Pfeifer Records Process

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Abstract

We establish an appropriate functional central limit theorem for partial sums of independent variables and use it to derive the asymptotic behaviour of the Pfeifer records process.

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1 Introduction

Let $\{X_0, X_1, \ldots X_n, \ldots\}$ be i.i.d. random variables with distribution function F. Let

$$C_0 = X_0, \quad L_n = \min\{j : X_j > C_{n-1}\} \text{ and } C_n = X_{L_n}.$$

Then $\{C_n\}$ is said to be the sequence of (upper) records (from F). We shall call these *classical records.* Chandler (1952) initiated the study of record values and documented many of the basic properties of records. Asymptotic behaviour of classical records and their partial sums have been obtained by several authors. See for example, Arnold and Villaseñor [2], Bose et al. [3], Resnick [8] and [9].

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The Pfeifer record model of Pfeifer [7] is a generalisation of the classical setup. Suppose $X_{0,1}$ has the cdf F_0 and for every $n \geq 1$, $\{X_{n,j},\ j \geq 1\}$ are i.i.d. random variables with cdf F_n. Let

$$R_0 = X_{0,1},\ \Delta_n = \min\{j : X_{n,j} > R_{n-1}\} \text{ and } R_n = X_{n,\Delta_n}.$$

Then $\{R_n\}$ is said to be the sequence of (upper) *Pfeifer records* (from $\{F_n\}$). Pfeifer mainly concentrated on the sequence of $\{F_n\}$ which satisfy

$$1 - F_n(x) = (1 - F_0(x))^{\alpha_n}. \tag{1.1}$$

If there is no chance of confusion we suppress its dependence on F_0 and $\{\alpha_n\}$. Motivated by an example of a shock model, Pfeifer assumed that $\{\alpha_n\}$ is a nondecreasing sequence of positive real numbers. We too assume that α_n is nondecreasing tending to ∞. $\{R_n\}$ has the following nice representation in terms of partial sums of independent exponential random variables and that proves very useful in studying its properties:

Suppose $Y_0, Y_1, \ldots$ are independent random variables and

$$Y_i \sim Exp(\alpha_i) \tag{1.2}$$

and for any distribution function F,

$$\psi_F(x) = F^{-1}(1 - e^{-x}). \tag{1.3}$$

Then

$$\big(R_0, R_1, \ldots R_n\big) \overset{\mathcal{D}}{=} \big(\psi_{F_0}(Y_0), \psi_{F_0}(Y_0 + Y_1), \ldots \psi_{F_0}(Y_0 + Y_1 + \ldots Y_n)\big) \text{ for all } n.$$

Observe that the classical records have the above representation with $\alpha_n = 1$ for all n so that $\{Y_i\}$ are i.i.d. $Exp(1)$ random variables.

Arnold and Villaseñor [1] show that $\{R_n\}$, properly normalized, converges in distribution to normal or log-normal distributions under appropriate conditions on the relative growth rates of $s_n(:= (\sum_{i=1}^{n}(1/\alpha_n^2))^{1/2})$, $\nu_n(:= \sum_{i=1}^{n}(1/\alpha_n))$ and ψ^{-1}.

Our goal is to investigate the limiting behaviour of the process version of Pfeifer records under appropriate conditions. Towards that end, we consider the following framework. For any sequence of random variables $\{X_i\}$, let $\{\mu_i = \mathbb{E}(X_i)\}$ and $\{\sigma_i^2 = \mathbf{Var}(X_i)\}$ be their mean and variance sequence. Define

$$X_i - \mu_i =: Y_i,\ S_n := \sum_{i=1}^{n} Y_i \quad \nu_n := \sum_{i=1}^{n} \mu_i, \quad \text{and} \quad s_n^2 := \sum_{i=1}^{n} \sigma_i^2.$$

For all n, let

$$Z_n(0) = 0 \ \text{ and } \ Z_n(t) = \frac{S_{[nt]}}{s_n}, \ \text{for} \ \ t \in (0, 1]. \tag{1.4}$$

We first establish a suitable functional central limit theorem for $\{Z_n(\cdot)\}$ under general conditions on $\{X_i\}$ and then apply it to our situation with appropriate conditions.

