Noncrossing partitions, Catalan words and the semicircle law

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Abstract

As is well known, the joint limit distribution of independent Wigner matrices is free with the marginals being semicircular. This freeness is very special to Wigner matrices and is intimately tied to noncrossing pair partitions or, what are known as Catalan words, each of which contributes one to the limit moments.

We investigate the following questions. Consider a sequence of patterned matrices: (i) when do *only* Catalan words contribute (one), so that we get the semicircle limit? (ii) when does each Catalan word contribute one (with possible nonzero contribution from non-Catalan words)? (iii) for what matrix models do Catalan words not necessarily contribute one each and non-semicircle limits arise, even when non-Catalan words have zero contribution?

In particular we show that in a general sense, the semicircle law serves as a lower bound for possible limits. Further, there is a large class of non-Wigner matrices whose limit is the free semicircle. This may be viewed as robustness of the semicircle law. Similarly, there is a large class of block matrices whose limit is not semicircular.

Keywords: Asymptotic freeness, Catalan words, divisor function, eigenvalues, empirical spectral distribution (ESD), limiting spectral distribution, moment method, noncrossing partitions, semicircular law, Wigner matrix.

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1 Introduction

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of a real symmetric matrix $A_{n\times n}$ then its *Empirical Spectral Distribution (ESD)* F^{A_n} is as defined below. The *Limiting Spectral Distribution (LSD)* of $\{A_n\}$ is the weak limit of $\{F^{A_n}\}_{n=1}^{\infty}$, if it exists, either almost surely or in probability.

$$F^{A_n}(x) = n^{-1}\sum_{i=1}^{n} \mathbb{I}(\lambda_i \leq x). \tag{1.1}$$

Wigner [8] showed that the semicircle law arises as LSD of Wigner matrices. See for example [2] and [1] for such results and their variations. It is also well known that the trace of finite product of independent Wigner

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matrices converge. This is tied to the idea of free independence developed by Voiculescu (see Voiculescu [9]). This freeness in the limit is very special to the Wigner matrices. In particular it is known that each limit moment is the total number of noncrossing pair partitions (see for example [1]). Equivalently, it is also the total number of so called *Catalan words* (see [3]).

The Wigner matrix is a special example of a (symmetric) patterned matrix. LSD of other pattern matrices have also been studied. For examples, see [2], [4] and [5]. [3] studies the joint convergence of symmetric patterned matrices. In particular, for independent copies of the Toeplitz, Hankel, symmetric circulant and reverse circulant matrices, the tracial limits exist for any monomial formed with these independent copies.

Perhaps surprisingly, for the above matrices, each Catalan word contributes one to the limit. We are thus led to the following questions for general pattern matrices: (i) when do *only* Catalan words contribute (one), so that we get the semicircle limit? (ii) when does each Catalan word contribute one (with possible nonzero contribution from non-Catalan words)? (iii) for what matrix models do Catalan words not necessarily contribute one each and non-semicircle limits arise, even when non-Catalan words have zero contribution? The goal of this paper is to investigate these issues.

Let d be a positive integer. Let $\mathbb{Z}$ be the set of all integers and let $\mathbb{Z}_+^d$ denote the d-dimensional lattice of nonnegative integers. Let $L_n : \{1, 2, \ldots n\}^2 \to \mathbb{Z}_+^d,\ n \geq 1$ be a sequence of functions such that $L_{n+1}(i,j) = L_n(i,j)$ whenever $1 \leq i,j \leq n$. We shall write $L_n = L$ and call it the **link** function. By abuse of notation we write $\mathbb{Z}^2$ as the common domain of $\{L_n\}$. In some of the cases we consider, L_n depends on n, but this still falls under the general framework as we deal with combinatorics for fixed n. For our purposes, d is either 1 or 2. Let $\{x(i); i \geq 0\}$ or $\{x(i\ j); i,j \geq 1\}$ be a sequence of random variables (the *input sequence*). Then $A_n = ((x(L(i,j)))$ is a sequence of patterned matrices. We consider only real symmetric matrices, so that L is symmetric in its arguments. The input sequence is taken to be independent with further assumptions on their moments as required later on. For the moment assume that they are uniformly bounded. Some well known matrices and their link functions are:

(i) Wigner matrix W_n with $L : \mathbb{Z}_+^2 \to \mathbb{Z}_\geq^2$ where $L(i,j) = (\min(i,j), \max(i,j))$.

(ii) Symmetric Toeplitz matrix T_n with $L : \mathbb{Z}_+^2 \to \mathbb{Z}_\geq$ where $L(i,j) = |i-j|$.

(iii) Symmetric Hankel matrix H_n with $L : \mathbb{Z}_+^2 \to \mathbb{Z}_\geq$ where $L(i,j) = i+j$.

(iv) Reverse circulant matrix RC_n with $L : \mathbb{Z}_+^2 \to \mathbb{Z}_\geq$ where $L(i,j) = (i+j) \mod n$.

(v) Symmetric circulant matrix SC_n with $L : \mathbb{Z}_+^2 \to \mathbb{Z}_\geq$ where $L(i,j) = n/2 - |n/2 - |i-j||$.

Without any further assumptions, the limiting spectral distribution need not exist. We introduce the following properties of the link function:

Property B: $\Delta(L) = \sup_n \sup_{t \in \mathbb{Z}_+^d} \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n,\ L(k,l) = t\} < \infty$.

In particular, $\Delta(L) = 2$ for T_n, SC_n, and $\Delta(L) = 1$ for W_n, H_n and RC_n. An L which does not satisfy Property B is $L(i,j) = \min(i,j)$. [4] has shown that under this property, the sequence of moments of the empirical spectral distribution of $\{n^{-1/2} A_n\}$ is tight.

Define

$$\alpha_n = \max_k \#\{(i,j) : L_n(i,j) = k,\ 1 \leq i,\ j \leq n\}.$$

Note that for Wigner matrices, $\alpha_n = 2\ (= o(n))$. For RC_n, SC_n, T_n and H_n, α_n is of the order n.

Let $M_{ij}^{(n)} = \{1 \leq k \leq n : L(k,i) = L(k,j)\}$ be the **"match set"** between rows i and j at stage n.

Property P: $M^* = \sup_n \sup_{i,j \le n} |M_{ij}^{(n)}| < \infty$.

We may now broadly summarize our results for the spectral distribution of $\{n^{-1/2}A_n\}$:

(A) If $\alpha_n = o(n)$ and Property B hold, then non-Catalan words do not contribute to the limit and the essential support of any subsequential limit $\subseteq [-2\sqrt{\Delta}, 2\sqrt{\Delta}]$. Further, if $\Delta = 1$, then each Catalan word contributes 1 and the LSD is the semicircular law. The Wigner matrix is a special case. See Theorem 1.

