{P}_n , we need $(W_{n1}^*(a, b), \dots, W_{nn}^*(a, b), \mathbf{0}, \dots)$ to be chosen at random from the set of sequences

$$\{(W_{n1}(a, b), \dots, W_{nn}(a, b), 0, \dots), (a, b) \in T_n\}.$$

Let

$$W_{nj}^* = (U_{nj}^*, V_{nj}^*).$$

So \tilde{P}_n produces the random variable

$$Y_n^*(\omega) = \sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*)(B_{nj}(\omega)Z_{nj}(\omega) - \mathbb{E}[B_{nj}Z_{nj}I(Z_{nj} \leq 1)]).$$

To derive the convergence of $W_{nj}^* = (U_{nj}^*, V_{nj}^*)$ we apply the second part of Lemma 4.1. That the anti-ranks satisfy its conditions is ensured by the following.

Lemma 4.2. *Let π_{nj} be the anti-ranks as defined in the previous section. Then $k^s \frac{\pi_{nj}}{n} \pmod{1}$ converges in distribution to the uniform distribution on $(0, 1)$, where $s = 0, 1, 2, \dots, g-1$. Further,*

$$\left(\frac{\pi_{n1}}{n}, \dots, k^{g-1} \frac{\pi_{n1}}{n} \pmod{1}, \dots, \frac{\pi_{nn}}{n}, \dots, k^{g-1} \frac{\pi_{nn}}{n} \pmod{1}\right) \xrightarrow{d} (\tilde{U}_{1,1}, \dots, \tilde{U}_{g,1}, \tilde{U}_{1,2}, \dots, \tilde{U}_{g,2}, \dots).$$

Proof. The proof goes along the same line as the proof of Lemma 4.1 and easily follows when one considers equation (4.3). We skip the details. \square

Proof of Theorem 3.1. The proof is quite similar to the proof of Theorem 5 of Knight (1991) and hence we just indicate the main steps. Let

$$Y^*(\omega) = \sum_{j=1}^{\infty} h(U_j^*, V_j^*)(B_j(\omega)Z_j(\omega) - \mathbb{E}[B_jZ_jI(Z_j \leq 1)]).$$

We show that $Y_n^* \xrightarrow{d} Y^*$.

Let \tilde{P} and \tilde{E} be the probability and expectation induced by the randomness of $\tilde{P}_n(\cdot, \omega)$ (or equivalently induced by W_{nj}^*).

Now note that $Y_n^*(\omega)$ can be broken into two parts as

$$\sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*)(B_{nj}(\omega)Z_{nj}(\omega)I(Z_{nj}(\omega) > \epsilon) - \mathbb{E}[B_{nj}Z_{nj}I(\epsilon < Z_{nj} \leq 1)]) \quad (4.5)$$

and

$$\sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*)(B_{nj}(\omega)Z_{nj}(\omega)I(Z_{nj}(\omega) \leq \epsilon) - \mathbb{E}[B_{nj}Z_{nj}I(Z_{nj} \leq \epsilon)]). \quad (4.6)$$

We show that the expression in (4.5) converges in distribution almost surely (with respect to the probability measure on Ω) and the expression in (4.6) goes to zero in L_2 in probability (with respect to the probability measure on Ω).

Since h is a bounded function it follows from Lemma 2 of Knight (1991) that,

$$\left\| \sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*)B_{nj}(\omega)Z_{nj}(\omega)I(Z_{nj} > \epsilon) - \sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*)B_j(\omega)Z_j(\omega)I(Z_j > \epsilon) \right\| \rightarrow 0 \quad a.s.$$

and

$$\begin{aligned} & \left\| \sum_{j=1}^{\infty} h(U_{n_j}^*, V_{n_j}^*) \mathbb{E} [(B_{n_j}(w) Z_{n_j}(w) I(\epsilon < Z_{n_j} \leq 1))] \right. \\ & \quad \left. - \sum_{j=1}^{\infty} h(U_{n_j}^*, V_{n_j}^*) \mathbb{E} [(B_j(w) Z_j(w) I(\epsilon < Z_j \leq 1))] \right\| \rightarrow 0. \end{aligned}$$