Assumption **I**: $\{X_i\}$ is an independent sequence such that

(a) $\nu_n \in RV_\beta$ and $s_n \in RV_\alpha$ and $0 < \alpha < \beta$ and

(b) $\{X_i\}$ satisfies Lindeberg condition: For any $\epsilon > 0$,

$$\lim_{n \longrightarrow \infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}[X_i^2 I_{|X_i| > \epsilon s_i}] = 0.$$

In Proposition 1, under Assumption **I** we show that $\{Z_n(t)\}$ converges to a Gaussian process as a sequence of random elements of $D[0,\ 1]$.

Note that in case of Pfeifer records if $\alpha_n \in RV_s$ and $0 < s < 1/2$, then clearly Assumption **I**(a) is satisfied. In this case, Assumption **I**(b) is also satisfied (see Arnold and Villaseñor [1]).

In Theorem 1(i), we study the limiting behaviour of $V_n(t) := \psi(\sum_{i=1}^{[nt]} X_i)$ where ψ is a regularly varying function but not necessarily of the form given in (1.3). Clearly, functional limit of Pfeifer records is a particular case. In Theorem 1(ii), we consider the partial sums of $V_n(t)$.

Theorem 2(i) and 2(ii) deal with $V_n(t)$ and its partial sums when ψ is a rapidly varying function so that $(\log \psi)'(\nu_n) = o(\frac{1}{s_n})$. In this case, their respective limiting behaviour is same as in the case of regularly varying ψ. In Theorem 2(iii) we consider the case when $(\log \psi)'(\nu_n) = O(\frac{1}{s_n})$. Here $V_n(t)$ converges in distribution to exponential function of Brownian motion. But in this case, there does not seem to be any appropriate normalization (that depends on n and is free of t) for the partial sum process of $V_n(t)$ to converge to a nice process.

2 Main results

2.1 Regularly varying ψ

We need the following functional limit theorem on sums of independent random variables. It does not seem to be available in the literature in exactly this form. This is the basis of our arguments for obtaining functional limits of $V_n(t)$.

Proposition 1. *Under Assumption **I**, $Z_n(t)$ converges in distribution to $t^{\alpha-1/2}B(t)$, for $t \in [0, 1]$ where B is a standard Brownian motion.*

Assumption **II** ψ is increasing and differentiable with derivative $\psi' \in RV_{\rho-1}$.

In the sequel, throughout,

$$\gamma := \alpha - 1/2 + \beta(\rho - 1).$$

Theorem 1. *Suppose Assumptions **I** and **II** hold.*

(i) Then

$$\xi_n(t) := \frac{\psi(\sum_{i=1}^{[nt]} X_i) - \psi(\nu_{[nt]})}{s_n \psi'(\nu_n)}, \quad t \in (0,1]$$

is a D-valued process which converges weakly to the continuous process $t^\gamma B(t)$ in $D(0,1]$.

(ii) Assume further that $\gamma > -(3/2)$. We also assume that

$$\frac{\sum_{i=1}^n X_i}{\nu_n} \xrightarrow{a.s.} 1. \tag{2.1}$$

Then

$$\frac{\sum_{j=1}^{[nt]} \left[\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)\right]}{n s_n \psi'(\nu_n)} \xrightarrow{\mathcal{D}} \int_0^t s^\gamma B(s) ds. \tag{2.2}$$

Remark 1 (i) Theorem 1(ii) extends Theorem 2 of Bose et al. [3] for partial sum process of classical records. There it was assumed that $a(x) := \psi(x + \sqrt{x}) - \psi(x)$ is regularly varying. In their arguments, the central role is played by necessary and sufficient condition (due to Resnick) for convergence of classical records to normal distribution and $a(n)$ is the natural scaling there. Here regular variation of $\psi(x)$ along with other suitable restrictions work as sufficient conditions for our results to hold.