(B) Symmetry and Property B ensures a contribution of at least one from each Catalan word, and thus, any subsequential limit has its moments greater than or equal to those of the semicircular law. So, the semicircular law is a "lower bound". See part (i) of Theorem 2.

(C) If Properties B and P hold, then each Catalan word contributes one to the limit. In particular, this is true for Toeplitz, Hankel, Symmetric circulant and the Reverse circulant matrices. See part (ii) of Theorem 2.

(D) If $\alpha_n = o(n)$ and Properties B and P hold then the LSD is the semicircular law. These may be grouped together as **Wigner type** matrices. See part (iii) of Theorem 2. We provide some interesting examples in this category.

(E) Non-semicircular limits arise if matches between rows are increased, say by **blocking**. In Subsection 3.1, we prove a general existence theorem for the LSD of block matrices, extending Oraby's result ([7]), and showing an explicit dependence of the LSD on the block structure. Using this result, we establish his result that for *symmetric circulant* block structure, the LSD is a mixture of two scaled semicircular laws. This proof continues to hold for other block structures having some common characteristics with symmetric circulant. The joint convergence of the Wigner type matrices to the free semicircle law also holds.

2 PRELIMINARIES

Define

$$k_n = \#\{L_n(i,j) : 1 \le i, j \le n\}.$$

Condition A: $\{x_i\}$ or $\{x_{ij}\}$ is independent with mean zero and variance one, and either (a) uniformly bounded or (b) identically distributed.

The following has been shown in [4]. Suppose for every bounded, mean zero and variance one i.i.d. input sequence, $F^{n^{-1/2}A_n}$ converges to some fixed nonrandom distribution F a.s. If $k_n \to \infty$, and $k_n\alpha_n = O(n^2)$, then the same limit continues to hold if the input sequence is i.i.d. with variance one.

Thus when dealing with an input sequence satisfying *Condition A*, we may assume that it is uniformly bounded with mean zero variance one. Then we may use the method of moments: suppose $\{A_n\}$ is a sequence of patterned random matrices, and let $\beta_h(A_n)$, for $h \ge 1$, denote the h-th moment of the ESD of A_n. Suppose there is a sequence of nonrandom $\{\beta_h\}_{h=1}^{\infty}$ such that,

(M1) For every $h \ge 1$, $\mathrm{E}[\beta_h(A_n)] \to \beta_h$

(M2) $\mathrm{Var}[\beta_h(A_n)] \to 0$.

Further assume that $\{\beta_h\}_{h=1}^{\infty}$ satisfies Carleman's condition

$$\sum_{h=1}^{\infty} \beta_{2h}^{-1/2h} = \infty \tag{2.1}$$

Then the LSD is identified by $\{\beta_h\}_{h=1}^{\infty}$ and the convergence to LSD holds in probability. This convergence may be strengthened to almost sure convergence by replacing (M2) by

(M4) $\mathrm{E}[\beta_h(A_n)-\mathrm{E}(\beta_h(A_n))]^4=\mathrm{O}(n^{-2})$.

We now mention some of the key concepts that we need. These are taken from [4].

Circuit: Any function $\pi:\{0,1,2,\cdots,h\}\to\{1,2,\cdots,n\}$ is said to be a *circuit* if $\pi(0)=\pi(h)$. The **length** $l(\pi)$ of π is taken to be h. A circuit depends on h and n but we will suppress this dependence.

Define $X_\pi=x_{L(\pi(0),\pi(1))}x_{L(\pi(1),\pi(2))}\cdots x_{L(\pi(h-2),\pi(h-1))}x_{L(\pi(h-1),\pi(h))}$.

Any function of the trace may be written in terms of X_π. For instance $\mathrm{E}[\beta_h(A_n)]=\mathrm{E}(X_\pi)$.

Matched Circuits: Any value $L(\pi(i-1),\pi(i))$ is said to be an *L-value* of π and π is said to have an **edge of order** e $(1\le e\le h)$ if it has an L-value repeated exactly e times. If π has at least one edge of order one then $E(X_\pi)=0$. Thus only those π with all $e\ge 2$ are relevant. Such a π is called **L-matched** (or *matched*).

Equivalence relation on circuits: Two circuits π_1 and π_2 (of same length) are said to be equivalent if their L values agree at exactly the same pairs (i,j). That is, iff $\big\{L(\pi_1(i-1),\pi_1(i))=L(\pi_1(j-1),\pi_1(j)) \Leftrightarrow L(\pi_2(i-1),\pi_2(i))=L(\pi_2(j-1),\pi(j))\big\}$. This defines an equivalence relation between the circuits.

Words: Equivalence classes may be identified with partitions of $\{1,2,\cdots,h\}$: to any partition we associate a **word** w of length $l(w)=h$ of letters where the first occurrence of each letter is in alphabetical order. For example, if $h=5$, then the partition $\{\{1,3,5\},\{2,4\}\}$ is represented by the word $ababa$.

The class $\Pi(w)$: Let $w[i]$ denote the i-th entry of w. The equivalence class corresponding to w is

$$\Pi(w)=\{\pi: w[i]=w[j]\Leftrightarrow L(\pi(i-1),\pi(i))=L(\pi(j-1),\pi(j))\}.$$

The number of partition blocks corresponding to w will be denoted by $|w|$. If $\pi\in\Pi(w)$, then clearly, $\#\{L(\pi(i-1),\pi(i)):1\le i\le h\}=|w|$.

The notion of order e edges, matching, nonmatching for π carries over to words. For instance, while $ababa$ is matched, $abcadbaa$ is nonmatched with edges of order 1, 2 and 4 and the corresponding partition is $\{\{1,4,7,8\},\{2,6\},\{3\},\{5\}\}$. As pointed out, it will be enough to consider only matched words. The total number of matched words having order 2 edges with $|w|=k$ equals $\frac{(2k)!}{2^k k!}$.

A pair matched word is said to be *Catalan* if succesive removal of double letters of the form xx reduces it to the empty word. This sets up a 1-1 correspondence with the noncrossing pair partitions. The number of Catalan words of length $2k$ equals $\frac{(2k)!}{k!(k+1)!}$.

