Mixed Markets in Bilateral Monopoly

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Abstract

Is there justification for any pattern in the sequencing of privatization of a public bilateral monopoly? To address this question, the paper analyzes the welfare implications of mixed markets in this setting. We conclude that merely observing cost savings from privatization upstream/downstream, does not dictate good policy. It ignores societal priorities on the role of public and private profit and the strategic significance of upstream/downstream location. If public profit is relatively insignificant in welfare, then only relative cost savings matter. However, if society sufficiently prioritizes public profit, then privatization downstream will maximize welfare if it is as (or more) cost effective than privatization upstream. Moreover, welfare may be higher with privatization downstream even if there is a relative cost advantage with upstream privatization. We also find that the cost differentials in the vertical stream have to be significantly in favor of privatization upstream to make that the welfare maximizing policy. For a given cost differential, privatization upstream is more likely to maximize welfare if society places a relatively higher (lower) priority on the role of private (public) profit.

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1 Introduction

In many countries, core infrastructure sectors such as electricity, steel and natural gas and downstream sectors like rail and air transportation are dominated by firms with significant

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market power. In these markets it is well known that vertical restraints can either increase or decrease welfare. For instance, in a vertically integrated bilateral monopoly, welfare unambiguously increases from the elimination of double marginalization.[1] On the other hand, in a vertical framework with downstream retail competition, an upstream monopoly manufacturer can increase profit by using vertical restraints such as exclusive territories or franchise fees and the welfare effects are ambiguous.[2] This strand of research almost uniformly assumes that firms in both sectors, either manufacturers of an intermediate and final good or in a manufacturer/retailer relationship, are private firms.

However in many countries across Asia, Latin America and Western Europe, public firms operate as a bilateral monopoly. For instance, until recently, British Electric (upstream) and British Rail (downstream) were both state owned firms. There has been limited entry into the (upstream) steel industry in India but Indian Railways (downstream) remains a public monopoly.

While privatization of state owned firms has been extremely popular in recent decades, it remains politically sensitive, and simultaneously privatizing both sectors in a bilateral public monopoly can be problematic. We think it is important to investigate whether there is any pattern in the sequencing of privatization of the upstream and downstream sectors. Are upstream (downstream) sectors privatized first / more frequently than downstream (upstream) firms? Empirical evidence does not offer a clear answer. In Latin America for example, Chile has privatized water utilities while water privatization in Brazil has been relatively limited. While British Gas and British Telecom (both arguably upstream sectors) were privatized, many Asian economies (albeit with exceptions such as Korea, Malaysia and Singapore) have retained state ownership and control of their major upstream infrastructure sectors.

In view of this, if only one sector in a bilateral public monopoly can be privatized, which should it be? To the best of our knowledge, this is among the first papers to analyze mixed markets and the optimal sequencing of privatization from a social welfare perspective.

There is an extensive literature on mixed oligopolies, where public welfare maximizing firms compete with private profit maximizing firms. This area has its roots in the early pioneering work of Merrill and Schneider (1966). The reader may consult Harris and Wiens (1980), and DeFraja and Delbono (1990) for surveys. Since then, authors have studied a variety of interesting questions in mixed oligopoly models.

In a one sector model, Matsumura (1998) studies a quantity setting mixed oligopoly, where the private firm maximizes profit and the public firm considers both profit and social welfare. He finds that neither full privatization nor full nationalization is optimal under moderate conditions. Fershtman (1990) analyzes the interdependence between ownership status and the market structures in which firms operate. He shows that in a duopoly, nationalization of an inefficient firm may decrease welfare and that a partly nationalized firm may realize higher profits than its private profit maximizing competitor. Matsumura and Matsushima (2004) study a mixed duopoly with product differentiation, where the production costs are endogenized. They show that strategic interactions between firms yield a higher production

[1]The literature on mergers in vertically related industries is very well developed. For example, Salinger (1988, 1989) has discussion on effects of vertical mergers.

[2]See Tirole (1988) for discussion.

cost to the public firm than to the private firm. Pal (1998) studies an endogenous order of moves in mixed oligopoly and finds results that are very different from a market with all profit maximizing firms.[3] Ichida and Matsushima (2009) analyze whether public firm employees should be allowed to bargain collectively. Two notable examples of mixed market monopolistic competition are Anderson et al. (1997) and Matsumura and Kanda (2005), which study mixed markets with free entry.

The novelty of our framework is a mixed market in a bilateral monopoly. We are familiar with only one other paper that uses a similar model. In a three sector model with an upstream firm, a downstream firm and a retailer, Glaeser and Scheinkman (1996) assume that privatization creates information benefits. Interestingly, they show that if only one sector can be privatized and demand uncertainty is high, it is possible that the optimal policy involves privatizing upstream. In general, they conclude that across industries, those with large amounts of cost and/or demand uncertainties should be privatized first.[4]

We focus on the welfare implications of a mixed market in a bilateral monopoly and are interested in whether public policy should adopt a hands off approach to privatization or whether an argument exists for policy directed at privatizing specifically the upstream or the downstream sector.

It is accepted that a benefit from privatization is higher productivity and we retain this assumption in the analysis.[5] Further, much of the literature on mixed markets assumes that the public firm maximizes welfare. In our framework, the public firm maximizes a weighted average of consumer surplus, public and private profit.[6] We also study aggregate surplus which is the usual sum of consumer and producer surplus. A distinction is made between welfare (the public firm's payoff) and aggregate surplus, and both are derived and compared across three alternative versions of the vertical model, which are discussed later. The results highlight the significance of cost differentials between public and private firms, unequal weights for public and private profit in the public firm's payoff, and the strategic role of upstream / downstream location in a vertical framework.

It should be noted that in the mixed oligopoly literature, a public bilateral monopoly is seldom discussed, perhaps because it is trivially first best to have a single public firm. Since the focus of our work is on mixed markets in a bilateral monopoly and we assume that privatization increases productive efficiency, the private bilateral monopoly sets the benchmark for welfare comparison. The optimum prices, profit, welfare and aggregate surplus with privatization in the upstream (downstream) sector are compared to those with privatization in the downstream (upstream) sector. Both are also compared with private–private welfare

[3]In a related strand of research, Fjell and Pal (1996) study mixed oligopoly with foreign private firms. Matsushima and Matsumura (2006) study a spatial model with location choice in a circular city in the presence of foreign private firms.

[4]A recent work by Wen and Yuan (2010) is on privatization of a vertically integrated regulated public utility. They view the optimal restructuring plan from a public finance perspective. Their work has a different focus and examines how the optimal restructuring depends on the cost of public funds in a setting where the government can simultaneously privatize all sectors and choose the numbers of firms in each sector.

[5]In an oligopoly model, De Fraja and Delbono (1989) show that welfare might actually be higher if the public firm maximizes profit but they do not consider the effect of higher productivity with privatization. They argue that privatization is better even if it does not increase productivity.

[6]Cremer et al. (1989) assume that the public firm maximizes total surplus but state that "..clearly a better welfare index would include firm and consumer specific weights".

and aggregate surplus.

We conclude that merely observing cost savings from privatization upstream/downstream, ignores societal priorities on the role of public and private profit and the strategic significance of upstream/downstream location in the vertical monopoly. We find that if public profit is relatively insignificant in welfare, then only relative cost savings matter. However, if society cares sufficiently about public profit, then privatization downstream will maximize welfare if it is as (or more) cost effective than privatization upstream. Moreover, welfare may be higher with privatization downstream even if there is a relative cost advantage with upstream privatization. This is because the upstream location gives the public firm a strategic first mover advantage. We also find that the cost differentials in the vertical stream have to be significantly in favor of privatization upstream to make that the welfare maximizing policy. For a given cost differential, privatization upstream is more likely to maximize welfare if society places a relatively higher (lower) priority on the role of private (public) profit. Hence, in general, relative cost savings and the relative importance of public and private profit in welfare should both play a role in public policy toward privatization.

While the main focus of this paper is on welfare comparisons, there are some additional results of interest in this work. Different societies place varying priorities on public and private profit. We find that the relative significance of public and private profit is very important in determining how input and output prices and resulting profit and welfare, respond to varying societal priorities with downstream privatization. If the upstream firm is privatized, then its strategic location allows it to dominate optimal outcomes in favor of private profit and the downstream public firm chooses marginal cost pricing.