(ii) Putting $t = 1$, from the above Theorem we conclude that

$$\frac{\sum_{j=1}^{n} \left[\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)\right]}{n s_n \psi'(\nu_n)} \xrightarrow{\mathcal{D}} N(0,\ g(\gamma)) \text{ where } g(\gamma) = \frac{1}{(2\gamma+3)(\gamma+2)}.$$

(iii) Recall that for Pfeifer records, $X_i = (1/\alpha_i) X_i^*$, where $\alpha_i \in RV_s$, $0 < s < 1/2$ and $\{X_i^*\}$ is i.i.d. Exp(1). By Theorem 2 of Etemadi [6], condition (2.1) is then satisfied.

2.2 Rapidly varying ψ

We now assume ψ is rapidly varying.

Assumption **III**: $\psi(x) = \exp(f(x))$ where $f \in RV_\rho$, $\rho > 0$. f is differentiable and $f' \in RV_{\rho-1}$.

Theorem 2. *Suppose Assumptions **I** and **III** hold.*

(i) If $\gamma > -(3/2)$ and

$$s_n f'(\nu_n) \xrightarrow{\mathbb{P}} 0 \text{ as } n \longrightarrow \infty \tag{2.3}$$

then

$$\frac{\psi(\sum_{i=1}^{[nt]} X_i) - \psi(\nu_{[nt]})}{s_n \psi(\nu_{[nt]}) f'(\nu_n)} \xrightarrow{\mathcal{D}} t^\gamma B(t).$$

(ii) If $\gamma > -(3/2)$ *and*

$$(\sum_{i=1}^{n} X_i - \nu_n) f'(\nu_n) \xrightarrow{a.s.} 0 \tag{2.4}$$

holds, then

$$\frac{1}{n s_n f'(\nu_n)} \sum_{j=1}^{[nt]} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{\psi(\nu_j)} \xrightarrow{\mathcal{D}} \int_0^t s^{\gamma} B(s) ds.$$

(iii) If

$$s_n f'(\nu_n) \longrightarrow c, \ \ c \textit{ is a constant } 0 < c < \infty, \tag{2.5}$$

then

$$\frac{\psi(\sum_{i=1}^{[nt]})}{\psi(\nu_{[nt]})} \xrightarrow{\mathcal{D}} \exp(cB_t \cdot t^{\gamma}) \ \ \textit{in} \ \ D(0,1].$$

For case (iii) above, we do not have non-trivial limit for partial sums as there is no appropriate linear normalization to control the growth. Similar difficulty arises in sum process for classical records. See discussion in p.45 Bose et al [3].

Remark 2 (i) Observe that (2.4) implies (2.3).

(ii) If there exists $M > 0$ such that $\mathbb{E}|X_i - \mu_i|^4 < M, \ \ \forall i$ and $[f'(\nu_n)]^4 = O(n^{-(3+\delta)})$, for some $\delta > 0$, then Condition (2.4) holds.

(iii) Consider the special case of Pfeifer records, where $\alpha_n \in RV_s$, and $\psi_{F_0} := F_0^{-1}(1-e^{-x}) \sim \exp(f(x))$ with $f' \in RV_{\rho-1}$. If we take $s > 0$ and $\rho > 0$ sufficiently small then it is easy to see that all the conditions in Remark 2(ii) are satisfied.

3 Proofs

3.1 Proof of Proposition 1

Lemma 1. *Assume that* $s_n \in RV_\alpha$, $\alpha > 0$, *where* RV_α *is the space of regularly varying function with index* α. *Then* $\{Z_n(t)\}$ *is tight.*

The proof is essentially same as that given in Billingsley [4] (Theorem 15.5), only slight modification is necessary. For the sake of completeness we present it below.

