SPECTRAL NORM OF CIRCULANT TYPE MATRICES WITH HEAVY TAILED ENTRIES

Arup Bose [*]　　Rajat Subhra Hazra　　Koushik Saha[†]

Statistics and Mathematics Unit
Indian Statistical Institute
203 B. T. Road, Kolkata 700108
INDIA

Abstract

We first study the probabilistic properties of the spectral norm of scaled eigenvalues of large dimensional Toeplitz, circulant and symmetric circulant matrices when the input sequence is independent and identically distributed with appropriate heavy tails.

When the input sequence is a stationary two sided moving average process of infinite order, we scale the eigenvalues by the spectral density at appropriate ordinates and study the limit for their maximums.

Keywords Large dimensional random matrix, eigenvalues, Toeplitz matrix, circulant matrix, symmetric circulant matrix, reverse circulant matrix, spectral norm, moving average process, power transfer function.

AMS 2000 Subject Classification 15A52, 60F99, 60F05.

1 Introduction

Matrices with suitable patterned random inputs where the dimension tends to infinity, are known as large dimensional random matrices. In this article we focus on the (symmetric) Toeplitz, circulant, reverse circulant and symmetric circulant matrices defined as follows: Let $\{x_0, x_1, \ldots\}$ be a sequence of real random variables, which will be called the *input sequence*.

Toeplitz (T_n). This $n \times n$ symmetric matrix with input $\{x_i\}$ is the matrix with (i, j)-th entry $x_{|i-j|}$ for all i, j.

Reverse circulant (RC_n). This $n \times n$ symmetric matrix has its (i, j)-th entry as $x_{(i+j-2) mod\ n}$.

Circulant (C_n). In this case (i, j)-th entry is $x_{(j-i+n) mod\ n}$. This is not a symmetric matrix.

Symmetric circulant (SC_n). This symmetric version of the circulant matrix has the (i, j)-th element as $x_{n/2-|n/2-|i-j||}$.

---

[*] Research supported by J.C.Bose Fellowship, Dept. of Science and Technology, Govt. of India.

[†] Research supported by CSIR Fellowship, Govt. of India.

The spectral norm $\|A\|$ of a matrix A with complex entries is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A:

$$\|A\| = \sqrt{\lambda_{max}(A^*A)}$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n \times n$ real symmetric matrix or A is a normal matrix then

$$\|A\| = \max_{1 \le i \le n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A.

For the existing limit results on spectral norms, see Silverstein (1994), Adamczak (2008), Bose and Sen (2007), Meckes (2007), Bryc and Sethuraman (2009) and Bose, Hazra and Saha (2009). All these works are for the situation when the entries have finite moment of at least order two.

Soshnikov (2004) shows distributional convergence of the maximum eigenvalue of appropriately scaled Wigner matrix with heavy tailed entries $\{x_{ij}\}$ satisfying $\mathrm{P}(|x_{ij}| > x) = h(x)x^{-\alpha}$ where h is slowly varying function at infinity and $0 < \alpha < 2$. The limiting distribution is $\Phi_\alpha(x) = \exp(-x^{-\alpha})$. A similar result was proved for sample covariance matrices in Soshnikov (2006). These results were extended in Auffinger, Ben Arous and Peche (2008) to $2 \le \alpha < 4$.

We focus on the above listed four matrices when the input sequence is heavy tailed and $0 < \alpha < 1$. We establish distributional convergence of the spectral norm of the three circulant matrices. Though we are unable to obtain the exact limit in the Toeplitz case, we provide upper and lower bounds. Our approach is to exploit the structure of the matrices and use existing methods on the study of maximum of periodograms for heavy tailed sequences.

It seems to be a nontrivial problem to derive properties of the spectral norm in the case of moving average process inputs. We resort to scaling each eigenvalue by the power transfer function (defined in Section 3) at the appropriate ordinate as described below and then consider their maximums. For any of the above mentioned matrix A_n, we define $\mathrm{M}(A_n, f) = \max_{1 \le k \le n} \frac{|\lambda_k|}{\sqrt{2\pi f(\omega_k)}}$ where f is the power transfer function corresponding to $\{x_n\}$ and $\{\lambda_k\}$ are eigenvalues of A_n. Similar scaling has been used in the study of periodograms (see Davis and Mikosch (1999), Lin and Liu (2009), Mikosch, Resnick and Samorodnitsky (2000)). We show the distributional convergence of $\mathrm{M}(A_n, f)$ for the three circulant matrices.