The class $\Pi^*(w)$: Define for any (matched) word w,

$$\Pi^*(w)=\{\pi: w[i]=w[j]\Rightarrow L(\pi(i-1),\pi(i))=L(\pi(j-1),\pi(j))\}\supseteqq\Pi(w).$$

The classes $C(w)$ **and** $\Gamma_{ij}(w)$: We shall have occasion to deal with $(2k+1)$ tuple π satisfying all the constraints imposed by a pair matched word w of length $2k$ except that it is not necessarily a circuit (that is, $\pi(0)=\pi(2k)$). Define

$$\begin{aligned}
C(w) &= \{\pi:\pi \text{ is a } 2k+1 \text{ tuple } \ni w[i]=w[j]\Rightarrow L(\pi(i-1),\pi(i))=L(\pi(j-1),\pi(j))\}\\
\Gamma_{ij}(w) &= \{\pi\in C(w):\pi(0)=i,\pi(2k)=j\},\ (1\le i,j\le n),\ \ \gamma_{ij}(w)=|\Gamma_{ij}(w)|.
\end{aligned}$$

Clearly, $|\Pi^*(w)|=\displaystyle\sum_{i=1}^n\gamma_{ii}(w)$.

Vertex and generating vertex: Given π, any $\pi(i)$ (or i) is a **vertex**. It is *generating* if either $i=0$ or $w[i]$

is the position of the *first* occurrence of a letter. For example, if $w = abbcab$ then $\pi(0), \pi(1), \pi(2), \pi(4)$ are generating vertices. By Property B, a circuit is completely determined, *up to a finitely many choices* by its generating vertices. The number of generating vertices in π is $|w| + 1$. Hence $|\Pi^*(w)| = \mathrm{O}(n^{|w|+1})$.

It is further known from [4] that $\frac{1}{n^{1+k}}|\Pi(w)|$ and $\frac{1}{n^{1+k}}|\Pi^*(w)|$ have same limit behaviour. Define, for each fixed matched word w of length $2k$ with $|w| = k$,

$$p(w) = \lim_n \frac{1}{n^{1+k}}|\Pi^*(w)| = \lim_n \frac{1}{n^{1+k}}|\Pi(w)| \tag{2.2}$$

whenever any of the two limit exists. Then the limiting $(2k)$-th moment of $n^{-1/2}A_n$ is the finite sum

$$\beta_{2k} = \sum_{w:|w|=k,\, l(w)=2k} p(w), \quad \text{where } \ p(w) \leq \Delta^k \ \text{ for all } \ w$$

and the odd moments are zero. Further, $(M4)$ holds.

3 Main results

Theorem 1 *(i) Suppose $\alpha_n = o(n)$. If w is pair matched non-Catalan word of length $2k$, then $|\Pi^*(w)| = O(n^k\alpha_n)$. Hence $p(w) = 0$ and any subsequential limit has essential support bounded in $[-2\sqrt{\Delta}, 2\sqrt{\Delta}]$.*

(ii) Suppose $\Delta = 1$. Then, for any Catalan word w of length $2k$, and any n, $|\Pi^*(w)| = n^{k+1}$. Consequently, $p(w) = 1$. If in addition, $\alpha_n = o(n)$, and the input sequence satisfies Condition A, then the LSD for $n^{-(1/2)}A_n$ is the semicircular law.

Proof of (i): Suppose w is not a Catalan word. Then there is $i < j < t < l$ such that $w[i] = w[t] = a$ (say) and $w[j] = w[l] = b$ (say). Without loss of generality, let i be the minimum such choice (for some j, t, l), and for this i, let j be the maximum such choice (for some t, l). Thus, all letters of w in $\{w[j+1], ..., w[t-1]\}$ have both copies in $\{i+1, ..., t-1\}$. We use the following algorithm for circuit π corresponding to w:

(1) First fill up $\pi(0), ..., \pi(j-1)$ from *left to right*. If the number of distinct letters in $w[1], ..., w[j-1]$ is x, then the number of generating vertices in $\{0, ..., j-1\}$ is $x + 1$.

(2) Now fill up $\pi(t), ..., \pi(j)$ *in this order*. Since $L(\pi(i-1), \pi(i)) = L(\pi(t-1), \pi(t))$, $(\pi(t-1), \pi(t))$ can be chosen in at most α_n ways. If $w[s]$ $(j < s < t)$ is the second occurrence of a letter in the filling sequence, then, having chosen up to $\pi(s)$, $\pi(s-1)$ has at most Δ choices. Therefore, if $w[i_1], ..., w[i_y](i_1 < i_2 < .. < i_y)$ are the first occurrences of the letters in the filling sequence occurring in $\{t, .., j+1\}$ (clearly $i_1 > j+1$), each vertex in $\{t-2, ..., j\} \cap \{i_y - 1, ..., i_1 - 1\}^c$ can be chosen in at most Δ ways, given all previous vertices in the filling sequence.

(3) Finally fill $\pi(t+1), ..., \pi(2k)$ from *left to right*. If the number of first occurrences in $w[t+1], ..., w[2k]$ is z, then the number of generating vertices is z.

So, the circuit is completely specified. The number of vertices other than $\pi(t)$ and $\pi(t-1)$ with choices more than Δ, given previously filled vertices in the filling sequence, is at most $(x+1) + y + z$. As the first occurrence of letter b is never counted among the y first occurrences in step (2), therefore, $x + (y+1) + z = k$. As a consequence, $|\Pi^*(w)| = O(n^k\alpha_n)$ and hence $p(w) = 0$.

For any subsequential limit, using Stirling's Approximation, $\beta_{2k} \leq \Delta^k \frac{(2k)!}{k!(k+1)!} \leq \Delta^k C 2^{2k}$, where C is a constant. The claim on the support follows as the L^p norm converges to the essential sup as $p \to \infty$.

Proof of (ii): The essential implication of $\Delta = 1$ is

$$L(\pi(i-1), \pi(i)) = L(\pi(i), \pi(i+1)) \Rightarrow \pi(i-1) = \pi(i+1). \tag{3.1}$$

For $k = 1$, choose $\pi(0)$ and $\pi(1)$ in n ways each, and for all such choices, $\pi(2)$ has exactly one choice by (3.1), i.e., $\pi(2) = \pi(0)$. Therefore, $|\Pi^*(w)| = n^2$.

Now we apply induction on k. Suppose the statement is true for some $k \geq 1$. Take any Catalan w of length $2k+2$. Then there is $i, i+1$, $i \leq 2k+1$ such that $w[i] = w[i+1]$. If w' is the Catalan word obtained from w by deleting $w[i]$ and $w[i+1]$, then any circuit π' in $\Pi^*(w')$ yields n distinct circuits $\pi_1, ..., \pi_n$ in $\Pi^*(w)$, where π_s is obtained by putting $\pi_s(j) = \pi'(j)\ \forall\ j \neq i, i+1$, $\pi_s(i) = s$ and $\pi_s(i+1) = \pi(i-1)$ by (3.1). Also, this process applied to all circuits in $\Pi^*(w')$, yielding the entire $\Pi^*(w)$. Therefore, $|\Pi^*(w)| = n|\Pi^*(w')|$. By induction hypothesis, $|\Pi^*(w)| = n.n^{k+1} = n^{k+2}$. $\square$

Theorem 2 *(i) Suppose L satisfies Property B. Then, for any Catalan word w of length $2k$, $|\Pi^*_{A_n}(w)| \geq n^{k+1}$. Hence if the input sequence satisfies Condition A, then for any subsequential LSD, $\beta_{2k} \geq \dfrac{(2k)!}{k!(k+1)!}$.*

(ii) Suppose L satisfies Properties B and P. Then for any Catalan word w, $p(w) = 1$.