As we mentioned earlier, there is an extensive literature on mixed oligopoly and monopolistic competition. There is also a well established body of work on vertical markets and effects of controls in these markets. However, the role of mixed markets in a bilateral monopoly where a private profit maximizing firm and welfare maximizing public firm interact in a vertical setting, has not been explored in the literature. Given that both core infrastructure and downstream (output) sectors are often (or have been) historically public entities in many places, this is a relevant and important framework that can provide policy guidance for privatization.

The paper is organized as follows: Section 2 presents the model. Section 3 provides the solutions while Section 4 discusses the comparative statics. Section 5 has the welfare comparisons and Section 6 concludes the paper. All proofs are in the Appendix.

2 Model

The market is a bilateral monopoly of a private firm p and a public firm g, one operating upstream and the other downstream. The upstream firm (u) produces the input used by the downstream firm (d) to produce the final output. The private firm p maximizes profit and the public firm g maximizes welfare, which is a weighted average of the sum of profits of both firms and the consumer surplus. This is given by:

$$W = \alpha_p \pi_p + \alpha_g \pi_g + \alpha_c CS$$

where

$$\begin{aligned}\pi_p &= \text{profit of private firm} && (2.1)\\ \pi_g &= \text{profit of public firm} && (2.2)\\ CS &= \text{consumer surplus} && (2.3)\end{aligned}$$

and α_p, α_g and $\alpha_c > 0$. The corresponding *aggregate surplus* is the sum of public and private profit and consumer surplus, and denoted by AS.

Following standard specifications in the bilateral monopoly literature, we assume that the technology is fixed proportions production with one unit of input required to produce one unit of output. The market demand downstream is linear and of the form: $q_d = 1 - p_d$. CS is then given by

$$CS = \int_{p_d}^{1} (1-t)dt = \frac{(1-p_d)^2}{2} \tag{2.4}$$

where p_d is the price charged by the downstream firm for the final output.

Three basic conditions are used throughout the paper. First, the welfare weights are normalized, so that $\alpha_p + \alpha_g + \alpha_c = 1$. Second, the public firm must earn a nonnegative profit, that is $\pi_g \geq 0$. Since tight budgets are frequently a principal motivation for privatization, this condition is easily justified. Third, the marginal cost of production must be less than one. This condition is a necessary (but not sufficient, as we will see later) condition for a positive level of production, given the demand specifications. The alternative, which will always mean no output, is trivial and therefore not considered.

Let c_i be the marginal cost of production for the public firm $i = u, d$. The marginal cost of production for a private firm is lower with cost savings from privatization. It is given by $\theta_i c_i$, where $0 < \theta_i < 1$, $i = u, d$ represents the productivity efficiency parameter.

The firms play a two stage game: in stage one, the upstream firm announces an input price, p_u. In the second stage, the downstream firm learns the input price and determines the downstream price, p_d. The market clears at the end of stage two.

Case 1 analyzes the bilateral monopoly with an upstream private firm and a downstream public firm. Case 2 is the reverse, with privatization downstream, and a public firm in the upstream input market.

There are of course two remaining cases. As discussed earlier, we disregard the public-public case. As a benchmark, we consider a third case where the upstream and downstream firms are both private profit maximizing firms. Incidentally, it is important to keep in mind that to compare outcomes with mixed markets in a bilateral monopoly, we retain the private–private bilateral monopoly.

We now describe the model for the three cases.

Case 1. In this case, the upstream firm is private and the downstream firm is public. Then,

$$\pi_g = \pi_g(p_u, p_d) = (p_d - p_u - c_d)(1 - p_d) \quad \text{and} \quad \pi_p = \pi_p(p_u, p_d) = (p_u - \theta_u c_u)(1 - p_d). \tag{2.5}$$

The downstream public firm will maximize welfare:

$$\max_{p_d}[W^{PU}(p_u, p_d)] = \max_{p_d}\left[\alpha_p \pi_p + \alpha_g \pi_g + \alpha_c CS\right]. \tag{2.6}$$

Let the solution be $p_d = p_d(p_u)$. The upstream private firm will then maximize profit:

$$\max_{p_u}[\pi_p] = \max_{p_u}[\pi_p(p_u, p_d)]. \tag{2.7}$$

W^{PU} will denote the maximized welfare and AS^{PU} will denote the value of $\pi_p + \pi_g + CS$ at the optimum (p_d, p_u).

Case 2. In this case, the upstream firm is public and the downstream firm is private. Then

$$\pi_g = \pi_g(p_u, p_d) = (p_u - c_u)(1 - p_d) \text{ and } \pi_p = \pi_p(p_u, p_d) = (p_d - p_u - \theta_d c_d)(1 - p_d). \tag{2.8}$$

The downstream firm will maximize profit

$$\max_{p_d}[\pi_g] = \max_{p_d} \pi_g(p_u, p_d). \tag{2.9}$$

Let the solution be $p_d = p_d(p_u)$. The upstream public firm will then maximize welfare:

$$\max_{p_u}[W^{PD}(p_u, p_d)] = \max_{p_u} \left[\alpha_p \pi_p + \alpha_g \pi_g + \alpha_c CS\right]. \tag{2.10}$$

W^{PD} will denote the maximized weighted welfare and AS^{PD} will denote the value of $\pi_p + \pi_g + CS$ at the optimum (p_d, p_u).

Case 3. In this case, the upstream and the downstream firms are both private. For clarity, we now adopt a slightly different notation for profit. Let the upstream and downstream profits be denoted by π_u and π_d respectively. Then they are given by:

$$\pi_u = \pi_u(p_u, p_d) = (p_u - c_u)(1 - p_d) \text{ and } \pi_d(p_u, p_d) = (p_d - p_u - \theta_d c_d)(1 - p_d). \tag{2.11}$$

The downstream private firm will maximize

$$\max_{p_d}[\pi_d]. \tag{2.12}$$

Let the solution be $p_d = p_d(p_u)$. The upstream private firm will maximize π_u:

$$\max_{p_u}[\pi_u] = \max_{p_u}[\pi_u(p_u, p_d)]. \tag{2.13}$$

The welfare W is not meaningful with both private firms. If (p_u, p_d) is the optimal solution, then the aggregate surplus AS^{PP} equals $\pi_p + \pi_g + CS$ at the optimum (p_d, p_u).

3 Results

3.1 Case 1: Privatization upstream

Proposition 1 summarizes the solution to the maximization problem with upstream privatization. As mentioned above, we assume that the marginal cost of production in the vertical stream, $0 < \theta_u c_u + c_d < 1$.

Proposition 1. *Consider a bilateral monopoly with a private firm upstream and a public firm downstream. The optimum is given by,*

$$p_u = \frac{1 - c_d + \theta_u c_u}{2} \tag{3.1}$$

$$p_d = \frac{1 + c_d + \theta_u c_u}{2} \tag{3.2}$$

$$\pi_g = 0 \tag{3.3}$$

$$\pi_p = \frac{1}{4}(1 - c_d - \theta_u c_u)^2 \tag{3.4}$$

$$CS^{PU} = \frac{1}{8}(1 - c_d - \theta_u c_u)^2 \tag{3.5}$$

$$W^{PU} = \frac{(1 + \alpha_p - \alpha_g)}{8}(1 - c_d - \theta_u c_u)^2 \tag{3.6}$$

$$AS^{PU} = \frac{3}{8}(1 - c_d - \theta_u c_u)^2. \tag{3.7}$$

When located upstream, the private firm has a first mover advantage over the downstream public firm. Further, since there are no subsidies available to the welfare maximizing public enterprise, it has to set at least a break even price. As a consequence, the public firm has no leverage and the upstream private firm effectively monopolizes the surplus in the vertical stream. At the optimal outcome, the public firm's (output) price equals marginal cost and it earns zero profit. Private profit is, expectedly, insensitive to the weights attached to the public and private profit in welfare. Further, since public profit is zero at the optimum, welfare is decreasing (increasing) in α_g (α_p).

3.2 Case 2: Privatization downstream

In Case 2, we assume that the marginal cost of production in the vertical stream, $0 < \theta_d c_d + c_u < 1$. Proposition 2 summarizes the solutions to the maximization problem.