Now note that if $g : \mathbb{R}^{2m} \rightarrow \mathbb{R}^4$ is a bounded continuous map having periodicity one in each coordinate then under the assumptions of Lemma 4.1 it follows that

$$\begin{aligned} \frac{1}{n} \sum_{(a,b) \in T_n} g\left(\left(a + b\right) \frac{\pi_{n1}}{n}, \left(b - a\right) \frac{\pi_{n1}}{n}, \dots, \left(a + b\right) \frac{\pi_{nm}}{n}, \left(b - a\right) \frac{\pi_{nm}}{n}\right) \\ \rightarrow \int_0^1 \cdots \int_0^1 g(x_1, \dots, x_{2m}) dx_1 \cdots dx_{2m}. \end{aligned}$$

Now we use

$$g(x_1, \dots, x_{2m}) = \sum_{j=1}^m h(x_{2j-1}, x_{2j}) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)])$$

and Lemma 4.2 to conclude that,

$$\begin{aligned} & \sum_{j=1}^m h(U_{n_j}^*, V_{n_j}^*) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)]) \\ & \xrightarrow{\tilde{d}} \sum_{j=1}^m h(U_j^*, V_j^*) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)]). \end{aligned}$$

Now using the bounded nature of h and the bounds used in the proof of Theorem 5 of Knight (1991) it follows that, as $n \rightarrow \infty$ and $m \rightarrow \infty$, we have

$$\left\| \sum_{j=m+1}^{\infty} h(U_{n_j}^*, V_{n_j}^*) B_j Z_j I(Z_j > \epsilon) \right\| \rightarrow 0 \quad \text{a.s.}$$

and

$$\left\| \sum_{j=m+1}^{\infty} h(U_{n_j}^*, V_{n_j}^*) \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)] \right\| \rightarrow 0.$$

Now

$$\sum_{j=1}^{\infty} h(U_j^*, V_j^*) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)])$$

is finite for almost all ω , with \tilde{P} probability 1. So, as $m \rightarrow \infty$, we have

$$\begin{aligned} & \sum_{j=1}^m h(U_j^*, V_j^*) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)]) \\ & \xrightarrow{\tilde{P}} \sum_{j=1}^{\infty} h(U_j^*, V_j^*) (B_j Z_j I(Z_j > \epsilon) - \mathbb{E} [B_j Z_j I(\epsilon < Z_j \leq 1)]). \end{aligned}$$

The rest of the arguments are similar to the proof of Theorem 5 in Knight (1991). In fact that the second part (4.6) goes to zero in probability follows from the bound on h and regularly varying property of the distribution of X_j 's. This completes the proof of first part of the theorem.

The proof the second part can be written along lines similar to the proof of the second part of Theorem 5 in Knight (1991). We indicate the main steps. First, following arguments similar to those given in the part (a),

$$\sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} > \epsilon) \xrightarrow{\tilde{d}} \sum_{j=1}^{\infty} h(U_j^*, V_j^*) B_j Z_j I(Z_j > \epsilon)$$

as $n \rightarrow \infty$. Proof of the next step requires extra care. To show

$$\tilde{P} \left[\left\| \sum_{j=1}^{\infty} h(U_j^*, V_j^*) (B_j Z_j I(Z_j \leq \epsilon)) \right\| > \gamma \right] \xrightarrow{p} 0$$

as $n \rightarrow \infty$ and $\epsilon \rightarrow 0$, observe that

$$\begin{aligned} & \tilde{P} \left[\left\| \sum_{j=1}^{\infty} h(U_{nj}^*, V_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} \leq \epsilon) \right\| > \gamma \right] \\ & \leq \tilde{P} \left[\left| \sum_{j=1}^{\infty} \cos(2\pi U_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} \leq \epsilon) \right| > \frac{\gamma}{2} \right] + \cdots + \tilde{P} \left[\left| \sum_{j=1}^{\infty} \sin(2\pi V_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} \leq \epsilon) \right| > \frac{\gamma}{2} \right]. \end{aligned} \quad (4.7)$$