Proof: By a Lemma in Billingsley [4], p.69, we have the following inequality: For $\lambda > 0$ large enough,

$$\mathbb{P}(\max_{i \le m} |S_i| \ge \lambda s_m) \ \ \le 2\mathbb{P}(|S_m| \ge \tfrac{\lambda}{2} s_m). \tag{3.1}$$

By our assumption $\{X_n\}$ satisfies Lindeberg condition. Hence,

$$2\mathbb{P}(\frac{|S_m|}{s_m} \ge \frac{\lambda}{2}) \ \ \longrightarrow 2\mathbb{P}(|N| \ge \tfrac{\lambda}{2}) \ \ \le 2\frac{\mathbb{E}|N|^{\frac{1}{\alpha}+1}}{(\lambda/2)^{\frac{1}{\alpha}+1}}. \tag{3.2}$$

Given $\epsilon > 0$ we choose $\lambda > 1$ so large that the last quantity in the above inequality (2.2), becomes less than $\frac{\epsilon}{\lambda^{\frac{1}{\alpha}}}$ so that we have

$$2\mathbb{P}(\frac{|S_m|}{s_m} \geq \frac{\lambda}{2}) \leq \frac{\epsilon}{\lambda^{\frac{1}{\alpha}}} \text{ for all large } \lambda. \tag{3.3}$$

Now to show that $\{Z_n(t)\}$ is tight we have to show that (Billingsley [4], p.59), for all $\epsilon > 0$ and for all $\eta > 0$ there exists δ, $0 < \delta < 1$ and integer $n_0 \geq 1$ such that

$$\frac{1}{\delta}\mathbb{P}(\sup_{t \leq s \leq t+\delta} |Z_n(s) - Z_n(t)| \geq \epsilon) \leq \eta. \tag{3.4}$$

Since $Z_n(\cdot)$ has a polygonal path, it is enough to verify the condition for $t = \frac{k}{n}$ and $t + \delta = \frac{j}{n}$. Then (3.4) becomes

$$\frac{1}{\delta}\mathbb{P}(\max_{i \leq n\delta} \frac{|S_{k+i} - S_k|}{s_n} \geq \epsilon) \quad = \tfrac{1}{\delta}\mathbb{P}(\max_{i \leq n\delta} |\textstyle\sum_{j=k+1}^{k+i} Y_j| \geq \epsilon s_n) \quad \leq \eta. \tag{3.5}$$

By inequality (3.3), there exists $\lambda > 1$ such that

$$\mathbb{P}(\max_{i \leq n} |\sum_{j=k+1}^{k+i} Y_j| \geq \lambda s_n) \leq \frac{\epsilon}{\lambda^{\frac{1}{\alpha}}} \tag{3.6}$$

for all n sufficiently large and for any given $\epsilon > 0$.

Now fix $\epsilon > 0$ and $\eta > 0$. Without loss of generality we choose $0 < \epsilon, \eta < 1$. Let $\epsilon' = \eta(\frac{\epsilon}{1+\epsilon})^{\frac{1}{\alpha}}$. Recall that α is the index of the regularly varying function s_n. Now for this ϵ', there exists $\lambda > 1$ and a positive integer n_1 such that by (3.6)

$$\mathbb{P}(\max_{i \leq n} |\sum_{j=k+1}^{k+i} Y_j| \geq \lambda s_n) \leq \frac{\epsilon'}{\lambda^{\frac{1}{\alpha}}} \ \forall n \geq n_1. \tag{3.7}$$

Let $0 < \delta := \left[(\frac{\epsilon}{1+\epsilon})/\lambda\right]^{\frac{1}{\alpha}} < 1$. Let $n_0' > \frac{n_1}{\delta}$. Hence if $n > n_0'$ then $[n\delta] > n_1$. Therefore,

$$\mathbb{P}(\max_{i \leq [n\delta]} |\sum_{j=k+1}^{k+i} Y_j| \geq \lambda s_{[n\delta]}) \leq \frac{\epsilon'}{\lambda^{\frac{1}{\alpha}}} = \eta\big[\frac{\frac{\epsilon}{1+\epsilon}}{\lambda}\big]^{\frac{1}{\alpha}}. \tag{3.8}$$

Since by our assumption $s_n \in RV_\alpha$, $s_n = n^\alpha L(n)$ for some slowly varying function $L(\cdot)$, from which it follows, using (3.8)

$$\mathbb{P}(\max_{i \leq [n\delta]} |\sum_{j=k+1}^{k+i} Y_j| \geq \lambda [n\delta]^\alpha L([n\delta])) \leq \eta\delta. \tag{3.9}$$