2 Results for i.i.d. input

Let $\{Z_t,\ t \in \mathbb{Z}\}$ be a sequence of i.i.d random variables with common distribution F where F is in the domain of attraction of an α-stable random variable with $0 < \alpha < 1$. Thus, there exist $p, q \ge 0$ with $p + q = 1$ and a slowly varying function $L(x)$, such that

$$\lim_{x\to\infty} \frac{\mathrm{P}(Z_1 > x)}{\mathrm{P}(|Z_1| > x)} = p, \ \lim_{x\to\infty} \frac{\mathrm{P}(Z_1 \le -x)}{\mathrm{P}(|Z_1| > x)} = q \text{ and } \mathrm{P}(|Z_1| > x) \sim x^{-\alpha}L(x) \text{ as } x \to \infty. \tag{2.1}$$

A random variable Y_α is said to have a stable distribution $S_\alpha(\sigma, \beta, \mu)$ if there are parameters $0 < \alpha \le 2, \sigma \ge 0, -1 \le \beta \le 1$ and μ real such that its characteristic function has the form

$$\mathrm{E}[\exp(itY_\alpha)] = \begin{cases} \exp\{i\mu t - \sigma^\alpha |t|^\alpha (1 - i\beta \,\mathrm{sgn}(t) \tan(\pi\alpha/2))\}, & \text{if } \alpha \ne 1, \\ \exp\{i\mu t - \sigma|t|(1 + (2i\beta/\pi) \,\mathrm{sgn}(t) \ln|t|)\}, & \text{if } \alpha = 1. \end{cases}$$

If $\beta = \mu = 0$, then Y_α is symmetric α-stable ($S\alpha S$). For details on stable processes see Samorodnitsky and Taqqu (1994).

In the description of our results, we shall need the following: $\{\Gamma_j\}$, $\{U_j\}$ and $\{B_j\}$, are three independent sequences defined on the same probability space where $\{\Gamma_j\}$ is the arrival sequence of a unit rate poisson process on $\mathbb{R}$, U_j are i.i.d $U(0,1)$ and B_j's are i.i.d satisfying

$$\mathrm{P}(B_1 = 1) = p \text{ and } \mathrm{P}(B_1 = -1) = q, \tag{2.2}$$

where p and q are defined in (2.1). We also define

$$Y_\alpha = \sum_{j=1}^{\infty} \Gamma_j^{-1/\alpha} \sim S_\alpha(C_\alpha, 1, 0) \text{ where } C_\alpha = \left(\int_0^\infty x^{-\alpha} \sin x dx\right)^{-1}.$$

Let $b_n = n^{1/\alpha}$. The eigenvalues of $b_n^{-1} RC_n$ are given by (see Bose and Mitra (2002)):

$$\begin{cases} \lambda_{n,x}(\omega_0) & = b_n^{-1} \sum_{t=0}^{n-1} x_t \\ \lambda_{n,x}(\omega_{n/2}) & = b_n^{-1} \sum_{t=0}^{n-1} (-1)^t x_t, \quad \text{if } n \quad \text{is even} \\ \lambda_{n,x}(\omega_k) = -\lambda_{n,x}(\omega_{n-k}) & = \sqrt{I_{n,x}(\omega_k)}, \ 1 \le k \le [\frac{n-1}{2}]. \end{cases} \tag{2.3}$$

where

$$I_{n,x}(\omega_k) = \frac{1}{b_n} |\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 \text{ and } \omega_k = \frac{2\pi k}{n}.$$

The eigenvalues of $b_n^{-1} C_n$ are given by

$$\lambda_{n,x}(\omega_j) = b_n^{-1} \sum_{t=1}^{n} x_j e^{it\omega_j}, \quad 1 \le j \le n.$$

Note that $\{|\lambda_{n,x}(\omega_k)|^2;\ 1 \le k < n/2\}$ is the periodogram of $\{x_i\}$ at the frequencies $\{\omega_k = \frac{2\pi k}{n};\ 1 \le k < n/2\}$. From the eigenvalue structure of C_n and RC_n, it is clear that $\|b_n^{-1} C_n\| = \|b_n^{-1} RC_n\|$ and therefore they have identical limiting behavior.

Theorem 1. *Assume that the input sequence is i.i.d.* $\{Z_t\}$ *satisfying (2.1). Then for* $\alpha \in (0,1)$, $\|b_n^{-1} C_n\| \Rightarrow Y_\alpha$ *and* $\|b_n^{-1} RC_n\| \Rightarrow Y_\alpha$.

The eigenvalues of $b_n^{-1} SC_n$ are given by:

(i) for n odd:

$$\begin{cases} \lambda_{n,x}(\omega_0) & = b_n^{-1} \big[x_0 + 2 \sum_{j=1}^{[n/2]} x_j\big] \\ \lambda_{n,x}(\omega_k) & = b_n^{-1} \big[x_0 + 2 \sum_{j=1}^{[n/2]} x_j \cos(\omega_k j)\big], \quad 1 \le k \le [n/2] \end{cases} \tag{2.4}$$

(ii) for n even:

$$\begin{cases} \lambda_{n,x}(\omega_0) & = b_n^{-1} \big[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j + x_{n/2}\big] \\ \lambda_{n,x}(\omega_k) & = b_n^{-1} \big[x_0 + 2 \sum_{j=1}^{\frac{n}{2}-1} x_j \cos(\omega_k j) + (-1)^k x_{n/2}\big], \quad 1 \le k \le \frac{n}{2} \end{cases} \tag{2.5}$$

with $\lambda_{n,x}(\omega_{n-k}) = \lambda_{n,x}(\omega_k)$ in both cases.

Theorem 2. *Assume that the input sequence is i.i.d.* $\{Z_t\}$ *satisfying (2.1). Then for* $\alpha \in (0,1)$, $\|b_n^{-1} SC_n\| \Rightarrow 2^{1-1/\alpha} Y_\alpha$.