Proposition 2. *Consider a bilateral monopoly with a public firm upstream and a private firm downstream.*
(i) If $3\alpha_g - \alpha_p - 1 \geq 0$ *then the following is the optimum:*

$$p_u^{(1)} = c_u + \frac{3\alpha_g - \alpha_p - 1}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) \tag{3.8}$$

$$p_d^{(1)} = 1 - \frac{\alpha_g}{5\alpha_g - \alpha_p - 1}(1 - c_u - \theta_d c_d) \tag{3.9}$$

$$\pi_p^{(1)} = \frac{\alpha_g^2}{(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \tag{3.10}$$

$$\pi_g^{(1)} = \frac{\alpha_g(3\alpha_g - \alpha_p - 1)}{(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \tag{3.11}$$

$$CS^{PD1} = \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \tag{3.12}$$

$$W^{PD1} = \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)}(1 - c_u - \theta_d c_d)^2 \quad (3.13)$$

$$AS^{PD1} = \frac{\alpha_g(9\alpha_g - 2\alpha_p - 2)}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2. \quad (3.14)$$

(ii) If $3\alpha_g - \alpha_p - 1 \leq 0$ *then optimum is given by:*

$$p_u^{(2)} = c_u \quad (3.15)$$

$$p_d^{(2)} = \frac{1 + c_u + \theta_d c_d}{2} \quad (3.16)$$

$$\pi_g^{(2)} = 0 \quad (3.17)$$

$$\pi_p^{(2)} = \frac{1}{4}(1 - c_u - \theta_d c_d)^2 \quad (3.18)$$

$$CS^{PD2} = \frac{1}{8}(1 - c_u - \theta_d c_d)^2 \quad (3.19)$$

$$W^{PD2} = \frac{(1 + \alpha_p - \alpha_g)}{8}(1 - c_u - \theta_d c_d)^2 \quad (3.20)$$

$$AS^{PD2} = \frac{3}{8}(1 - c_u - \theta_d c_d)^2. \quad (3.21)$$

With downstream privatization, the optimal outcomes are sensitive to the weights attached to private and public profit in welfare. Observe in (ii) that if α_g is sufficiently small, the upstream public firm will set price equal to marginal cost and earn zero profit, similar to Proposition 1. However, very different incentives drive these similar results. With upstream privatization, the strategic location for the private firm effectively allows it to capture all the profits in the vertical stream, and the downstream welfare maximizing firm finds itself pricing at marginal cost to break even, in the face of a binding budget constraint. With downstream privatization, the upstream public firm has the first mover advantage. We find that when α_g is sufficiently small (likewise implies α_p is relatively large), the public firm finds it welfare maximizing to price at marginal cost and concede positive profits to the private firm. On the other hand, if α_g is above a minimum threshold value, the optimal solutions in (i) will hold. Public profit is now sufficiently significant in welfare. For example, $\alpha_g = 1/3$, $\alpha_p = 1/3$, satisfies (ii) and the public firm earns zero profit whereas for $\alpha_g = 0.4$, $\alpha_p = 0.15$ or $\alpha_g = 0.5$, $\alpha_p = 0.3$, the sufficient condition in (i) is satisfied and public profit is positive.

3.3 Case 3: Upstream and Downstream Privatization

In Case 3, there are private profit maximizing firms upstream and downstream. The optimum solution follows easily from standard arguments for private–private bilateral monopoly profit maximization and are given in the following Proposition. Assume that $0 < \theta_d c_d + \theta_u c_u < 1$.

Proposition 3. *The solution to the optimization problem and the corresponding optimum* *values are given by:*

$$\begin{aligned} p_u &= \frac{1 - \theta_d c_d + \theta_u c_u}{2} & (3.22) \\ p_d &= \frac{3 + \theta_d c_d + \theta_u c_u}{4} & (3.23) \\ \pi_u &= \frac{(1 - \theta_d c_d - \theta_u c_u)^2}{8} & (3.24) \\ \pi_d &= \frac{1}{16}(1 - \theta_d c_d - \theta_u c_u)^2 & (3.25) \\ CS^{PP} &= \frac{1}{32}(1 - \theta_d c_d - \theta_u c_u)^2 & (3.26) \\ AS^{PP} &= \frac{7}{32}(1 - \theta_d c_d - \theta_u c_u)^2. & (3.27) \end{aligned}$$

4 Comparative statics

The comparative statics of the optimal solutions with respect to α_g and α_p, are discussed in this section. This will highlight the tradeoffs between welfare weights and cost differentials and also explain the role of varying priorities in the optimal payoffs. We will see the contrasting outcomes from privatizing upstream/downstream under different conditions on α_g and α_p and cost differentials, and focus on the strategic incentives with upstream/downstream privatization.

4.1 Case 1: Privatization upstream

First consider privatization upstream. As discussed earlier, the strategic first mover advantage for the private firm coupled with the binding budget constraint for the public firm, leads to an optimal outcome with marginal cost pricing and zero profit for the public firm. Hence recall from Proposition 1 that the optimal prices and profits are independent of the weights and therefore their comparative statics are irrelevant.

The behavior of optimal welfare with respect to the welfare weights is given in the Corollary below. It is worth noting that for a small increase in α_g welfare will decrease since public profit is zero. Likewise, a small increase in α_p will increase welfare because private profit is a significant component of welfare. Further detailed discussion of the welfare findings is presented in Section 5.

Corollary 1.

(i) $\dfrac{\partial W^{PU}}{\partial \alpha_g} < 0.$

(ii) $\dfrac{\partial W^{PU}}{\partial \alpha_p} > 0.$

4.2 Case 2: Privatization downstream

When α_g is relatively small (Proposition 2(ii)), we have found that optimal prices are independent of the weights. We know that the public firm sets price equal to marginal cost, just as it did with privatization upstream and so as before, the comparative statics for prices and profit are irrelevant and not reported. For α_g above a minimum threshold, (Proposition 2(i)), the properties of the optimum prices, profit and welfare are studied further to understand the strategic differences between the optima with privatization upstream/downstream.

The Corollary below states these relationship between the optimum prices and α_g and α_p. With the public firm upstream, input and output price is positively related to α_g. So, when the profit of the public firm is marginally more important in the welfare function, the public firm increases its price. When the upstream price increases, the price of the downstream private firm has to increase to pay the higher input cost.

It is interesting to compare this to the effect of a marginal increase in the importance of the private profit in the welfare function. With the public firm upstream, we find that input and output price is inversely related to α_p. So, when the profit of the private firm is marginally more important, the welfare maximizing public firm (upstream) decreases its input price, thus enabling the private firm (downstream) to also lower its price in response to lower input costs. To what extent and whether welfare weights will play a role in the optimal price, profit and welfare, therefore also depends on whether privatization occurs upstream or downstream. These contrasting results highlight the important role of firm location in the vertical stream. They are summarized below.

Corollary 2. *When the public firm is upstream,*

(i) $\dfrac{\partial p_u^{(1)}}{\partial \alpha_g} > 0$ *and* $\dfrac{\partial p_u^{(1)}}{\partial \alpha_p} < 0.$

(ii) $\dfrac{\partial p_d^{(1)}}{\partial \alpha_g} > 0$ *and* $\dfrac{\partial p_d^{(1)}}{\partial \alpha_p} < 0.$

The relationships between the private firm's optimal profit and α_p, α_g are given in the next Corollary. We note that if the share of the public (private) firm's profit in welfare is marginally higher, then as expected, the private firm's profit falls (increases).

Corollary 3. *When the public firm is upstream,*

(i) $\dfrac{\partial \pi_p^{(1)}}{\partial \alpha_g} < 0.$

(ii) $\dfrac{\partial \pi_p^{(1)}}{\partial \alpha_p} > 0.$

The final result in this section gives the relationship between the optimum welfare and α_g, α_p. Recall from Corollary 1 that if the share of public firm's profit in welfare is marginally higher, then welfare decreases with privatization upstream because $\pi_g = 0$. However, here we find that welfare may increase or decrease with privatization downstream.

Corollary 4. *When the public firm is upstream*

(i) $\dfrac{\partial W^{PD1}}{\partial \alpha_g} > 0$ *if* $5\alpha_g - 2\alpha_p - 2 > 0.$

(ii) $\dfrac{\partial W^{PD1}}{\partial \alpha_g} < 0$ *if* $5\alpha_g - 2\alpha_p - 2 < 0.$

(iii) $\dfrac{\partial W^{PD1}}{\partial \alpha_p} > 0.$

(iv) $\dfrac{\partial W^{PD2}}{\partial \alpha_g} < 0.$

(v) $\dfrac{\partial W^{PD2}}{\partial \alpha_p} > 0.$

In general, if α_g is above a minimum threshold, then the strategic upstream location of the public firm positively impacts welfare. That is, for a small increase in α_g, we find that welfare will typically increase (decrease) if the weight of the public firm's profit is relatively large (small). Intuitively, this is because the effect of higher α_g on welfare is driven by the relatively higher (smaller) weight of the public firm's profit in the welfare function itself. Welfare increases with α_p for upstream (Corollary 1) and downstream (Corollary 4) privatization because the private firm's incentives are to maximize private profit, whereas the public firm maximizes welfare and so is sensitive to the relative weights on public and private profit, given its location in the bilateral monopoly.