(iii) Suppose L satisfies Properties B and P, $\alpha_n = o(n)$ and the input sequence satisfies Condition A. Then the LSD of $\{n^{-1/2}A_n\}$ is the semicircle law.

Matrices satisfying Properties B, P and $\alpha_n = o(n)$ will be called **Wigner type** matrices.

Before we prove the theorem, we wish to make the following remark on the joint convergence.

Remark 1 *Let $\{A_{in}, 1 \leq i \leq p\}$ be p independent sequences of $n \times n$ random matrices with a common link function that satisfies Property B. Suppose, for any pair matched word w of length $2s$, $\lim_n \dfrac{|\Pi(w)|}{n^{s+1}} = 1$ or 0, depending on whether or not w is Catalan. Further, assume that the input sequence $\{x^{(n)}_{i,j}\}$ $(1 \leq j \leq k_n)$ satisfies $Ex^{(n)}_{i,j} = 0$, $E|x^{(n)}_{i,j}|^2 = 1$ and for all $m \in \mathbb{N}$, $\sup_n \sup_{1 \leq i \leq p} \sup_{1 \leq j \leq k_n} E|x^{(n)}_{i,j}|^m \leq c_m < \infty$. Then $n^{-1/2}A_{in}, 1 \leq i \leq p$ jointly converges to the freely independent semicircle law. In particular, this holds for Wigner type matrices. All this can be proved by using the above results and the developments in [3]. We omit the details.*

Proof (i) To ease notation, suppress dependency on n and write, for example A for A_n. Let L_W be the Wigner link function and for any Catalan w, let $\Pi^*_W(w)$ be the corresponding $\Pi^*(w)$. If $\pi \in \Pi^*_W(w)$, then $w[i] = w[j] \Rightarrow L_W(\pi(i-1), \pi(i)) = L_W(\pi(j-1), \pi(j)) \Rightarrow L_A(\pi(i-1), \pi(i)) = L_A(\pi(j-1), \pi(j))$ (as A is symmetric) $\Rightarrow \pi \in \Pi^*_A(w)$. Therefore, $\Pi^*_W(w) \subseteq \Pi^*_A(w)$. As $|\Pi^*_W(w)| = n^{k+1}$, the result (i) follows.

(ii) We will prove this part by induction. Consider the following induction statement S_k: For any Catalan w of length $2k$, $\exists M_k > 0$ such that

$$\gamma_{ij}(w) \leq M_k n^{k-1} \text{ for all } i \neq j \text{ and } \frac{1}{n}\sum_{i=1}^{n} |\frac{\gamma_{ii}(w)}{n^k} - 1| = O(1/n).$$

Note that if S_k is true, then $p(w) = 1$ for w Catalan.

For $k = 1$, $\gamma_{ij}(w) = |M^{(n)}_{ij}| \leq M^*$ for all $i \neq j$ by Property P, and $\gamma_{ii}(w) = n$. Therefore S_1 is true.

Let S_l be true for all $l \leq k-1$ $(k \geq 2)$. We shall now prove that S_k is true. Let w be a Catalan word of length $2k$. Then the following two cases arise:

Case (i). $w = w_1 w_2$, where w_1 and w_2 are Catalan words of length $2k_1$ and $2k_2$ respectively (say) where

$k_1 \leq k-1$ and $k_2 \leq k-1$. Then for $i \neq j$, using induction hypothesis and Property B:

$$\begin{aligned}
\gamma_{ij}(w) &= \sum_{l=1}^{n} \gamma_{il}(w_1)\gamma_{lj}(w_2) = \sum_{l \neq i,j} \gamma_{il}(w_1)\gamma_{lj}(w_2) + \gamma_{ii}(w_1)\gamma_{ij}(w_2) + \gamma_{ij}(w_1)\gamma_{jj}(w_2) \\
&\leq (M_{k_1} n^{k_1-1})(M_{k_2} n^{k_2-1})(n) + (\Delta^{k_1-1} n^{k_1})(M_{k_2} n^{k_2-1}) + (\Delta^{k_2-1} n^{k_2})(M_{k_1} n^{k_1-1}) \\
&= (M_{k_1} M_{k_2} + \Delta^{k_1-1} M_{k_2} + \Delta^{k_2-1} M_{k_1}) n^{k-1}.
\end{aligned}$$

Now we prove the other part of the induction statement S_k:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \left|\frac{\gamma_{ii}(w)}{n^k} - 1\right| &= \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\sum_{l \neq i} \gamma_{il}(w_1)\gamma_{li}(w_2) + \gamma_{ii}(w_1)\gamma_{ii}(w_2)}{n^k} - 1\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\gamma_{ii}(w_1)}{n^{k_1}} \frac{\gamma_{ii}(w_2)}{n^{k_2}} - 1\right| + \frac{1}{n}\sum_{i=1}^{n}\sum_{l \neq i} \frac{\gamma_{il}(w_1)\gamma_{li}(w_2)}{n^k} = T_1 + T_2 \text{ (say)}.
\end{aligned}$$

By using the induction hypothesis,

$$T_2 \leq \frac{1}{n}\sum_{i=1}^{n}\sum_{l \neq i} \frac{M_{k_1} M_{k_2} n^{k-2}}{n^k} \leq M_{k_1} M_{k_2} \frac{n^k}{n^{k+1}} = \frac{M_{k_1} M_{k_2}}{n} = O(1/n).$$

By induction hypothesis and Property B,

$$\begin{aligned}
T_1 &\leq \frac{1}{n}\sum_{i=1}^{n} \frac{\gamma_{ii}(w_1)}{n^{k_1}} \left|\frac{\gamma_{ii}(w_2)}{n^{k_2}} - 1\right| + \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\gamma_{ii}(w_1)}{n^{k_1}} - 1\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \Delta^{k_1-1} \left|\frac{\gamma_{ii}(w_2)}{n^{k_2}} - 1\right| + \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\gamma_{ii}(w_1)}{n^{k_1}} - 1\right| = O(1/n).
\end{aligned}$$

This takes care of case (i).