Resolving the question of the exact limit of the Toeplitz spectral norm seems to very difficult. Here we provide a good upper and lower bound in the distribution sense.

Theorem 3. *Suppose that the input sequence is i.i.d.* $\{Z_t\}$ *satisfying (2.1). Then for* $\gamma > 0$,

$$P\big(2\sum_{j=1}^{\infty}(1-U_j)\Gamma_j^{-1/\alpha} > \gamma\big) \leq \liminf_n P\big(b_n^{-1}\|T_n\| > \gamma\big) \leq \limsup_n P\big(b_n^{-1}\|T_n\| > \gamma\big) \leq P\big(2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} > \gamma\big).$$

3 RESULTS FOR DEPENDENT INPUTS

Now suppose the input sequence is a linear process $\{X_t, t \in \mathbb{Z}\}$ given by

$$X_t = \sum_{j=-\infty}^{\infty} a_j Z_{t-j}, \quad t \in \mathbb{Z}, \text{ where } \sum_{j=-\infty}^{\infty} |a_j|^{\alpha-\epsilon} < \infty \text{ for some } 0 < \epsilon < \alpha. \tag{3.1}$$

Let

$$\psi(x) = \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi x j), \quad x \in [0,1]$$

be the *transfer function* of the linear filter $\{a_j\}$ and $f_X(x)$ be the *power transfer function* of $\{X_t\}$. Then

$$f_X(x) = |\psi(x)|^2.$$

Define

$$\mathrm{M}(RC_n, f_X) = \max_{1\leq k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}}, \ \mathrm{M}(C_n, f_X) = \max_{1\leq k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}}, \ \mathrm{M}(SC_n, f_X) = \max_{1\leq k<\frac{n}{2}} \frac{|\lambda_k|}{\sqrt{f_X(k/n)}}$$

where in each case $\{\lambda_k\}$ are the eigenvalues of the corresponding matrix. Also from the eigenvalue structure of C_n and SC_n, $\mathrm{M}(C_n, f_X) = \mathrm{M}(RC_n, f_X)$.

Theorem 4. *Assume that* $\{X_n\}$ *and* $\{a_j\}$ *satisfy (3.1) and* $\{Z_t\}$ *is i.i.d satisfying (2.1). Suppose* f_X *is strictly positive on* $[0, 1/2]$. *Then*

(a) $M(b_n^{-1}C_n, f_X) \Rightarrow Y_\alpha$ *and* $M(b_n^{-1}RC_n, f_X) \Rightarrow Y_\alpha$.

(b) Further, if $a_j = a_{-j}$, *then* $M(b_n^{-1}SC_n, f_X) \Rightarrow 2^{1-1/\alpha}Y_\alpha$.

4 PROOFS OF RESULTS

Main idea of the proofs is taken from Mikosch, Resnick and Samorodnitsky (2000) who show weak convergence of maximum of the periodogram based on heavy tailed sequence for $\alpha < 1$. Let $\epsilon_x(\cdot)$ denote the point measure which gives unit mass to any set containing x and let $E = [0,\ 1] \times ([-\infty, \infty]\backslash\{0\})$. Let $M_p(E)$ be the set of point measures on E, topologized by vague convergence. The following convergence result follows from Proposition 3.21 of Resnick (1987):

$$N_n := \sum_{k=1}^{n} \epsilon_{(k/n, Z_k/b_n)} \Rightarrow N := \sum_{j=1}^{\infty} \epsilon_{(U_j, B_j\Gamma_j^{-1/\alpha})} \ \text{ in } \ M_p(E). \tag{4.1}$$

Suppose f is a bounded continuous complex valued function defined on $\mathbb{R}$ and without loss of generality assume $|f(x)| \leq 1$ for all $x \in \mathbb{R}$. Now pick $\eta > 0$ and define $T_\eta : M_p(E) \longrightarrow C[0,\infty)$ as follows:

$$(T_\eta m)(x) = \sum_j v_j 1_{\{|v_j|>\eta\}} f(2\pi x t_j)$$

if $m = \sum_j \epsilon_{(t_j,v_j)} \in M_p(E)$ and v_j's are finite. Elsewhere, set $(T_\eta m)(x) = 0$. The following Lemma was proved by Mikosch, Resnick and Samorodnitsky (2000) using the function $f(x) = \exp(-ix)$. Same proof works in our case.