5 Welfare and Aggregate Surplus Comparison

In this section we compare welfare with privatization upstream versus downstream. The results highlight the interaction between welfare weights, cost tradeoffs and the strategic positioning upstream of the private (or public) firm. Similar analysis is presented for aggregate surplus.

5.1 Comparison of welfare

From Propositions 1 and 2, recall that

$$W^{PU} = \frac{(1+\alpha_p-\alpha_g)}{8}(1-c_d-\theta_u c_u)^2 \tag{5.1}$$

$$W^{PD1} = \frac{\alpha_g^2}{2(5\alpha_g-\alpha_p-1)}(1-c_u-\theta_d c_d)^2 \text{ if } 3\alpha_g-\alpha_p-1 \geq 0 \tag{5.2}$$

$$W^{PD2} = \frac{(1+\alpha_p-\alpha_g)}{8}(1-c_u-\theta_d c_d)^2 \text{ if } 3\alpha_g-\alpha_p-1 \leq 0. \tag{5.3}$$

To compare welfare, suppose for some β,

$$\beta(1-c_u-\theta_d c_d)^2 = (1-c_d-\theta_u c_u)^2. \tag{5.4}$$

Note that if β equals 1, the cost savings from privatization upstream are equal to the savings from privatization downstream (or, the marginal cost of production in the vertical stream is identical with privatization upstream/downstream). If β is less (greater) than 1, then the cost savings with privatization downstream are greater (less) than with privatization upstream.

Corollary 5. *Suppose* $3\alpha_g - \alpha_p - 1 \leq 0$. *Then* $\dfrac{W^{PU}}{W^{PD2}} = \beta$.

This finding highlights the role of welfare weights that reflect societal priorities in welfare maximization. We find that if public profit is sufficiently insignificant (α_g is relatively small) then only the relative cost savings will determine whether upstream/downstream privatization will maximize welfare. Welfare is higher with privatization upstream (downstream), if it is relatively more cost effective to privatize upstream (downstream). Therefore, if public profit is sufficiently low priority in welfare, the location upstream/downstream of the private/public firm does not have any effect on the comparison of optimal welfare.

However, as Corollary 6 will show, if society places sufficiently high priority on public profit (α_g is relatively large), then the comparisons are quite different. We now find that if it is relatively cost effective to privatize downstream, then welfare is always maximized with downstream privatization ((ii)). Privatization upstream will maximize welfare if it is sufficiently more cost effective relative to downstream privatization ((iii)). That is, even if privatization upstream is more cost effective than downstream privatization, that does not necessarily translate into welfare maximization. This is because, when located upstream, the private firm has a strategic first mover advantage and monopolizes the surplus in the vertical stream. Hence unless it is sufficiently cost effective, downstream privatization will continue to maximize welfare.

Corollary 6. *Suppose* $3\alpha_g - \alpha_p - 1 \geq 0$. *Then*

(i) $\beta\dfrac{(1+\alpha_p-\alpha_g)}{2\alpha_g} \leq \dfrac{W^{PU}}{W^{PD1}} \leq \beta$

As a consequence, by finding upper and lower bounds for β *from the expression above,*

(ii) $W^{PU} < W^{PD1}$ *if* $\beta < 1$ *and*

(iii) $W^{PU} > W^{PD1}$ *if* $\beta > \frac{2\alpha_g}{1+\alpha_p-\alpha_g}$.

Hence comparing Corollaries 5 and 6, we note that welfare weights, firm location in the vertical stream and relative cost savings all play a role in the comparative analysis of welfare maximization. The example below highlights the result in Corollary 6 (iii). Even if the marginal cost with privatization upstream is less than with privatization downstream ($\beta > 1$), welfare with privatization downstream may dominate. The upstream/downstream welfare tradeoffs depend on the relative weights of the firms' profits. The contrasting results with relatively low priority on public profit versus its alternative (Corollaries 5 and 6), very precisely set the stage for policy discussions.

Example. From Corollary 6, it can be easily checked that $W^{PU} < W^{PD1}$ in the following cases.

(a) $\alpha_g = 0.70$, $\alpha_p = 0.25$, $\beta < 1.59$.

(b) $\alpha_g = 0.60$, $\alpha_p = 0.25$, $\beta < 1.26$.

(b) $\alpha_g = 0.50$, $\alpha_p = 0.20$, $\beta < 1.10$.

(b) $\alpha_g = 0.50$, $\alpha_p = 0.25$, $\beta < 1.07$.

(b) $\alpha_g = 0.50$, $\alpha_p = 0.30$, $\beta < 1.04$.

(b) $\alpha_g = 0.40$, $\alpha_p = 0.15$, $\beta < 1.004$.

From a policy perspective, this analysis conclusively shows that merely observing cost savings from privatization does not translate into good policy. It ignores societal priorities on the role of public and private profit and the strategic location of a firm (public or private) in the vertical stream. If public profit is relatively insignificant, then only relative cost savings matter. If that is not the case and society cares sufficiently about public profit, then cost savings may or may not matter depending on the location of the public and private firm in the bilateral monopoly. If privatization downstream is as (or more) cost efficient than privatization upstream, then privatization downstream will generally maximize welfare. But if that is not true, then the policy implication from mere observation of cost savings are ambiguous. The cost differentials in the vertical stream have to be significantly in favor of privatization upstream, to make that the welfare maximizing policy. Moreover, for any given cost differential, privatization upstream is more likely to maximize welfare if society places a relatively higher (lower) priority on private (public) profit.

5.2 Comparison of aggregate surplus

The aggregate surplus comparison provides a useful complement to the welfare comparison results. It underlines further, the role of private and public profit and relative cost savings from privatization upstream/downstream. These results are presented below. Recall the following expressions from Propositions 1, 2 and 3.

$$AS^{PU} = \frac{3}{8}(1 - c_d - \theta_u c_u)^2 \tag{5.5}$$

$$AS^{PD1} = \frac{\alpha_g(9\alpha_g - 2\alpha_p - 2)}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \tag{5.6}$$

$$AS^{PD2} = \frac{3}{8}(1 - c_u - \theta_d c_d)^2. \tag{5.7}$$

Corollary 7. *Suppose* $3\alpha_g - \alpha_p - 1 \geq 0$. *Then*

$$\beta \leq \frac{AS^{PU}}{AS^{PD1}} \leq \frac{12}{7}\beta.$$

Corollary 8. *Suppose* $3\alpha_g - \alpha_p - 1 \leq 0$. *Then* $\dfrac{AS^{PU}}{AS^{PD2}} = \beta$.

From Corollary 8 we see the following: If public profit is sufficiently insignificant in welfare, then relative cost savings drive aggregate surplus maximization. Given the relative priorities on public and private profit, neither of them matter any further and nor does the location of the private/public firm have any effect on whether aggregate surplus is maximized with privatization upstream or downstream. If it is relatively cheaper to privatize upstream (downstream) then privatization upstream (downstream) will maximize aggregate surplus. This is identical to Corollary 5 with welfare maximization.

However, if public profit is sufficiently important in welfare, then Corollary 7 shows that equal cost savings do not imply equal aggregate surplus. We find that if privatization upstream is relatively more cost effective, then aggregate surplus (in contrast to welfare) is also higher with upstream privatization. However, (and again in contrast to welfare) if privatization downstream is relatively more cost effective, aggregate surplus may be maximized with privatization upstream/downstream. It will be maximized with privatization downstream if the latter is significantly cheaper ($\beta < 7/12$) than upstream privatization. Otherwise, privatization upstream may generate higher aggregate surplus even though it is relatively cheaper to privatize downstream ($7/12 < \beta < 1$).

Aggregate surplus maximization results provide an interesting contrast to welfare maximization. In the latter case we saw that privatization downstream may maximize welfare even if it is relative more cost effective to privatize upstream. With aggregate surplus comparison, we have the reverse: privatization upstream may maximize aggregate surplus even though downstream privatization is more cost effective. This difference occurs because public profit and private profit are not weighted in aggregate surplus. Hence, the strategic advantage from locating upstream may go to the private firm if that maximizes surplus with or without a cost advantage.