But $[n\delta]^\alpha \leq n^\alpha \delta^\alpha = n^\alpha \frac{\epsilon}{1+\epsilon}\frac{1}{\lambda}$ and since $L(\cdot)$ is slowly varying, there exists a positive integer n_0 such that $\frac{L([n\delta])}{L(n)} \leq 1+\epsilon$ for all $n > n_0$. Now simplifying (3.9) we have, for all $n > \max\{n_0', n_0\}$,

$$\mathbb{P}(\max_{i \leq [n\delta]} |\sum_{j=k+1}^{k+i} Y_j| \geq \epsilon s_n) \leq \eta\delta. \tag{3.10}$$

Since $Z_n(0) = 0$ for all n, $\{Z_n(\cdot)\}$ is tight. □

Lemma 2. *For any fixed k tuples* $\{t_1, t_2, \dots t_k\} \in (0,1]^k$, *with* $t_1 < t_2 < \dots < t_k$ *where* $k \geq 1$, $\{Z_n(t_1), Z_n(t_2), \dots Z_n(t_k)\}$ *converges jointly in distribution to* $\{W_\alpha(t_1), W_\alpha(t_2), \dots W_\alpha(t_k)\}$ *where* $W_\alpha(t) \sim N(0, t^\alpha)$ *and* $W_\alpha(t_{i+1}) - W_\alpha(t_i)$ *is a* $N(0, t_{i+1}^\alpha - t_i^\alpha)$ *variable, independent of* $W_\alpha(t),\ t \leq t_i$.

Proof. The proof is similar to that given in Billingsley [4]. □

The proof of Proposition 1 now follows from Lemma 1 and 2. Observe that since $\alpha > 0$, $t^{\alpha-1/2}B(t) \xrightarrow{a.s} 0$, as $t \longrightarrow 0$. So the limit function is continuous at zero.

3.2 Proof of Theorem 1

(i) Let us fix any $\epsilon > 0$. By Resnick [9], p.205, it is enough to establish the convergence on $D[\epsilon,\ 1]$ for every $0 < \epsilon < 1$. For a fixed ω and $t \in [\epsilon, 1]$, we use the mean value theorem to get,

$$\frac{\psi(\sum_{i=1}^{[nt]} X_i) - \psi(\nu_{[nt]})}{s_n \psi'(\nu_n)} = \frac{(\sum_{i=1}^{[nt]} X_i - \nu_{[nt]})\psi'(Y_{nt}^*)}{s_n \psi'(\nu_n)},$$

where Y_{nt}^* is a $D[\epsilon,\ 1]$-valued function of ω, lying between $\sum_{i=1}^{[nt]} X_i$ and $\nu_{[nt]}$. Observe that for any fixed ω, $Y_{[nt]}^*(\omega)$ lies between $\sum_{i=1}^{[nt]} X_i(\omega)$ and ν_{nt} and by definition, $\forall t \in [\frac{k}{n}, \frac{k+1}{n})$, $k = 1, 2, \dots n-1$, $\sum_{i=1}^{[nt]} X_i(\omega)$ and $\nu_{[nt]}$ do not change and hence $Y_{nt}(\omega)$ does not change on $[\frac{k}{n}, \frac{k+1}{n})$, $\forall k = 1, 2, \dots n-1$. So paths of $Y_{[nt]}^*$ are right continuous and have left limits and hence are elements of D. By the same argument, for any fixed ω $\psi'(Y_{nt}^*)(\omega)$ is an element of D and $\psi'(Y_{nt}^*)$ is a measurable function of ω, for any fixed $t \in [\epsilon, 1]$. Therefore, $\{\psi'(Y_{nt}^*)\}_n$ is a sequence of $D[\epsilon,\ 1]$-valued processes. Now,

$$\frac{\sum_{i=1}^{[nt]} X_i - \nu_{[nt]}}{s_n} \xrightarrow{\mathcal{D}} t^{\alpha-1/2}B(t).$$

Claim: $d\big(\frac{\psi'(Y_{nt}^*)}{\psi'(\nu_n)}, t^{\beta(\rho-1)}\big)$ converges to 0, in probability, where $d(\cdot,\cdot)$ is the Skorohod metric.