Lemma 1. *$T_\eta : M_p(E) \longrightarrow C[0,\infty)$ is continuous a.s. with respect to the distribution of N.*

Lemma 2. *For $0 < \alpha < 1$, as $n \to \infty$ the following convergence holds in $C[0,\infty)$:*

$$J_{n,Z}(x/n) := \sum_{j=1}^{n} \frac{Z_j}{b_n} f(2\pi x j/n) \Rightarrow J_\infty(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j), \quad 0 \le x < \infty.$$

Proof. Applying Lemma 1 on (4.1) we have

$$\begin{aligned} J_{n,Z}^{(\eta)}(x/n) &:= \sum_{j=1}^{n} \frac{Z_j}{b_n} f(2\pi x j/n) 1_{\{|Z_j| > \eta b_n\}} \\ &\Rightarrow \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j) 1_{\{\Gamma_j^{-1/\alpha} > \eta\}} := J_\infty^{(\eta)}(x) \text{ in } C[0,\infty). \end{aligned}$$

Also, as $\eta \to 0$ by dominated convergence theorem we have

$$J_\infty^{(\eta)}(x) \Rightarrow J_\infty(x) := \sum_{j=1}^{\infty} B_j \Gamma_j^{-1/\alpha} f(2\pi x U_j).$$

So using Theorem 4.2 of Billingsley (1968), the proof will be complete if for any $\epsilon > 0$,

$$\lim_{\eta \to 0} \limsup_{n \to \infty} \mathrm{P}\big(\|J_{n,Z}^{(\eta)} - J_{n,Z}\| > \epsilon\big) = 0, \tag{4.2}$$

where $\|x(\cdot) - y(\cdot)\|$ is the metric distance in $C[0,\infty)$ given by

$$\|x(\cdot) - y(\cdot)\| = \sum_{n=1}^{\infty} \frac{1}{2^n} \left[\|x(\cdot) - y(\cdot)\|_n \wedge 1\right], \quad \text{where } \|x(\cdot) - y(\cdot)\|_n = \sup_{t \in [0,n]} |x(t) - y(t)|.$$

Now

$$\begin{aligned} \lim_{\eta \to 0} \limsup_{n \to \infty} \mathrm{P}\big(\|J_{n,Z}^{(\eta)} - J_{n,Z}\| > \epsilon\big) &\le \lim_{\eta \to 0} \limsup_{n \to \infty} \mathrm{P}\Big(\sum_{j=1}^{n} |\frac{Z_j}{b_n}| 1_{\{|Z_j| \le \eta b_n\}} > \epsilon\Big) \\ &\le \lim_{\eta \to 0} \limsup_{n \to \infty} n \epsilon^{-1} \mathrm{E}\Big(|\frac{Z_1}{b_n}| 1_{\{|Z_j| \le \eta b_n\}}\Big), \end{aligned}$$

and using an application of Karamata's theorem (see Resnick (1987) Exercise 0.4.2.8) we get that

$$n \mathrm{E}\Big(|\frac{Z_1}{b_n}| 1_{\{|Z_j| \le \eta b_n\}}\Big) \sim \frac{\alpha}{1-\alpha} n \eta \mathrm{P}(|Z_1| > \eta b_n) \sim \frac{\alpha}{1-\alpha} \eta^{1-\alpha} \to 0 \text{ as } \eta \to 0.$$

This completes the proof of the lemma. □

Proof of Theorem 1. We use Lemma 1 and Lemma 2 with $f(x) = \exp(-ix)$. It is immediate that

$$b_n^{-1}\|C_n\| \leq b_n^{-1}\sum_{t=1}^{n}|Z_t|. \tag{4.3}$$

It is well known [cf. Feller (1971)] that

$$b_n^{-1}\sum_{t=1}^{n}|Z_t| \Rightarrow Y_\alpha = \sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} \sim S_\alpha(C_\alpha^{-1/\alpha}, 1, 0). \tag{4.4}$$

Hence it remains to show that for $\gamma > 0$,

$$\liminf_{n\to\infty} \mathrm{P}(b_n^{-1}\|C_n\| > \gamma) \geq \mathrm{P}(Y_\alpha > \gamma). \tag{4.5}$$

Now observe that for any integer K and sufficiently large n,

$$\mathrm{P}\big(\sup_{j=1,\ldots,[n/2]}|J_{n,Z}(j/n)| > \gamma\big) \geq \mathrm{P}\big(\sup_{j=1,\ldots,K}|J_{n,Z}(j/n)| > \gamma\big).$$

Now from Lemma 2 we have

$$\big(J_{n,Z}(j/n), 1 \leq j \leq K\big) \Rightarrow \big(J_\infty(j), 1 \leq j \leq K\big)$$

in $\mathbb{R}^k$, hence

$$\sup_{j=1,\ldots,K}|J_{n,Z}(j/n)| \Rightarrow \sup_{j=1,\ldots,K}|J_\infty(j)|$$

and so letting $K \to \infty$,

$$\liminf_{n\to\infty}\mathrm{P}\big(\sup_{j=1,\ldots,[n/2]}|J_{n,Z}(j/n)| > \gamma\big) \geq \mathrm{P}\big(\sup_{j=1,\ldots,\infty}|J_\infty(j)| > \gamma\big).$$

Now the theorem follows from Lemma 3. □

Lemma 3.

$$\sup_{j=1,\ldots,\infty}|J_\infty(j)| = \sup_{j=1,\ldots,\infty}\Big|\sum_{t=1}^{\infty}B_t\Gamma_t^{-1/\alpha}\exp(-2\pi ijU_t)\Big| = Y_\alpha \quad a.e.$$

Proof. Define

$$\Omega_0 = \Big\{\omega \in \Omega : \sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}(\omega) < \infty \text{ and for all } m \geq 1, (U_1(\omega), \ldots, U_m(\omega)) \text{ are rationally independent}\Big\}.$$