Finally, we compare aggregate surplus for the benchmark private–private case with privatization upstream/downstream. Let

$$\beta_1 = \left(\frac{1-\theta_d c_d - \theta_u c_u}{1-c_d-\theta_u c_u}\right)^2 \quad \text{and} \quad \beta_2 = \left(\frac{1-\theta_d c_d - \theta_u c_u}{1-c_u-\theta_d c_d}\right)^2.$$

Observe that $\beta_1 > 1$ and $\beta_2 > 1$. We then have the following conclusions:

Corollary 9.

(i) $\dfrac{AS^{PP}}{AS^{PU}} = \dfrac{7}{12}\beta_1$.

(ii) $\dfrac{7}{12}\beta_2 \leq \dfrac{AS^{PP}}{AS^{PD1}} \leq \beta_2$.

(iii) $\dfrac{AS^{PP}}{AS^{PD2}} = \dfrac{7}{12}\beta_2$.

Not surprisingly, we find that even with private firms upstream and downstream, higher cost savings do not necessarily translate into greater aggregate surplus. If it is significantly more cost effective to privatize both sectors (and not just merely more cost effective), then AS^{PP} will dominate over AS with upstream/downstream privatization. Otherwise, the role of the welfare maximizing public firm's payoff in AS is significant and the private - private case may not yield a surplus greater than privatization upstream ((i)) or privatization downstream ((ii, iii)).

6 Conclusions

The paper analyzes the welfare implications of a mixed market in a bilateral monopoly, to determine whether specific policies directed at privatizing either upstream or downstream find theoretical support.

We conclude that if public profit is relatively insignificant in welfare, then only the relative cost savings from privatization upstream/downstream matter. That is, higher cost savings with privatization upstream (downstream) maximize welfare with privatization upstream (downstream). However, if society prioritizes public profit sufficiently highly, then the welfare comparisons are quite different. Equal cost savings no longer imply identical welfare. Privatization downstream will maximize welfare if it is as (or more) cost effective than privatization upstream. Moreover, welfare may be higher with privatization downstream even if there is a relative cost advantage with upstream privatization. This is because the upstream location gives the public firm a strategic first mover advantage. We also find that the cost differentials in the vertical stream have to be significantly in favor of privatization upstream to make that the welfare maximizing policy. For a given cost differential, privatization upstream is more likely to maximize welfare if society places a relatively higher (lower) priority on the role of private (public) profit. Hence, in general, relative cost savings and the relative importance of public and private profit in welfare should both play a role in public policy toward privatization.

Aggregate surplus maximization provides an interesting contrast to the welfare comparison analysis. In contrast to the above results, we find that privatization upstream may maximize aggregate surplus even when downstream privatization is more cost effective. Since public and private profit are not weighted separately in aggregate surplus, the strategic advantage from locating upstream may go to the private firm if that maximizes the surplus, with or without a cost advantage.

Certainly, while the specificity of the model serves to generate closed form solutions, it may raise the question of how broadly the conclusions are applicable. We believe that based on our findings, the interpretations can be applied in a broader context. The strategic advantage from upstream location of the public firm is clearly welfare maximizing when public profit is sufficiently high priority, and this may hold even if there are higher cost savings with upstream privatization. If that is not true and public profit is not sufficiently high priority in welfare, then cost savings drive relative welfare maximization. The contrast of this conclusion with the aggregate surplus comparison, further highlights the role of social priorities towards private and public profit and is relevant and broadly applicable from a policy perspective.

An extension of our work would be to consider government policy that allows entry at either the upstream or the downstream stage. The question then is, which should it do? Our results suggest that depending on social priorities, private competition downstream may be the welfare maximizing policy, at least under a wide range of relative cost savings with privatization upstream/downstream. The framework can be extended to study the effect of entry by foreign firms on mixed markets in this vertical framework. Depending on societal priorities, whether it will be welfare maximizing to allow foreign entry upstream or downstream remains a topic for future research.

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7 Appendix

Proof of Proposition 1.

Recall that

$$\begin{aligned}\pi_g &= (p_d - p_u - c_d)(1 - p_d)\\ \pi_p &= (p_u - \theta_u c_u)(1 - p_d)\\ W &= \alpha_p \pi_p + \alpha_g \pi_g + \alpha_c \frac{(1 - p_d)^2}{2}.\end{aligned}$$

The optimization problem is to first find $p_d = p_d(p_u)$ that maximizes W for fixed p_u. Then we need to find p_u that maximizes π_p.

Fix p_u and consider all possible values of p_d. Note that $\pi_g \geq 0$ implies either $p_d = 1$ (which implies $\pi_g = \pi_p = W = 0$ and hence is ruled out) or $p_d \geq p_u + c_d$. So the range of possible values of p_d (given p_u) is

$$p_u + c_d \leq p_d \leq 1. \tag{7.1}$$

Now,

$$\begin{aligned}\frac{\partial W}{\partial p_d} &= \alpha_g(1-p_d) - [\alpha_p(p_u-\theta_u c_u) + \alpha_g(p_d - p_u - c_d)] - \alpha_c(1-p_d) \\ &= (1-p_d)(\alpha_g - \alpha_c) - [\alpha_p(p_u-\theta_u c_u) + \alpha_g(p_d-p_u-c_d)]\,. \end{aligned} \tag{7.2}$$

Let $p_d = p_d(p_u)$ be the solution of $\dfrac{\partial W}{\partial p_d} = 0$. It may be noted that in general, this may or may not be a valid value of p_d that satisfies (7.1).

Further,

$$\frac{\partial^2 W}{\partial p_d^2} = -(\alpha_g - \alpha_c) - \alpha_g = 1 - 3\alpha_g - \alpha_p. \tag{7.3}$$

We break the proof into two subcases, Case (a) and Case (b).

Case (a): $3\alpha_g + \alpha_p - 1 \leq 0$. Hence by (7.3), W is convex in p_d. So for fixed p_u the maximum of W occurs at one of the end points given by (7.1). At one end point $p_d = 1$, we get $\pi_g = \pi_p = CS = W = 0$, $p_u =$ any value. At the other end point $p_d = p_u + c_d$. Then W equals

$$W = \alpha_p(1-p_u-c_d)(p_u-\theta_u c_u) + 0 + \frac{1}{2}\alpha_c(1-p_u-c_d)^2. \tag{7.4}$$

Now we turn to maximizing π_p. Since $p_d = p_u + c_d$,

$$\pi_p = (p_u - \theta_u c_u)(1-p_u-c_d). \tag{7.5}$$

This implies

$$\frac{\partial \pi_p}{\partial p_u} = (1-p_u-c_d) - (p_u - \theta_u c_u), \tag{7.6}$$

$$\frac{\partial^2 \pi_p}{\partial p_u^2} = -2 < 0. \tag{7.7}$$

Thus by concavity, the maximum value of π_p is obtained by solving (7.6) which gives

$$p_u = \frac{1 - c_d + \theta_u c_u}{2}. \tag{7.8}$$

Note that this p_u satisfies (7.1) provided $c_d + \theta_u c_u \leq 1$. Now using $p_d = p_u + c_d$ and (7.8), all the expressions claimed in (ii) may be easily obtained. In particular, the maximized value of π_p is given by

$$\begin{aligned}\pi_p &= (p_u-\theta_u c_u)(1-p_u-c_d) \\ &= \left(\frac{1-c_d+\theta_u c_u}{2} - \theta_u c_u\right)\left(1 - \frac{1-c_d+\theta_u c_u}{2} - c_d\right) \\ &= \left(\frac{1-c_d-\theta_u c_u}{2}\right)^2. \end{aligned} \tag{7.9}$$

So Case (a) is complete.

Case (b): $3\alpha_g + \alpha_p - 1 > 0$. Then from (7.3), W is *concave* in p_d and further,

$$W \text{ is increasing for } p_d \leq p_d(p_u) \text{ and decreasing for } p_d \geq p_d(p_u). \tag{7.10}$$

So the following **three** subcases are possible. We shall find the optimum in each case and the best among these would provide the global optimum.

Case (b)(i): $p_d(p_u) \geq 1$. Then from (7.10), W is increasing in p_d for all $0 \leq p_d \leq 1$ and hence the maximum of W is reached at $p_d = 1$. That in turn implies $\pi_p = \pi_g = CS = W = 0$.

Case (b)(ii): $p_d(p_u) \leq p_u + c_d \leq p_d \leq 1$. Then from (7.10), W is decreasing in p_d and hence the maximum of W is reached at $p_d = p_u + c_d$. This yields the same solution as that obtained in Case (a).