First observe that $\tilde{P} \left[\left| \sum_{j=1}^{\infty} \cos(2\pi U_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} \leq \epsilon) \right| > \frac{\gamma}{2} \right] \xrightarrow{p} 0$. In fact, we have

$$\begin{aligned} & \tilde{P} \left[\left| \sum_{j=1}^{\infty} \cos(2\pi U_{nj}^*) B_{nj} Z_{nj} I(Z_{nj} \leq \epsilon) \right| > \frac{\gamma}{2} \right] \\ & = \frac{1}{|T_n|} \# \left\{ (a, b) : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) X_l I(|X_l| \leq a_n \epsilon) \right| > \frac{\gamma}{2} \right\} \\ & = \frac{1}{|T_n|} \# \left\{ (a, b) \in A_n : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) X_l I(|X_l| \leq a_n \epsilon) \right| > \frac{\gamma}{2} \right\} \\ & \quad + \frac{1}{|T_n|} \# \left\{ (a, b) \in A_n^c : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) X_l I(|X_l| \leq a_n \epsilon) \right| > \frac{\gamma}{2} \right\} \end{aligned}$$

where $A_n = \{(a, b) : \sum_{l=1}^n \cos(\frac{2\pi(ak+b)l}{n}) \neq 0\}$. Now $|A_n| = o(n)$ and $|T_n|/n \rightarrow 1$. So

$$\begin{aligned} & \frac{1}{|T_n|} \# \left\{ (a, b) : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) X_l I(|X_l| \leq a_n \epsilon) \right| > \frac{\gamma}{2} \right\} \\ & = \frac{o(n)}{n} + \frac{1}{|T_n|} \# \left\{ (a, b) \in A_n^c : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) X_l I(|X_l| \leq a_n \epsilon) \right| > \frac{\gamma}{2} \right\} \\ & = \frac{o(n)}{n} + \frac{1}{|T_n|} \# \left\{ (a, b) \in A_n^c : \left| a_n^{-1} \sum_{l=1}^n \cos \left(\frac{2\pi(ak+b)l}{n} \right) [X_l I(|X_l| \leq a_n \epsilon) - E(X_l I(|X_l| \leq a_n \epsilon))] \right| > \frac{\gamma}{2} \right\} \end{aligned}$$

$$\begin{aligned} &\leq \frac{o(n)}{n} + \frac{1}{|T_n|} \#\left\{ (a, b) : |a_n^{-1} \sum_{l=1}^n \cos\left(\frac{2\pi(ak+b)l}{n}\right) [X_l I(|X_l| \leq a_n \epsilon) - E(X_l I(|X_l| \leq a_n \epsilon))] | > \frac{\gamma}{2} \right\} \\ &= \frac{o(n)}{n} + \tilde{P} \left[\left| \sum_{j=1}^{\infty} \cos(2\pi U_{n_j}^*) (B_{n_j} Z_{n_j} I(Z_{n_j} \leq \epsilon) - E(B_{n_j} Z_{n_j} I(Z_{n_j} \leq \epsilon))) \right| > \frac{\gamma}{2} \right] \xrightarrow{P} 0, \end{aligned}$$

by the proof of Theorem 3.1(a). Now similar conclusion holds for the other three terms of (4.7). So, $\tilde{P} \left[\left| \sum_{j=1}^{\infty} h(U_{n_j}^*, V_{n_j}^*) (B_{n_j} Z_{n_j} I(Z_{n_j} \leq \epsilon)) \right| > \gamma \right] \xrightarrow{P} 0$ as $n \rightarrow \infty$ and $\epsilon \rightarrow 0$. So, this completes the proof for both part (a) and part (b) of Theorem 3.1. \square

4.2 Proof of Theorems 3.2 and 3.3

The proof of Theorem 3.2 is similar to the proof of Theorem 3.1, so we skip it.

To prove Theorem 3.3, first observe that, we can ignore $a_n^{-1} X_1$ from the eigenvalues formula (also $a_n^{-1} X_{n/2+1}$ if n even) since $a_n^{-1} X_1 \rightarrow 0$ in probability. Now the results follows from Theorem 5(a) of Knight (1991).

Remark 4.1. Our proof heavily rely on the nice structure obtained for the eigenvalue partition in $n = k^g \pm 1$ case. For general k and n , the eigenvalue partitions have different sizes and varied compositions, making it highly intractable.

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Arup Bose, Suman Guha and Rajat Subhra Hazra
 Statistics and Mathematics Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata 700108, India. bosearu@gmail.com, sumanguha_r@isical.ac.in and rajat_r@isical.ac.in.

Koushik Saha
 Bidhannagar Government College, Sector I, Salt Lake, Kolkata 700064, India. koushiksaha877@gmail.com.