Then $\mathrm{P}(\Omega_0) = 1$. Let $\overline{x}$ denote the fractional part of x. For any $\omega \in \Omega_0$, by Weyl (1916),

$$\big(\overline{nU_1(\omega)}, \ldots, \overline{nU_m(\omega)}\big),\ n \in \mathbb{N}$$

is dense in $[0,1]^m$. Fix any $\omega \in \Omega_0$ and $\epsilon > 0$. Then there exist an $N \in \mathbb{N}$ such that $\sum_{j=N+1}^{\infty}\Gamma_j^{-1/\alpha}(\omega) < \epsilon$ and from Weyl's result there exist a $N_0 \in \mathbb{N}$ such that

$$\mathrm{Real}\big(B_j\exp(-2\pi iN_0U_j)\big) \geq 1 - \frac{\epsilon}{N\Gamma_j^{-1/\alpha}},\ \ j = 1, \ldots, N.$$

Then we have

$$\begin{aligned}\sup_{j=1,\ldots,\infty}\Big|\sum_{t=1}^{\infty}B_j\Gamma_j^{-1/\alpha}\exp(-2\pi ijU_t)\Big| &\geq \sup_{j=1,\ldots,\infty}\Big|\sum_{t=1}^{N}B_j\Gamma_j^{-1/\alpha}\exp(-2\pi ijU_t)\Big| - \sum_{t=N+1}^{\infty}\Gamma_t^{-1/\alpha}\\ &\geq \Big|\sum_{t=1}^{N}B_t\Gamma_t^{-1/\alpha}\exp(-2\pi iN_0U_t)\Big| - \epsilon\\ &\geq \mathrm{Real}\Big(\sum_{t=1}^{N}B_t\Gamma_t^{-1/\alpha}\exp(-2\pi iN_0U_t)\Big) - \epsilon\\ &\geq \sum_{t=1}^{N}\big(1-\frac{\epsilon}{N\Gamma_t^{-1/\alpha}}\big)\Gamma_t^{-1/\alpha} - \epsilon = \sum_{t=1}^{N}\Gamma_t^{-1/\alpha} - 2\epsilon.\end{aligned}$$

Letting first $N \to \infty$ and then $\epsilon \to 0$, we get $\sup_{j=1,\ldots,\infty}|J_\infty(j)| \geq Y_\alpha$. Trivially $\sup_{j=1,\ldots,\infty}|J_\infty(j)| \leq Y_\alpha$. This completes the proof. □

Proof of Theorem 2. The proof is similar to the proof of Theorem 1. We provide the proof for n odd and for n even, the changes needed are minor. Define

$$J_{n,Z}(x) := 2b_n^{-1}\sum_{t=1}^{q}Z_t\cos(2\pi xt) \quad\text{and}\quad M_{n,Z} := \max_{0\leq k\leq q}\big|J_{n,Z}(k/n)\big|, \tag{4.6}$$

where $q = q_n = [\frac{n}{2}]$. Since $\big|\|b_n^{-1}SC_n\| - M_{n,Z}\big| \to 0$ almost surely, it is enough to show $M_{n,Z} \Rightarrow 2^{1-1/\alpha}Y_\alpha$. Now (4.1) holds with [0, 1] replaced by [0, 1/2], and letting $N_n = \sum_{k=1}^{q}\epsilon_{(k/n, Z_k/b_q)}$, $N = \sum_{j=1}^{\infty}\epsilon_{(U_j, B_j\Gamma_j^{-1/\alpha})}$ and U_j to be i.i.d. $U[0,1/2]$. Now following the argument given in Lemma 1, Lemma 2 and taking $f(x) = \cos(x)$ it is easy to establish that

$$J_{n,Z}(x/n) = 2b_n^{-1}\sum_{k=1}^{q}Z_k\cos\frac{2\pi kx}{n} \Rightarrow 2^{1-1/\alpha}\sum_{j=1}^{\infty}B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j) := J_\infty(x). \tag{4.7}$$

It is obvious that

$$M_{n,Z} \leq 2b_n^{-1}\sum_{t=1}^{q}|Z_t| \Rightarrow 2^{1-1/\alpha}\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} = 2^{1-1/\alpha}Y_\alpha.$$

It remains to show that for $\eta > 0$, $\liminf_{n\to\infty}\mathrm{P}(M_{n,Z} > \eta) \geq \mathrm{P}(2^{1-1/\alpha}Y_\alpha > \eta)$. Now following the arguments given to prove (4.5), we can establish the last relation. This completes the proof of the theorem. □

Proof of Theorem 3. Following Meckes (2007), T_n is a submatrix of the infinite Laurent matrix

$$L_n = \big[Z_{|j-k|}1_{|j-k|\leq n-1}\big]_{j,k\in\mathbb{Z}}$$

so $\|T_n\| \leq \|L_n\|$, where $\|L_n\|$ denotes the operator norm of L_n acting in the standard way on $l_2(\mathbb{Z})$. If we use the Fourier basis to identify $l_2(\mathbb{Z})$ with $L_2[0,1]$, it turns out that L_n corresponds to a multiplication operator, with the multiplier

$$g(x) = \sum_{j=-(n-1)}^{n-1}Z_{|j|}e^{2\pi ijx} = Z_0 + 2\sum_{j=1}^{n-1}\cos(2\pi jx)Z_j.$$