Case (ii). $w = aw_1a$, where w_1 is a Catalan word. Let $i \neq j$. Then

$$\gamma_{ij}(w) = \sum_{l \in M_{ij}^{(n)}} \gamma_{ll}(w_1) + \sum_{l=1}^{n} \sum_{t \neq l : L(i,l) = L(j,t)} \gamma_{lt}(w_1) = A_1 + A_2 \text{ (say)}.$$

The first sum is for those $\pi \in C(w)$, $\pi(0) = i, \pi(2k) = j$ for which $\pi(1) = \pi(2k-1)$. The second sum handles $\pi \in C(w), \pi(0) = i, \pi(2k) = j$ for which $\pi(1) \neq \pi(2k-1)$. Observe that

$$\begin{aligned}
A_1 &\leq \Delta^{k-2} n^{k-1} M^* \text{ (by Property P)}, \\
A_2 &\leq M_{k-1} n^{k-2}.n\Delta = M_{k-1}\Delta n^{k-1} \text{ (by induction hypothesis and Property B)}.
\end{aligned}$$

This proves the first part of the assertion. To prove the second half, we proceed as in the previous case. By Property P and induction hypothesis:

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} \left|\frac{\gamma_{ii}(w)}{n^k} - 1\right| &= \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\sum_{l=1}^{n} \gamma_{ll}(w_1) + \sum_{l,t : l \neq t, i \in M_{lt}^{(n)}} \gamma_{lt}(w_1)}{n^k} - 1\right| \\
&\leq \frac{1}{n}\sum_{i=1}^{n} \left|\frac{\sum_{l=1}^{n} \gamma_{ll}(w_1)}{n.n^{k-1}} - 1\right| + \frac{1}{n}\sum_{i=1}^{n} \sum_{l,t : l \neq t, i \in M_{lt}^{(n)}} \frac{\gamma_{lt}(w_1)}{n^k}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{n}\sum_{l=1}^{n}|\frac{\gamma_{ll}(w_1)}{n^{k-1}}-1| + \frac{1}{n^{k+1}}\sum_{l,t:l\neq t}\sum_{i:i\in M_{ij}^{(n)}}\frac{\gamma_{lt}(w_1)}{n^k}\\
&\leq \frac{1}{n}\sum_{l=1}^{n}|\frac{\gamma_{ll}(w_1)}{n^{k-1}}-1| + \frac{1}{n^{k+1}}\sum_{l,t:l\neq t}M^*.M_{k-1}n^{k-2}\\
&\leq \frac{1}{n}\sum_{l=1}^{n}|\frac{\gamma_{ll}(w_1)}{n^{k-1}}-1| + \frac{M^*M_{k-1}}{n} = O(1/n).
\end{aligned}
$$

Therefore, the above induction statement is true for all k.

(iii) Part (i) of Theorem 1 and part (ii) of Theorem 2 together imply this. □

The Toeplitz, Hankel, Symmetric circulant and Reverse circulant matrices all satisfy the assumptions of part (ii) of the above theorem.

Remark 2 *A careful look at the proof of part (i) of Theorem 2 shows that for any Catalan word w of length $2k$, $\Pi^*(w) \supseteq \Pi^*_W(w)$. By part (ii) of Theorem 2, $\lim_n \frac{|\Pi^*(w)|}{n^{k+1}} = \lim_n \frac{|\Pi^*_W(w)|}{n^{k+1}} = 1$. Therefore*

$$\lim_n \frac{|\Pi^*(w) - \Pi^*_W(w)|}{n^{k+1}} = 0.$$

This will be used crucially in Section 3.1.

Example 1. Suppose $L(i,j) = |P(i,j)|$, where P is a symmetric polynomial in two variables with integer coefficients. Then, it can be easily checked that the conditions of part (ii) of Theorem 2 are satisfied. However, the condition $\alpha_n = o(n)$ may not be satisfied, as in the Toeplitz matrix where $L(i,j) = |i-j|$.

Example 2. Let us now consider a natural subclass of the above. Let $L(i,j) = |P(ij)|$, where P is a polynomial of degree d (say) in one variable with integer coefficients. Let $u_l = \#\{(i,j) : L(i,j) = l\}$. Then $\alpha_n = \sup_{l\in\{|P(ij)|:1\leq i,j\leq n\}} u_l$. Clearly, $|P(ij)| = l$ has at most $2d$ non-negative integer solutions of ij. For each such solution z (say) in $\{1,2,...,n^2\}$, $\#\{(i,j) : 1 \leq i,j \leq n, ij = z\} \leq d(z)$, where $d(z)$ is the number of positive divisors of z. It is well known that (see [6]) $\limsup_n \frac{\log d(n)}{\log n/\log\log n} = \log 2$. In particular, for any $\epsilon > 0$, $d(n) = o(n^\epsilon)$. Choose $\epsilon < \frac{1}{2}$ and $C > 0$ such that $d(m) < Cm^\epsilon$ for all positive integers m. Therefore,

$$\alpha_n = \sup_{l\in\{|P(ij)|:1\leq i,j\leq n\}} u_l \leq (2d)[\sup_{m\leq n^2} d(m)] \leq (2d)Mn^{2\epsilon} = o(n).$$

Hence by part (iii) of Theorem 2 the LSD is the semicircular law. This also continues to be the LSD if $L(i,j) = |P(ij)| \mod m_n$ where m_n is a sequence of positive integers satisfying $\sup \frac{n^{2d}}{m_n} < \infty$.

Example 3. Wigner type matrices are easily obtained by *coloring*. For instance, take a triangular input sequence $\{x_{i,j}\}_{(0\leq i\leq n-1,\ 1\leq j\leq r_n)}$ satisfying Condition A. Consider the *Toeplitz link function* L_T and a *color link function*

$$L_c(i,j) = c_k \text{ if } l_k \leq i+j < l_{k+1}\ (1 \leq k \leq r_n - 1)$$

where c_k's are colors and $2 = l_1 < l_2 < ... < l_{r_n} = 2n+1$ satisfy $\sup_{k\leq r_n-1}(l_{k+1} - l_k) = o(n)$. Consider the *colored Toeplitz matrix* A_n with link function $L(i,j) = (L_T(i,j), L_c(i,j))$. Then it is easy to see that

$\alpha_n = \sup_{k \le r_n - 1} (l_{k+1} - l_k) = o(n)$ and Properties B, P are satisfied. Hence, this is a *Wigner type* matrix. A general treatment of colored matrices will be made in a forthcoming article.

3.1 Finite symmetric block matrices with large Wigner type blocks

A finite symmetric block structure $B_k = B_k(a_1, a_2, ..., a_t)$ is a $k \times k$ symmetric matrix whose entries are defined by $(B_k)_{ij} = a_{L'(i,j)}$, where L' is the link function for B_k.