Case (b)(iii). Now consider the remaining possible p_u values $\{p_u : p_u + c_d \leq p_d(p_u) \leq 1\}$. Note that now $p_d(p_u)$ is a valid p_d and recall that W is maximized at $p_d(p_u)$.

Hence $\pi_p = (p_u - \theta_u c_u)(1 - p_d(p_u))$. Without loss we restrict attention to $p_u \geq c_u$. Since $p_d(p_u) \geq p_u + c_d$,

$$\pi_p \leq (p_u - \theta_u c_u)(1 - p_u - c_d) = \hat{\pi}_p(p_u) \text{ say.} \tag{7.11}$$

On the other hand, from (7.5), we know that $\hat{\pi}_p$ is also maximized at p_u given in (7.8) and the maximum value is that given in (7.9). Thus by (7.11), the maximized value of π_p in this case is smaller than or equal to that in (7.9). Combining all this, we have shown that in Case (b), the optimum values are those obtained in Case b(ii) and are the same as obtained in Case (a). ▲

Proof of Proposition 2. Recall that $\pi_p = (p_d - p_u - \theta_d c_d)(1 - p_d)$. To solve for the downstream p_d first, fix p_u. Then

$$\frac{\partial \pi_p}{\partial p_d} = 1 - p_d - [p_d - p_u - \theta_d c_d].$$

Thus

$$\frac{\partial \pi_p}{\partial p_d} > 0 \Leftrightarrow p_d < \frac{1 + \theta_d c_d + p_u}{2}. \tag{7.12}$$

(1) Consider the *restricted* optimum on the set $S = \{p_u : p_u \geq 1 - \theta_d c_d\}$. This implies $\frac{1+\theta_d c_d + p_u}{2} \geq 1$. Then from (7.12), $\frac{\partial \pi_p}{\partial p_d} \geq 0$ for all values of p_d. Consequently, the optimum values are: $p_d = 1$ and hence $\pi_p = 0, \pi_g = 0$, $W = 0$, $p_u = 1 - \theta_d c_d$.

(2) Note that since $\pi_g \geq 0$, we must have either $p_u \geq c_u$ or $p_d = 1$. The latter case gives us back the answers in (1). Hence it remains to consider $T = \{p_u : c_u \leq p_u \leq 1 - \theta_d c_d\}$. Fix any p_u in T. Note that $\frac{\partial^2 \pi_p}{\partial p_d^2} = -2 < 0$ and so π_p is concave. Hence the optimum value of p_d is given by (see (7.12)),

$$p_d = \frac{1 + \theta_d c_d + p_u}{2} \quad \text{(clearly } 0 \leq p_d \leq 1 \ \forall p_u \in T). \tag{7.13}$$

The corresponding value of π_p is then given by

$$\pi_p(p_u) = \left[\frac{1+\theta_d c_d + p_u}{2} - p_u - \theta_d c_d\right]\left[1 - \frac{1+\theta_d c_d + p_u}{2}\right] = \frac{(1-p_u-\theta_d c_d)^2}{4}. \qquad (7.14)$$

Then with $p_d = p_d(p_u)$ satisfying (7.13),

$$W = W(p_u, p_d(p_u)) = (1-p_d)\alpha_g(p_u - c_u) + \alpha_p(1-p_d)^2 + \alpha_c\frac{(1-p_d)^2}{2}. \qquad (7.15)$$

We now maximize W with respect to p_u. Using (7.15) and the fact that $\frac{\partial p_d}{\partial p_u} = \frac{1}{2}$,

$$\begin{aligned}
\frac{\partial W}{\partial p_u} &= \alpha_g(1-p_d) + \alpha_g(p_u - c_u)(-\frac{1}{2}) + (\alpha_p + \frac{\alpha_c}{2})2(1-p_d)(-\frac{1}{2}) \\
&= (1-p_d)\left[\alpha_g - \alpha_p - \frac{\alpha_c}{2}\right] - \frac{\alpha_g}{2}(p_u - c_u) \\
\frac{\partial^2 W}{\partial p_u^2} &= \frac{1}{4}\left[-5\alpha_g + \alpha_p + 1\right] \quad (\text{using } \alpha_g + \alpha_p + \alpha_c = 1). \qquad (7.16)
\end{aligned}$$

Note that W is concave if $5\alpha_g - \alpha_p - 1 > 0$ and is convex otherwise.

Use the notation p_u^0 to denote the solution in (3.8). Note that an alternative expression for it equals

$$p_u^0 = \frac{(1-\theta_d c_d)(3\alpha_g - \alpha_p - 1) + 2\alpha_g c_u}{5\alpha_g - \alpha_p - 1}.$$

Claim. (i) If $5\alpha_g - \alpha_p - 1 > 0$, then $\frac{\partial W}{\partial p_u} > 0 \Leftrightarrow p_u < p_u^0$.

(ii) If $5\alpha_g - \alpha_p - 1 < 0$ $\frac{\partial W}{\partial p_u} > 0 \Leftrightarrow p_u > p_u^0$.

(iii) If $5\alpha_g - \alpha_p - 1 = 0$, then $\frac{\partial W}{\partial p_u} > 0\ \forall p_u \Leftrightarrow (1-\theta_d c_d)(3\alpha_g - \alpha_p - 1) + 2\alpha_g c_u > 0$.

(iv) If $5\alpha_g - \alpha_p - 1 = 0$, then $\frac{\partial W}{\partial p_u} \leq 0\ \forall p_u \Leftrightarrow (1-\theta_d c_d)(3\alpha_g - \alpha_p - 1) + 2\alpha_g c_u \leq 0$.

Proof of Claim. First assume $5\alpha_g - \alpha_p - 1 > 0$. Then

$$\begin{aligned}
& \frac{\partial W}{\partial p_u} > 0 \\
\Leftrightarrow \quad & (1-p_d)(\alpha_g - \alpha_p - \frac{\alpha_c}{2}) > \frac{\alpha_g}{2}(p_u - c_u) \\
\Leftrightarrow \quad & \frac{1-p_u-\theta_d c_d}{2}\left[3\alpha_g - \alpha_p - 1\right] > \alpha_g(p_u - c_u) \quad \text{using (7.13)} \\
\Leftrightarrow \quad & p_u(5\alpha_g - \alpha_p - 1) < (1-\theta_d c_d)(3\alpha_g - \alpha_p - 1) + 2\alpha_g c_u \qquad (7.17) \\
\Leftrightarrow \quad & p_u < \frac{(1-\theta_d c_d)(3\alpha_g - \alpha_p - 1) + 2\alpha_g c_u}{5\alpha_g - \alpha_p - 1} = p_u^0.
\end{aligned}$$

This proves (i). Proof of (ii) is similar and (iii) and (iv) follow from (7.17).

Lemma.

A. $3\alpha_g - \alpha_p - 1 < 0$ and $5\alpha_g - \alpha_p - 1 < 0$ implies $p_u^0 > (1 - \theta_d c_d)$.

B. $3\alpha_g - \alpha_p - 1 < 0$ and $5\alpha_g - \alpha_p - 1 > 0$ implies $p_u^0 < c_u$.

The above lemma can be easily proved and we skip the details. We will now use it to complete the proof of the proposition.

Proof of Proposition 2 (ii): $3\alpha_g - \alpha_p - 1 < 0$. From the Claim above, we know that

$5\alpha_g - \alpha_p - 1 > 0$ implies W is concave and $\frac{\partial W}{\partial p_u} \geq 0 \Leftrightarrow p_u < p_u^0$.

$5\alpha_g - \alpha_p - 1 < 0$ implies W is concave and $\frac{\partial W}{\partial p_u} \geq 0 \Leftrightarrow p_u > p_u^0$

So we have the two possibilities.

In Case A, the valid range of p_u is $\{p_u : c_u \leq p_u \leq 1 - \theta_d < p_u^0\}$ and on this range W is a decreasing function. Hence W is maximized at $p_u = c_u$.