Therefore

$$\|T_n\| \le \|L_n\| = \|g\|_\infty = \sup_{0\le x\le 1} |g(x)|.$$

Hence as $n \to \infty$,

$$b_n^{-1}\|T_n\| \le b_n^{-1}\big[|Z_0| + 2\sum_{j=0}^{n-1}|Z_j|\big] \Rightarrow 2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha}$$

and we have for $\gamma > 0$

$$\limsup_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \le \mathrm{P}\big(2\sum_{j=1}^{\infty}\Gamma_j^{-1/\alpha} > \gamma\big).$$

By another argument of Meckes (2007), we get the following estimate

$$\|T_n\| = \sup_{v\in\mathbb{C}^n\setminus\{0\}} \frac{\langle T_n v, v\rangle}{\langle v, v\rangle} \ge \sup_{0\le x\le 1} \frac{1}{n}|\langle T_n v_x, v_x\rangle|,$$

where $v_x \in \mathbb{C}^n$ is defined as $(v_x)_j = e^{2\pi i x j}$ for $j = 1, 2, \ldots, n$ and $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{C}^n$. Therefore

$$\begin{aligned}
\|T_n\| &\ge \frac{1}{n}\sup_{0\le x\le 1}\big|\sum_{j,k=1}^{n} Z_{|j-k|}e^{2\pi i(j-k)x}\big| \\
&= \frac{1}{n}\sup_{0\le x\le 1}\big|\sum_{j=-(n-1)}^{n-1}(n-|j|)Z_{|j|}e^{2\pi i j x}\big| \\
&= \sup_{0\le x\le 1}\big|Z_0 + 2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi j x)\big|.
\end{aligned}$$

Now

$$\begin{aligned}
\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) &\ge \liminf_n \mathrm{P}\big(b_n^{-1}\sup_{0\le x\le 1}\big|Z_0 + 2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi j x)\big| > \gamma\big) \\
&= \lim_n \mathrm{P}\big(b_n^{-1}\sup_{0\le x\le 1}\big|2\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi j x)\big| > \gamma\big)
\end{aligned}$$

To find the limit in the last expression, pick $\eta > 0$ and define $T_\eta : M_p(E) \longrightarrow C[0,\infty)$, as follows:

$$(T_\eta m)(x) = \begin{cases} \sum_j (1-t_j)v_j\cos(2\pi x t_j)1(|v_j| > \eta) & \text{if } m = \sum_j \epsilon_{(t_j,v_j)},\ \text{all } v_j's \text{ are finite} \\ 0 & \text{otherwise} \end{cases}$$

Following the argument given in Lemma 1, it is easy to see T_η is continuous a.s. with respect to the distribution of N and then using an argument from Lemma 2, we can show that for fixed x

$$2b_n^{-1}\sum_{j=1}^{n-1}(1-j/n)Z_j\cos(2\pi j x/n) \Rightarrow 2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi x U_j). \tag{4.8}$$

Now for any fixed T where $n > T > 0$,

$$\begin{aligned}\sup_{0\le x\le 1}\big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos(2\pi jx)\big| &= \sup_{0\le x\le n}\big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos\frac{2\pi jx}{n}\big| \\ &\ge \sup_{0\le x\le T}\big|2b_n^{-1}\sum_{j=1}^{n-1}\big(1-\frac{j}{n}\big)Z_j\cos\frac{2\pi jx}{n}\big| \\ &\Rightarrow \sup_{0\le x\le T}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big|,\end{aligned}$$

and using (4.8) we have

$$\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \ \ge\ \mathrm{P}\big(\sup_{0\le x\le T}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| > \gamma\big).$$

Since this is true for any T, we obtain

$$\liminf_n \mathrm{P}\big(b_n^{-1}\|T_n\| > \gamma\big) \ge \mathrm{P}\big(\sup_{0\le x<\infty}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| > \gamma\big).$$

Now to identify the distribution of $\sup_{0\le x\le\infty}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big|$ we follow Lemma 3. Here we use the fact $\big\{(\overline{xU_1(\omega)},\dots,\overline{xU_m(\omega)}),x\ge 0\big\}$ is dense in $[0,1]^m$ and we get

$$\sup_{0\le x<\infty}\big|2\sum_{j=1}^{\infty}(1-U_j)B_j\Gamma_j^{-1/\alpha}\cos(2\pi xU_j)\big| \Rightarrow \sum_{j=1}^{\infty}(1-U_j)\Gamma_j^{-1/\alpha}.$$

This completes the proof. □

Proof of Theorem 4(a). To prove the result it is enough to show that

$$\big|\mathrm{M}(b_n^{-1}C_n, f_X) - \|b_n^{-1}C_n\|\big| \xrightarrow{\mathcal{P}} 0.$$