Theorem 1 of [7] proves a general existence theorem for such block matrices with Wigner blocks. In Theorem 3, we supplement this result by relaxing his moment assumptions and also allowing *Wigner type blocks*. We are also able to provide additional information on the LSD and show its explicit dependence on the block structure B_k. Further, we use our result to demonstrate how a *large match set* can result in a non-semicircular LSD even if $\alpha_n = o(n)$. We need to introduce a few concepts and a few notation.

Let $W_1^{(n)}, W_2^{(n)}, ...$ be sequences of independent Wigner type matrices, with entries of color $1, 2, ..$ respectively and a common link function L^*. Then the $(i,j)th$ entry of $W_k^{(n)}$ may be written as by $x_{(k,\ L^*(i,j))}$, where k denotes the color. Replace the $(i,j)^{th}$ entry of B_k by $W^{(n)}_{L'(i,j)}$ to obtain the $kn \times kn$ matrix:

$$A_{n,k} = B_k(W_1^{(n)}, ..., W_t^{(n)}).$$

Denoting the greatest integer function by [ ], the link function L of $A_{n,k}$ is

$$L(i,j) = \left(L'([(i-1)/n] + 1, [(j-1)/n] + 1), L^*((i-1) \bmod n + 1, (j-1) \bmod n + 1)\right).$$

For a word w, define $C_{B_k}(w)$ and $\Pi^*_{B_k}(w)$ as the C and Π^* (defined in Section 2) for the matrix B_k. Let

$$\Delta_{B_k} = \sup_{t \in \text{Range}(L')} \sup_{1 \le i \le k} \#\{j : 1 \le j \le k,\ L'(i,j) = t\}.$$

Block address and address function: Let $U_i = \{(i-1)n + 1, ..., in\}$, $i = 1, ..., k$, and w be a word of length $2t$. The ***block address*** π' of a $2t + 1$ tuple $\pi \in C(w)$ is defined as follows:

$$\pi'(i) = j \text{ if } \pi(i) \in U_j.$$

Note that $L(\pi(i-1), \pi(i)) = L((\pi(j-1), \pi(j)) \Rightarrow L'(\pi'(i-1), \pi'(i)) = L'((\pi'(j-1), \pi'(j))$. Therefore $\pi' \in C_{B_k}(w)$. Also, as $\pi(0) = \pi(2t) \Rightarrow \pi'(0) = \pi'(2t)$, therefore, $\pi \in \Pi^*(w) \Rightarrow \pi' \in \Pi^*_{B_k}(w)$.

Define the **address function** $\phi : \Pi^*(w) \mapsto \Pi^*_{B_k}(w)$ by:

$$(\pi(0), ..., \pi(2t)) \mapsto (\pi'(0), ..., \pi'(2t)).$$

Phase: Let w be a word and $\pi \in C(w)$. Call $\pi(i)$ and $\pi(j)$ to be in the same ***phase*** (denote by $\pi(i) \sim \pi(j)$) if $\pi(i) \equiv \pi(j) \mod n$.

Claim 1: Let w be a Catalan word of length $2t$, and let $\pi \in C(w)$. Then $\pi(0) \sim \pi(2t)$.

Proof. We prove this inductively. Consider $t = 1$. Let $w = aa$ and $\pi' = (i, j, u)$ with $L'(i,j) = L'(j,u)$. As $\Delta = 1$ for the Wigner blocks and $(\pi(0), \pi(1))$ and $(\pi(2), \pi(1))$ both occur in the same column, therefore, $L(\pi(0), \pi(1)) = L(\pi(2), \pi(1)) \Rightarrow \pi(2) - \pi(0) = (u - i)n$. Therefore, $\pi(0) \sim \pi(2)$.

Let the hypothesis hold for $t \le m$. For $t = m + 1$, one of the following two cases arise:

Case 1. $w = w_1w_2$ where w_1 and w_2 are Catalan of length $2t_1$ and $2t_2$ respectively with $t_1 + t_2 = m + 1$. By induction hypothesis, $\pi(0) \sim \pi(2t_1)$ and $\pi(2t_1) \sim \pi(2(t_1 + t_2))$. Therefore, $\pi(0) \sim \pi(2m + 2)$.

Case 2. $w = aw_1a$, where w_1 is Catalan. By hypothesis, $\pi(1) \sim \pi(2m + 1)$. As $L(\pi(0), \pi(1)) = L(\pi(2m+2), \pi(2m+1)) \Rightarrow L'(\pi'(0), \pi'(1)) = L'(\pi'(2m+2), \pi'(2m+1))$, $\pi(1) \sim \pi(2m+1)$, and $\Delta = 1$ for the Wigner blocks, the only possibility is $\pi(2m+1) = \pi(1) + (\pi'(2m+1) - \pi'(1))n$ and $\pi(2m+2) = \pi(0) + (\pi'(2m+2) - \pi'(0))n$. Hence, $\pi(0) \sim \pi(2m+2)$ and our claim is proved. □

Claim 2: Let w be a Catalan word of length $2t$, and let $\pi' \in \Pi^*_{B_k}(w)$. Then

$$|\phi^{-1}(\pi')| = n^{t+1}$$

where $\phi^{-1}(\pi') = \{\pi \in \Pi^*(w) : \pi(i) \in U_{\pi'(i)} \forall i = 0, ..., 2t\}$.

Proof. First assume $t = 1$. Let $w = aa$ and $\pi' = (i, j, i)$. Choose $\pi(0)$ and $\pi(1)$ such that $\pi(0) \in U_i$ and $\pi(1) \in U_j$ in n^2 ways. As $\pi(2) = \pi(0)$, the claim is proved for $t = 1$.

Now assume $t > 1$. Suppose we choose any set of values for the $(t + 1)$ independent vertices of π under the only restriction that $\pi(i) \in U_{\pi'(i)}$ for each independent vertex $\pi(i)$. We claim that this uniquely determines the entire circuit π. For, suppose that $\pi(0), ..., \pi(l)$ are chosen and the vertex $\pi(l + 1)$ is a dependent one. Therefore, the $(l + 1)$-th place in the word corresponds to the second occurrence of a letter whose first occurrence is, say, at the u-th place ($u \leq l$). Then either $l = u$ or $(w[u + 1], ..., w[l])$ forms a Catalan word. In either case, $\pi(u) \sim \pi(l)$ (for the latter case, we use Claim 1). As $L(\pi(l + 1), \pi(l)) = L(\pi(u - 1), \pi(u))$, therefore, $L'(\pi'(l + 1), \pi'(l)) = L'(\pi'(u - 1), \pi'(u))$, and as $\pi(u) \sim \pi(l)$, therefore $\pi(l) = \pi(u) + (\pi'(l) - \pi'(u))n$ and $\pi(l + 1)$ has exactly one choice, namely, $\pi(l + 1) = \pi(u - 1) + (\pi'(l + 1) - \pi'(u - 1))n$. Thus, the entire $2t + 1$ tuple π is uniquely determined. To check that this is indeed a circuit, we note that $\pi'(2t) = \pi'(0)$ (as π' is a circuit). Also, $\pi(0) \sim \pi(2t)$. Therefore, $\pi(2t) = \pi(0) + (\pi'(2t) - \pi'(0))n = \pi(0)$. Therefore $|\phi^{-1}(\pi')| = n^{t+1}$ and the claim is proved. □