In case B, the valid range of p_u is $\{p_u : p_u^0 < c_u \leq p_u \leq 1 - \theta_d c_d\}$ and on this range W is again decreasing and hence again $p_u = c_u$ maximizes W. Thus in either case the optimum values are given by

$$\begin{aligned} p_u^{(2)} &= c_u \\ p_d^{(2)} &= \frac{1 + \theta_d c_d + c_u}{2} \end{aligned}$$

As a consequence,

$$\begin{aligned} \pi_g^{(2)} &= 0 \\ \pi_p^{(2)} &= (p_d - p_u - \theta_d c_d)(1 - p_d) \\ &= \left(\frac{1 + \theta_d c_d + c_u}{2} - c_u - \theta_d c_d\right)\left(1 - \frac{1 + \theta_d c_d + c_u}{2}\right) \\ &= \frac{1}{4}(1 - \theta_d c_d - c_u)^2 \end{aligned}$$

$$\begin{aligned} CS^{PD_2} &= \frac{(1 - pd)^2}{2} \\ &= \frac{1}{8}(1 - \theta_d c_d - c_u)^2 \\ W^{PD_2} &= \frac{\alpha_p}{4}(1 - \theta_d c_d - c_u)^2 + \frac{\alpha_c}{8}(1 - \theta_d c_d - c_u)^2 \\ &= \frac{1}{8}(1 - \theta_d c_d - c_u)^2[2\alpha_p + 1 - \alpha_p - \alpha_g] \\ &= \frac{1}{8}(1 + \alpha_p - \alpha_g)(1 - \theta_d c_d - c_u)^2 \\ AS^{PD_2} &= \frac{1}{4}(1 - \theta_d c_d - c_u)^2 + \frac{1}{8}(1 - \theta_d c_d - c_u)^2 \\ &= \frac{3}{8}(1 - \theta_d c_d - c_u)^2. \end{aligned}$$

This proves part (ii) of the proposition.

Proof of Proposition 2 (i). Now suppose $3\alpha_g - \alpha_p - 1 > 0$. Then automatically $5\alpha_g - \alpha_p - 1 > 0$. From (7.16), W is concave in p_u. Hence the maximizer with respect to p_u is given by:

$$\begin{aligned} (1-p_d)\left[\alpha_g - \frac{(2\alpha_p+\alpha_c)}{2}\right] &= \frac{\alpha_g(p_u-c_u)}{2} \\ \Rightarrow \qquad & (1-p_d)\,[3\alpha_g-\alpha_p-1] = \alpha_g(p_u-c_u). \end{aligned} \tag{7.18}$$

Using this and (7.15), and using $\alpha_g+\alpha_p+\alpha_c = 1$, the maximized W is given by

$$W = \frac{(1-p_d)^2}{2}\,[5\alpha_g-\alpha_p-1]\,. \tag{7.19}$$

Using the value of p_u from (7.13) in (7.18),

$$1-p_d^{(1)} = \frac{\alpha_g}{5\alpha_g-\alpha_p-1}(1-c_u-\theta_d c_d). \tag{7.20}$$

Thus $p_d^{(1)}$ is as in (3.9). Using this and (7.19), the expression (3.13) follows easily.

The value of $p_u^{(1)}$ is then easily obtained from (7.18) as

$$\begin{aligned} (1-p_d^{(1)})\,[3\alpha_g-\alpha_p-1] &= \alpha_g(p_u^{(1)}-c_u) \\ \Leftrightarrow \qquad & p_u^{(1)} = p_u^0. \end{aligned} \tag{7.21}$$

Now we establish the other expressions:

$$\begin{aligned} \pi_p^{(1)} &= (p_d^{(1)}-p_u^{(1)}-\theta_d c_d)(1-p_d^{(1)}) \\ &= \left(1-p_d^{(1)}\right)\left(1-\frac{\alpha_g(1-c_u-\theta_d c_d)}{5\alpha_g-\alpha_p-1} - c_u - \frac{3\alpha_g-\alpha_p-1}{5\alpha_g-\alpha_p-1}(1-c_u-\theta_d c_d) - \theta_d c_d\right) \\ &= \frac{\alpha_g^2}{(5\alpha_g-\alpha_p-1)^2}(1-c_u-\theta_d c_d)^2 \end{aligned}$$

$$\begin{aligned}
\pi_g^{(1)} &= (p_u^{(1)} - c_u)\left(1 - p_d^{(1)}\right) \\
&= \frac{\alpha_g}{5\alpha_g - \alpha_p - 1}\left(\frac{3\alpha_g - \alpha_p - 1}{5\alpha_g - \alpha_p - 1}\right)(1 - c_u - \theta_d c_d)^2 \\
CS^{PD_1} &= \frac{\left(1 - p_d^{(1)}\right)^2}{2} \\
&= \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \\
W^{PD_1} &= \alpha_p \pi_p^{(1)} + \alpha_g \pi_g^{(2)} + \alpha_c cS^{(1)} \\
&= \frac{\alpha_g}{(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \left\{\alpha_p\alpha_g + (3\alpha_g - \alpha_p - 1)\alpha_g + \frac{(1 - \alpha_p - \alpha_g)\alpha_g}{2}\right\} \\
&= \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \left[2\alpha_p + 6\alpha_g - 2\alpha_p - 2 + 1 - \alpha_p - \alpha_g\right] \\
&= \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}(1 - c_u - \theta_d c_d)^2 \\
AS^{PD_1} &= \left[\frac{\alpha_g^2}{(5\alpha_g - \alpha_p - 1)^2} + \frac{\alpha_g(3\alpha_g - \alpha_p - 1)}{(5\alpha_g - \alpha_p - 1)^2} + \frac{\alpha_g^2}{2(5\alpha_g - \alpha_p - 1)^2}\right](1 - c_u - \theta_d c_d)^2 \\
&= \frac{\alpha_g}{2(5\alpha_g - \alpha_p - 1)^2}\left[2\alpha_g + 6\alpha_g - 2\alpha_p - 2 + \alpha_g\right](1 - c_u - \theta_d c_d)^2 \\
&= \frac{\alpha_g}{2(5\alpha_g - \alpha_p - 1)^2}\left[9\alpha_g - 2\alpha_p - 2\right](1 - c_u - \theta_d c_d)^2.
\end{aligned}$$

▲

Proof of Proposition 3. This is a standard argument in a model with profit maximizing firms. We give it here in brief. By earlier calculations for Proposition 2, on the set $S = \{p_u : p_u \geq 1 - \theta_d c_d\}$, the optimum values are $p_d = 1$, $\pi_d = 0$, $\pi_u = 0$.

On the other hand, on the set $T = \{p_u : p_u \leq 1 - \theta_d c_d\}$, by earlier calculations, the optimum equals $p_d = \frac{1+\theta_d c_d + p_u}{2}$ and hence

$$\pi_d(p_u) = \left[\frac{1 + \theta_d c_d + p_u}{2} - p_u - \theta_d c_d\right]\left[1 - \frac{1 + \theta_d c_d + p_u}{2}\right] = \frac{(1 - p_u - \theta_d c_d)^2}{4}. \tag{7.22}$$

For the upstream profit $\pi_u = (p_u - \theta_u c_u)(1 - p_d)$, substituting the value of p_d,

$$\pi_u = (p_u - \theta_u c_u)\left[1 - \frac{1 + \theta_d c_d + p_u}{2}\right] = \frac{1}{2}\left(p_u - \theta_u c_u\right)\left(1 - p_u - \theta_d c_d\right). \tag{7.23}$$

Hence $\frac{\partial \pi_u}{\partial p_u} = \frac{1}{2}\left[1 - p_u - \theta_d c_d - (p_u - \theta_u c_u)\right] = \frac{1}{2}\left[1 - 2p_u - \theta_d c_d + \theta_u c_u\right]$ and as consequence, $\frac{\partial \pi_u}{\partial p_u} > 0$ if and only if $p_u < \frac{1-\theta_d c_d + \theta_u c_u}{2}$. Further $\dfrac{\partial^2 \pi_u}{\partial p_u^2} = -1 < 0$ so that π_u is concave.

Then the optima are:

$$
\begin{aligned}
p_u &= \frac{1-\theta_d c_d+\theta_u c_u}{2} \\
\pi_u(p_u) &= \frac{1}{2}\left[\frac{1-\theta_d c_d+\theta_u c_u}{2}-\theta_u c_u\right]\left[1-\frac{1-\theta_d c_d+\theta_u c_u}{2}-\theta_d c_d\right] \\
&= \frac{1}{8}\left[1-\theta_d c_d-\theta_u c_u\right]^2.
\end{aligned}
$$

The rest of the expressions follow easily from the above. ▲

Proof of Corollary 1. This is immediate from the expression for W^{PU}. ▲

Proof of Corollaries 2 and 3.