Let $J_{n,Z}(x) = b_n^{-1}\sum_{t=1}^{n} Z_t\exp(-i2\pi xt)$. Note

$$\begin{aligned}\big|\mathrm{M}(b_n^{-1}C_n, f_X) - \|b_n^{-1}C_n\|\big| &= \big|\sup_{1\le k\le n}(f_X(k/n))^{-1/2}|J_{n,X}(k/n)| - \sup_{1\le k\le n}|J_{n,Z}(k/n)|\big| \\ &\le \sup_{1\le k\le n}\big||\psi(k/n)J_{n,X}(k/n)| - |J_{n,Z}(k/n)|\big| \\ &\le \sup_{1\le k\le n}\big|\psi(k/n)J_{n,X}(k/n) - J_{n,Z}(k/n)\big|\end{aligned}$$

and

$$\begin{aligned}J_{n,X}(x) &= b_n^{-1}\sum_{t=1}^{n} X_t\exp(-i2\pi xt) \\ &= b_n^{-1}\sum_{j=-\infty}^{\infty} a_j\exp(-i2\pi xj)\big(\sum_{t=1}^{n} Z_t\exp(-i2\pi xt) + V_{n,j}\big)\end{aligned}$$

$$= \quad \psi(x) J_{n,Z}(x) + Y_n(x), \tag{4.9}$$

where

$$V_{n,j} = \sum_{t=1-j}^{n-j} Z_t \exp(-i2\pi xt) - \sum_{t=1}^{n} Z_t \exp(-i2\pi xt),\ Y_n(x) = b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) V_{n,j}.$$

Since f_X is bounded away from 0 and ∞ and in view of relation (4.9), it is enough to show that $\max_{1\le k\le n} |Y_n(k/n)| \xrightarrow{\mathcal{P}} 0$. Now

$$\begin{aligned} Y_n(x) &= b_n^{-1} \sum_{j=n+1}^{\infty} a_j \exp(-i2\pi xj) V_{n,j} + b_n^{-1} \sum_{j=1}^{n} a_j \exp(-i2\pi xj) V_{n,j} \\ & \quad b_n^{-1} \sum_{j=-\infty}^{-n-1} a_j \exp(-i2\pi xj) V_{n,j} + b_n^{-1} \sum_{j=-n}^{-1} a_j \exp(-i2\pi xj) V_{n,j} \\ &= S_1(x) + S_2(x) + S_3(x) + S_4(x). \end{aligned}$$

Now following similar argument given in the proof of Lemma (2.6) in Mikosch, Resnick and Samorodnitsky (2000), we can show that

$$\max_{1\le k\le n} |S_1(k/n)| \xrightarrow{\mathcal{P}} 0 \quad \text{and} \quad \max_{1\le k\le n} |S_2(k/n)| \xrightarrow{\mathcal{P}} 0.$$

Behavior of $S_3(x)$ and $S_4(x)$ are similar as $S_1(x)$ and $S_2(x)$ respectively. Therefore following similar argument we can show that $\max_{1\le k\le n} |S_3(k/n)| \xrightarrow{\mathcal{P}} 0$ and $\max_{1\le k\le n} |S_4(k/n)| \xrightarrow{\mathcal{P}} 0$. This completes the proof of part (a).

Proof of Theorem 4(b). In view of Theorem 2, it is enough to show that

$$\left| \mathrm{M}(b_n^{-1} SC_n, f_X) - \|b_n^{-1} SC_n\| \right| \xrightarrow{\mathcal{P}} 0.$$

Let

$$\begin{aligned} J_{n,Z}(x) &:= 2 b_n^{-1} \sum_{t=1}^{q} Z_t \cos(2\pi xt) \\ &= b_n^{-1} \sum_{t=1}^{q} Z_t \exp(i2\pi xt) - b_n^{-1} \sum_{t=1}^{q} Z_t \exp(-i2\pi xt). \end{aligned}$$

Then using $a_j = a_{-j}$ we have

$$\begin{aligned} J_{n,X}(x) &:= b_n^{-1} \sum_{t=1}^{q} X_t \exp(i2\pi xt) - b_n^{-1} \sum_{t=1}^{q} X_t \exp(-i2\pi xt) \\ &= b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) \big(\sum_{t=1}^{n} Z_t \exp(i2\pi xt) + U_{n,j} \big) \\ & \quad - b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) \big(\sum_{t=1}^{n} Z_t \exp(-i2\pi xt) + V_{n,j} \big) \\ &= \psi(x) J_{n,Z}(x) + Y_{1n}(x) - Y_{2n}(x), \end{aligned}$$

where

$$U_{n,j} = \sum_{t=1+j}^{q+j} Z_t \exp(i2\pi xt) - \sum_{t=1}^{q} Z_t \exp(i2\pi xt), \;\; V_{n,j} = \sum_{t=1-j}^{n-j} Z_t \exp(-i2\pi xt) - \sum_{t=1}^{n} Z_t \exp(-i2\pi xt),$$

$$Y_{1n} = b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) U_{n,j}, \quad Y_{2n} = b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) V_{n,j}.$$

Since f_X is bounded away from 0 and ∞, it is enough to show that

$$\sup_{1\le k\le q} \left| J_{n,X}(k/n) - \psi(k/n) J_{n,Z}(k/n) \right| \;\; \le \;\; \sup_{1\le k\le q} \left| Y_{1n}(k/n) \right| + \sup_{1\le k\le q} \left| Y_{2n}(k/n) \right| \xrightarrow{\mathcal{P}} 0.$$