Theorem 3 *Let B_k be any $k \times k$ symmetric block structure. Then the LSD of $\{(nk)^{-1/2}A_{n,k}\}$ (as $n \to \infty$) is μ_{B_k} whose odd moments are 0 and $2t$-th moments are given by*

$$\beta_{2t} = \frac{1}{k^{t+1}} \sum_{w \text{ Catalan}:|w|=t} |\Pi^*_{B_k}(w)|. \tag{3.2}$$

In particular, the LSD has support bounded in $[-2\sqrt{\Delta_{B_k}}, 2\sqrt{\Delta_{B_k}}]$.

Note that if B_k is a singleton structure, then the above reduces to the well known semicircular law.

Proof of Theorem 3. First we consider the case of Wigner blocks. As we deal with combinatorics for a fixed n, we suppress the superscript n, without loss of generality. Note that

$$|\Pi^*(w)| = \sum_{\pi' \in \Pi^*_{B_k}(w)} |\phi^{-1}(\pi')|.$$

By Claim 2, for each Catalan word w,

$$|\Pi^*(w)| = n^{t+1}|\Pi^*_{B_k}(w)| \text{ and hence } \lim_{n\to\infty} \frac{|\Pi^*(w)|}{(nk)^{t+1}} = \frac{|\Pi^*_{B_k}(w)|}{k^{t+1}}.$$

Also, from Theorem 1 (i), if w is not a Catalan word, the above limit is 0. (This is because there are finitely many blocks and each block contains at most 2 copies of each entry and hence $\alpha_n = O(1)$). Therefore, the $2t$-th limit moment is precisely the value claimed in (3.2).

The proof for Wigner type blocks follows easily from the above and Remark 2. We omit the details. $\square$

The $k \times k$ **symmetric circulant block structure** $SC_k(x_0, x_1, ..., x_{[\frac{n}{2}]})$ is defined by the link function $L(i,j) = k/2 - |k/2 - |i-j||$.

$$SC_k(x_0, x_1, ..., x_{[\frac{n}{2}]}) = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_2 & x_1 \\ x_1 & x_0 & x_1 & \dots & x_3 & x_2 \\ x_2 & x_1 & x_0 & \dots & x_2 & x_3 \\ & & & \vdots & & \\ x_1 & x_2 & x_3 & \dots & x_1 & x_0 \end{bmatrix}. \tag{3.3}$$

Oraby [7] proved that the LSD of block matrices with the symmetric circulant block structure and independent Wigner block is non-semicircular. To prove this, he used the explicit representation of the eigenvalues of circulant matrices. We provide an alternate combinatorial proof using, the basic properties of the block structure and the moment formula obtained in Theorem 3. This proof has the advantage of being extendable to other similar block structures. Let γ_{α,σ^2} denote the semicircular law with mean α and variance σ^2. The following result is due to Oraby[7]:

Proposition 1 *Let $\{W_i^{(n)}\}$ be independent $n \times n$ Wigner matrices. Then,*

$$\mu_{(nk)^{-1/2} SC_k(W_0^{(n)}, W_1^{(n)}, ..., W_{[\frac{k}{2}]}^{(n)})} \to \nu_k \quad a.s.$$

where

$$\nu_k = \frac{k-1}{k}\gamma_{0,\frac{k-1}{k}} + \frac{1}{k}\gamma_{0,\frac{2k-1}{k}} \quad \text{if } k \text{ is odd},$$
$$= \frac{k-2}{k}\gamma_{0,\frac{k-2}{k}} + \frac{2}{k}\gamma_{0,\frac{2k-2}{k}} \quad \text{if } k \text{ is even}.$$

Proof: First assume that k odd. It suffices to prove that the moments of the LSD satisfy:

$$\beta_{2t} = \frac{1}{k^{t+1}}[(k-1)^{t+1} + (2k-1)^t]\frac{(2k)!}{k!(k+1)!}.$$

By what we have proved in Theorem 3, it suffices to show that, for any Catalan word w of length $2t$,

$$|\Pi^*_{SC_k}(w)| = (k-1)^{t+1} + (2k-1)^t.$$

The only structural properties of the block structure that will be used in the proof are the following:

(P1). Out of $[\frac{k}{2}] + 1$ input variables in the block structure, x_0 occurs exactly once in each row (and column), and the remaining $[\frac{k}{2}]$ of them occur twice in every row (and column).

(P2). The *match set* between any two distinct rows (or columns) is a singleton.

To see things more clearly, we can view a circuit in $\Pi^*(w)$ as a path of the form $(\pi(1), \pi(2)) \to (\pi(3), \pi(2)) \to (\pi(3), \pi(4)) \to ... \to (\pi(2t-1), \pi(2t))$, where we first select a starting point, then move vertically in the column, then horizontally in the row,.., and so on, and finally back to the row we started from, maintaining the constraints imposed by the word.

Now we calculate $|C(w)|$. We can select $\pi(0)$ in k ways. Then fill up the vertices such that l of the t remaining independent vertices have their link values of corresponding edges $\neq 0$ and the rest $t-l$ have link values of corresponding edges $= 0$. For the former class, each independent vertex can be chosen in $k-1$ ways, and the corresponding dependent vertices can be filled up in 2 ways. For the latter, both these counts are 1 ($P1$). Therefore

$$|C(w)| = k.\sum_{l=1}^{t}\binom{t}{l}[(k-1).2]^l = k(2k-1)^t.$$

Consider the following induction statement S_t: For each Catalan word w of length $2t$, for all i,

(a)

$$\gamma_{ii}(w) = \frac{|\Pi^*_{SC_k}(w)|}{k} = \frac{1}{k}[(k-1)^{t+1} + (2k-1)^t]$$

and for all $i \neq j$,

(b)

$$\begin{aligned}\gamma_{ij} = \frac{1}{k(k-1)}[|C(w)| - |\Pi^*_{SC_k}(w)|] &= \frac{1}{k(k-1)}[k(2k-1)^t - \{(k-1)^{t+1} + (2k-1)^t\}] \\ &= \frac{1}{k}[(2k-1)^t - (k-1)^t]\end{aligned}$$