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_g}\left[\log\frac{3\alpha_g-\alpha_p-1}{5\alpha_g-\alpha_p-1}\right] &= \frac{3}{3\alpha_g-\alpha_p-1}-\frac{5}{5\alpha_g-\alpha_p-1} \\
&= 2(1+\alpha_p)>0. \\
\frac{\partial}{\partial \alpha_p}\left[\log\frac{3\alpha_g-\alpha_p-1}{5\alpha_g-\alpha_p-1}\right] &= \frac{1}{3\alpha_g-\alpha_p-1}+\frac{1}{5\alpha_g\alpha_p-1}<0.
\end{aligned}
$$

This proves (i) of Corollary 2.

$$
\frac{\partial}{\partial \alpha_g}\left[\log\frac{\alpha_g}{5\alpha_g-\alpha_p-1}\right]=\frac{1}{\alpha_g}-\frac{5}{5\alpha_g-\alpha_p-1}<0.
$$

This proves first part of Corollary 2 (ii). Second part is obvious. The same arguments prove Corollary 3 (i) and(ii). ▲

Proof of Corollary 4. It is enough to study the behavior of $f(\alpha_g)=2\log\alpha_g-\log(5\alpha_g-\alpha_p-1)$. Observe that

$$
\frac{\partial}{\partial \alpha_g}f(\alpha_g)=\frac{2}{\alpha_g}-\frac{5}{5\alpha_g-\alpha_p-1}=\frac{5\alpha_g-2\alpha_p-2}{\alpha_g(5\alpha_g-\alpha_p-1)}.
$$

▲

Proof of Corollary 6. Observe that

$$
\begin{aligned}
\frac{W^{PU}}{W^{PD_1}} &= \beta\frac{(1+\alpha_p-\alpha_g)}{8}\frac{2(5\alpha_g-\alpha_p-1)}{\alpha_g^2} \\
&= \beta\frac{(1+\alpha_p-\alpha_g)}{4\alpha_g^2}(5\alpha_g-\alpha_p-1). \qquad (7.24)
\end{aligned}
$$

Define

$$
f(\alpha_g)=\log\left[\frac{(1+\alpha_p-\alpha_g)(5\alpha_g-\alpha_p-1)}{4\alpha_g^2}\right].
$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial \alpha_g} f(\alpha_g) &= -\frac{1}{1+\alpha_p-\alpha_g} + \frac{5}{5\alpha_g-\alpha_p-1} - \frac{2}{\alpha_g} \\
&= -\frac{1}{1+\alpha_p-\alpha_g} - \frac{-5\alpha_g+10\alpha_g-2\alpha_p-2}{\alpha_g(5\alpha_g-\alpha_p-1)} \\
&= -\frac{1}{1+\alpha_p-\alpha_g} - \frac{5\alpha_g-2\alpha_p-2}{\alpha_g(5\alpha_g-\alpha_p-1)} = -\frac{N}{D}(\text{say})
\end{aligned}
$$

where

$$
\begin{aligned}
N &= \alpha_g(5\alpha_g-\alpha_p-1)+(5\alpha_g-2\alpha_p-2)(1+\alpha_p-\alpha_g) \\
&= 5\alpha_g^2-\alpha_p\alpha_g-\alpha_g+5\alpha_g-2\alpha_p-2+5\alpha_p\alpha_g-2\alpha_p^2-2\alpha_p-5\alpha_g^2+2\alpha_p\alpha_g-2\alpha_g \\
&= 6\alpha_g+6\alpha_p\alpha_g-4\alpha_p-2\alpha_p^2-2 \\
&= 2(3\alpha_g-\alpha_p-1)+2\alpha_p(3\alpha_g-\alpha_p-1)>0.
\end{aligned}
$$

So f is a decreasing function of α_g and minimum value of f is at $\alpha_g = \left(\frac{1+\alpha_p}{3}\right)$ and

$$
f\left(\frac{1+\alpha_p}{3}\right) = \log\left[\frac{\left(1+\alpha_p-\left(\frac{1+\alpha_p}{3}\right)\right)\left(\frac{5}{3}(1+\alpha_p)-(1+\alpha_p)\right)}{4\frac{(1+\alpha_p)^2}{9}}\right] = \log(1)=0. \tag{7.25}
$$

Hence using (7.24) and (7.25),

$$
\frac{W^{PU}}{W^{PD_1}} \leq \beta.
$$

This proves the right side inequality of (i). To prove the left side,

$$
\begin{aligned}
\frac{W^{PU}}{W^{PD_1}} &\geq \beta\frac{(1+\alpha_p-\alpha_g)4\alpha_g}{8\alpha_g^2} \quad (\text{since} \quad 3\alpha_g-\alpha_p-1\geq 0) \\
&= \beta\frac{(1+\alpha_p-\alpha_g)}{2\alpha_g}.
\end{aligned}
$$

Parts (ii) and (iii) now follow easily. ▲

Proof of Corollary 7. Note that

$$
\frac{AS^{PD1}}{AS^{PU}} = \frac{4}{3\beta}\frac{\alpha_g(9\alpha_g-2\alpha_p-2)}{(5\alpha_g-\alpha_p-1)^2} = \frac{4}{3\beta}f(\alpha_g) \quad (\text{say}). \tag{7.26}
$$

We first find a lower bound of f.

$$
\frac{\partial \log f(\alpha_g)}{\partial \alpha_g} = \frac{1}{\alpha_g} + \frac{9}{9\alpha_g-2\alpha_p-2} - \frac{2(5)}{5\alpha_g-\alpha_p-1} = \frac{N}{D} \quad (\text{say}), \text{ where } D>0.
$$

Now

$$
\begin{aligned}
N &= (5\alpha_g-\alpha_p-1)(9\alpha_g-\alpha_p-1)-5\alpha_g(9\alpha_g-\alpha_p-2) \\
&= (1+\alpha_p)(-4\alpha_g+1+\alpha_p)<0, \quad \text{since } 3\alpha_g>1+\alpha_p.
\end{aligned}
$$

So for fixed α_p, f is a *decreasing* function of α_g. Now $f(1) = \frac{9-2\alpha_p-2}{(5-\alpha_p-1)^2} = \frac{7-2\alpha_p}{(4-\alpha_p)^2} = t(\alpha_p)$ (say). Then $\frac{\partial \log t(\alpha_p)}{\partial \alpha_p} = -\frac{2}{7-2\alpha_p} + \frac{2}{4-\alpha_p} = \frac{2(3-\alpha_p)}{(7-2\alpha_p)(4-\alpha_p)} > 0$. Hence t is *increasing* in α_p. Since $t(0) = \frac{7}{16}$, we conclude that for all α_p and α_g,

$$f(\alpha_g) > f(1) > t(0) = \frac{7}{16}. \tag{7.27}$$

This implies $\frac{AS^{PD1}}{AS^{PU}} \geq \frac{4}{3\beta}\left(\frac{7}{16}\right) = \frac{7}{12\beta}$ proving the right side of the inequality.

We now need to find an appropriate upper bound for f.

$$\begin{aligned} f(\alpha_g) &= \frac{\alpha_g(9\alpha_g - 2\alpha_p - 2)}{(5\alpha_g - \alpha_p - 1)^2} \\ &= 2\frac{\alpha_g}{5\alpha_g - \alpha_p - 1} - \frac{\alpha_g^2}{(5\alpha_g - \alpha_p - 1)^2} = 2x - x^2 = s(x) \quad \text{(say)}. \end{aligned}$$

It is easily seen that s is an increasing function for $x \leq 1$. Now since $3\alpha_g > 1 + \alpha_p$, $\frac{1}{x} = 5 - \frac{1+\alpha_p}{\alpha_g} > 5 - \frac{3\alpha_g}{\alpha_g} = 2$. Hence $x \leq \frac{1}{2}$ and as a consequence

$$f(\alpha_g) \leq s(1/2) = 2\frac{1}{2} - \left[\frac{1}{2}\right]^2 = \frac{3}{4}. \tag{7.28}$$

Hence $\frac{AS^{PD1}}{AS^{PU}} \leq \frac{4}{3\beta}\left(\frac{3}{4}\right) = \frac{1}{\beta}$. This proves the left side of the inequality and the proof of the Corollary is complete. ▲

Proof of Corollary 9. Using Propositions 1, 2 and 3, $\frac{AS^{PP}}{AS^{PU}} = \frac{7}{12}\beta_1$ and $\frac{AS^{PP}}{AS^{PD2}} = \frac{7}{12}\beta_2$. These proves (i) and (iii).

Further $\frac{AS^{PD1}}{AS^{PP}} = \frac{16}{7\beta_2}f(\alpha_g)$ where f is as in (7.26). We also know from (7.27) and (7.28) that $\frac{7}{16} < f(\alpha_g) < \frac{3}{4}$ and (ii) follow easily from this. ▲