Now

$$\begin{aligned} Y_{1n}(x) &= b_n^{-1} \sum_{j=-\infty}^{\infty} a_j \exp(-i2\pi xj) U_{n,j} \\ &= b_n^{-1} \sum_{j=q+1}^{\infty} a_j \exp(-i2\pi xj) U_{n,j} + b_n^{-1} \sum_{j=1}^{q} a_j \exp(-i2\pi xj) U_{n,j} \\ &\quad b_n^{-1} \sum_{j=-\infty}^{-q-1} a_j \exp(-i2\pi xj) U_{n,j} + b_n^{-1} \sum_{j=-q}^{-1} a_j \exp(-i2\pi xj) U_{n,j} \\ &= S_1(x) + S_2(x) + S_3(x) + S_4(x). \end{aligned}$$

Again following similar argument of of Lemma (2.6) in Mikosch, Resnick and Samorodnitsky (2000), we can show that $\sup_{1\le k\le q} \left| S_i(k/n) \right| \xrightarrow{\mathcal{P}} 0$ for $= 1, \ldots, 4$. Hence $\sup_{1\le k\le q} \left| Y_{1n}(k/n) \right| \xrightarrow{\mathcal{P}} 0$. Similarly $\sup_{1\le k\le q} \left| Y_{2n}(k/n) \right| \xrightarrow{\mathcal{P}} 0$. This completes the proof of part (b). □

5 CONCLUDING REMARKS

All the results stated in Section 2 require $\alpha \in (0,1)$ but the case when $\alpha \in [1,2)$ is a very non trivial problem. In the reverse circulant case we saw that the eigenvalue structure is similar to the square root of the periodogram and the maximum of the periodogram is not tight with the scaling $n^{1/\alpha}$ when $\alpha \ge 1$. Instead it is tight with a different scaling (see Mikosch, Resnick and Samorodnitsky (2000), Section 3 for details). We are actively trying to see what happens in this case for other matrices.

REFERENCES

Adamczak, Radosław. A few remarks on the operator norm of random Toeplitz matrices. *Available at http://arxiv.org/abs//0803.3111.* To appear in J. Theoret. Probab. (DOI 10.1007/s10959-008-0201-7).

Auffinger, Antonio, Ben Arous, Gerard and Peche, Sandrine. Poisson convergence for the largest eigenvalues of Heavy Tailed Random Matrices. *arXiv:0710.3132.* To appear in Annales de l'Institut Henri Poincare.

Bose, Arup, Hazra, Rajat Subhra, and Saha, Koushik. Spectral norm of circulant type matrices. *Technical Report No.R7/2009, Stat-Math Unit, Indian Statistical Institute, Kolkata, May 09, 2009.* Submitted for publication. To appear in J. Theoret. Probab.

Bose, Arup and Mitra, Joydip. Limiting spectral distribution of a special circulant. *Statist. Probab. Lett.*, 60(1):111–120, 2002.

Bose, Arup and Sen, Arnab. Spectral norm of random large dimensional noncentral Toeplitz and Hankel matrices *Electronic Communications in Probability*, 12: 29–35(electronic), 2007.

Bryc, Wlodek and Sethuraman, Sunder. A remark on maximum eigenvalue for circulant matrices, 2009. *To appear in IMS volume High Dimensional Probability Luminy conference proceedings.*

Davis, Richard A. and Mikosch, Thomas. The maximum of the periodogram of a non-Gaussian sequence *Ann. Probab.*, 27:no. 1, 522–536, 1999.

Lin, Zhengyan and Liu, Weidong. On maxima of periodograms of stationary processes. *Ann. Statist.*, 37, no. 5B, 2676-2695., 2009.

Meckes, Mark W. On the spectral norm of a random Toeplitz matrix *Electron. Comm. Probab.* 12, 315–325, 2007.

Mikosch, Thomas and Resnick, Sidney and Samorodnitsky, Gennady. The maximum of the periodogram for a heavy-tailed sequence. *Ann. Probab.*, 28:no. 2, 885–908, 2000.

Resnick, Sidney I. *Extreme values, regular variation, and point processes.* Applied Probability. A Series of the Applied Probability Trust. Springer-Verlag, New York, 1987.

Samorodnitsky, Gennady and Taqqu, Murad S. *Stable non-Gaussian random processes.*. Chapman & Hall, New York, 1994. ISBN 0-412-05171-0.

Silverstein, Jack W. The spectral radii and norms of large-dimensional non-central random matrices. *Comm. Statist. Stochastic Models*, 10, No. 3, 525–532, 1994.

Soshnikov, Alexander. Poisson statistics for the largest eigenvalues of Wigner random matrices with heavy tails. *Electron. Comm. Probab*, 9, 82–91 (electronic), 2004. ISSN 1083-589X.

Soshnikov, Alexander. Poisson statistics for the largest eigenvalues in random matrix ensembles. *Mathematical physics of quantum mechanics*, 351–364. *Lecture Notes in Phys.*, 690, Springer, Berlin, 2006.