For $t = 1$, trivially, $\gamma_{ii}(w) = k = \frac{1}{k}[(k-1)^2 + (2k-1)]$, proving assertion (i). Also, as for each $i \neq j$, the match set $M_{ij}^{(k)}$ is a singleton ($P2$), $\gamma_{ij}(w) = 1 = \frac{1}{k}[(2k-1) - (k-1)]$, proving assertion (ii). Therefore, our statement is true for $t = 1$. Suppose the statement is true for $t \leq s-1$. We prove it for $t = s$. The following two cases arise:

Case (i) $w = w_1w_2$, where w_1 and w_2 are Catalan of length s_1 and s_2 respectively, where, $s_1,\ s_2 \leq s-1$. Then by induction hypothesis,

$$\begin{aligned}\gamma_{ii}(w) &= \sum_{l=1}^{k}\gamma_{il}(w_1)\gamma_{li}(w_2) = \sum_{l\neq i}\gamma_{il}(w_1)\gamma_{li}(w_2) + \gamma_{ii}(w_1)\gamma_{ii}(w_2) \\ &= \frac{(k-1)}{k^2}[(2k-1)^{s_1} - (k-1)^{s_1}][(2k-1)^{s_2} - (k-1)^{s_2}] \\ &\quad + \frac{1}{k^2}[(k-1)^{s_1+1} + (2k-1)^{s_1}][(k-1)^{s_2+1} + (2k-1)^{s_2}].\end{aligned}$$

Simplifying the above expression and using the fact $s_1+s_2 = s$, we get $\gamma_{ii}(w) = \dfrac{1}{k}[(k-1)^{s+1} + (2k-1)^s]$ proving assertion (a). Similarly, for $i \neq j$,

$$\begin{aligned}\gamma_{ij}(w) &= \sum_{l=1}^{k}\gamma_{il}(w_1)\gamma_{lj}(w_2) = \sum_{l\neq i,j}\gamma_{il}(w_1)\gamma_{li}(w_2) + \gamma_{ii}(w_1)\gamma_{ij}(w_2) + \gamma_{ij}(w_1)\gamma_{jj}(w_2) \\ &= \frac{(k-2)}{k^2}[(2k-1)^{s_1} - (k-1)^{s_1}][(2k-1)^{s_2} - (k-1)^{s_2}] \\ &\quad + \frac{1}{k^2}[(k-1)^{s_1+1} + (2k-1)^{s_1}][(2k-1)^{s_2} - (k-1)^{s_2}] \\ &\quad + \frac{1}{k^2}[(2k-1)^{s_1} - (k-1)^{s_1}][(k-1)^{s_2+1} + (2k-1)^{s_2}] \text{ (by induction hypothesis)} \\ &= \frac{1}{k}[(2k-1)^s - (k-1)^s]\end{aligned}$$

proving assertion (b).

Case (ii). $w = aw_1a$, where w_1 is a Catalan word of length $2s-2$. To prove assertion (a), suppose $\pi \in \Pi^*_{SC_k}(w)$ such that $\pi(0) = \pi(2s) = i$. Then $\pi = (i \;\; \pi_1 \;\; i)$ where $\pi_1 \in C(w_1)$ and $L(i, \pi_1(0)) = L(i, \pi_1(2s-2))$. Note that $\pi(1) = \pi_1(0)$ and $\pi(2s-1) = \pi_1(2s-2)$. There are the following two possibilities:

Case (iia). $\pi(1) \neq \pi(2s-1)$. We can choose the value of $(\pi(1), \pi(2s-1)) = (i^*, j^*)$ (say), such that $L(i, i^*) = L(i, j^*)$ in $(k-1)$ ways, by $P1$. Then the circuit π is completely determined by choosing $\pi_1 \in C(w_1)$ such that $\pi_1(0) = i^*, \pi_1(2s-2) = j^*$ in $\gamma_{i^*j^*}(w_1)$ ways.

Case (iib). $\pi(1) = \pi(2s-1)$. We can choose $\pi(1) = \pi(2s-1) = i^*$ (say), in k ways, and then the circuit π is completely determined by choosing π_1 in $\gamma_{i^*i^*}(w_1)$ ways. Thus, by induction hypothesis,

$$\begin{aligned}\gamma_{ii}(w) &= \frac{(k-1)}{k}[(2k-1)^{s-1} - (k-1)^{s-1}] + \frac{k}{k}[(k-1)^s + (2k-1)^{s-1}] \\ &= \frac{1}{k}[(k-1)^{s+1} + (2k-1)^s].\end{aligned}$$

To prove assertion (b), suppose $\pi \in C(w)$ such that $\pi(0) = i, \pi(2s) = j$, $(i \neq j)$. Then $\pi = (i \;\; \pi_1 \;\; j)$ where $\pi_1 \in C(w_1)$ and $L(i, \pi_1(0)) = L(j, \pi_1(2s-2))$. As before, there are the following two possibilities:

Case (I) $\pi(1) \neq \pi(2s-1)$. We can choose the value of $(\pi(1), \pi(2s-1)) = (i^*, j^*)$ (say), such that $L(i, i^*) = L(j, j^*)$, in $(k-2).2 + 1 + 1 = 2k-2$ ways. Suppose $M^{(k)}_{ij} = \{m\}$ ($P2$). Clearly, as x_0 occurs exactly once in each row and each column ($P1$), $m \neq 1$. Let $L(i, u) = L(j, v) = 0$ ($u \neq v$). Then for $i^* \neq m, u$, there are 2 choices for j^*. For $i^* = m$ and $i^* = u$, j^* has exactly one choice). Then, π is completely determined by choosing π_1 in $\gamma_{i^*j^*}(w_1)$ ways.

Case (II) $\pi(1) = \pi(2s-1)$. Then, there can be exactly one choice for $\pi(1) = \pi(2s-1)$, namely, the element of $M^{(k)}_{ij}$, say m. Then, π is completely determined by choosing π_1 in $\gamma_{mm}(w_1)$ ways.

Thus, by induction hypothesis,

$$\begin{aligned}\gamma_{ii}(w) &= \frac{(2k-2)}{k}[(2k-1)^{s-1} - (k-1)^{s-1}] + \frac{1}{k}[(k-1)^s + (2k-1)^{s-1}] \\ &= \frac{1}{k}[(2k-1)^s - (k-1)^s].\end{aligned}$$

Hence, both assertions (a) and (b) have been proved by induction for k odd. When k is even, similar combinatorial arguments and application of Theorem 3 yields the result. We omit the details. □

Remark 3 *The LSD for the case k odd remains the same for any block structure satisfying the Properties (P1) and (P2) given in the proof above. The above type of Catalan induction may also be useful to obtain insight into effect of structure on the LSD for other patterns.*

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