Proof of the claim. Since $t^{\beta(\rho-1)}$ is a uniformly continuous deterministic function on $[\epsilon,\ 1]$,

$$d\left(\frac{\psi'(Y_{nt}^*)}{\psi'(\nu_n)}, t^{\beta(\rho-1)}\right) = \sup_{t\in[\epsilon,1]} |\frac{\psi'(Y_{nt}^*)}{\psi'(\nu_n)} - t^{\beta(\rho-1)}|.$$

First note that since by our assumption, $\nu_n \in RV_\beta,\ \ \beta > 0$, if n is large enough, we have

(a) $\frac{\nu_{[nt]}}{\nu_{[n\epsilon]}} \geq C_1(t/\epsilon)^{\frac{\beta}{2}} > C_1$, and

(b) $\frac{\nu_{[nt]}}{\nu_{[n\epsilon]}} \leq C_2(t/\epsilon)^{\frac{3\beta}{2}}$, for $t \in [\epsilon, 1]$ where $C_1, C_2 > 0$ are constants.

Now, by Billingsley [5], p.287, for some $K > 0$,

$$\mathbb{P}\big[\sup_{\epsilon\leq t\leq 1} |\frac{\sum_{i=1}^{[nt]}(X_i - \mu_i)}{\nu_{[nt]}}| > \theta\big] \ \leq\ \mathbb{P}\big[\sup_{\epsilon\leq t\leq 1} |\sum_{i=1}^{[nt]}(X_i - \mu_i)| \geq C_1\nu_{[n\epsilon]}\theta\big] \qquad (3.11)$$

$$\leq \quad \frac{s_n^2}{C_1^2\theta^2\nu_{[n\epsilon]}^2} \leq K\frac{s_n^2}{\nu_n^2}\frac{\nu_n^2}{\nu_{[n\epsilon]}^2}. \tag{3.12}$$

The right-hand side converges to zero by (b) above and by our assumption $\frac{s_n}{\nu_n} \longrightarrow 0$ as $n \longrightarrow \infty$. Since Y_{nt}^* lies between $\sum_{i=1}^{[nt]} X_i$ and $\nu_{[nt]}$,

$$\{\omega : \sup_{\epsilon\leq t\leq 1} |\frac{Y_{nt}^*}{\nu_{[nt]}} - 1| > \theta\} \subset \{\omega : \sup_{\epsilon\leq t\leq 1} |\frac{\sum_{i=1}^{[nt]} X_i}{\nu_{[nt]}} - 1| > \theta\}.$$

Therefore, $\frac{Y_{nt}^*}{\nu_{[nt]}} \overset{\mathbb{P}}{\longrightarrow} 1$ in Skorohod metric. Also, $\frac{\nu_{[nt]}}{\nu_n} \longrightarrow t^\beta$ uniformly on $[\epsilon, 1]$ by uniform convergence theorem for regularly varying functions. Hence, using $\psi' \in RV_{\rho-1}$ and applying uniform convergence theorem also for ψ', we have

$$\frac{\psi'(Y_{nt}^*)}{\psi'(\nu_n)} \quad = \frac{\psi'(\nu_n \frac{\nu_{[nt]}}{\nu_n}\frac{Y_{nt}^*}{\nu_{[nt]}})}{\psi'(\nu_n)} \overset{\mathbb{P}}{\longrightarrow} t^{\beta(\rho-1)}, \text{ in } D[\epsilon,\ 1]. \tag{3.13}$$

Therefore,

$$\frac{\psi(\sum_{i=1}^{[nt]} X_i) - \psi(\nu_{[nt]})}{s_n\psi'(\nu_n)} \overset{\mathcal{D}}{\longrightarrow} t^{\alpha-1/2+\beta(\rho-1)}B(t), \text{ in } D[\epsilon,\ 1]$$

where $B(t)$ is a standard Brownian motion. □

(ii) As in (i) above, using the mean value theorem we have,

$$\frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{s_j\psi'(\nu_j)} = \frac{\sum_{i=1}^{j} X_i - \nu_j}{s_j}\frac{\psi'(Y_j^*)}{\psi'(\nu_j)},$$

where Y_j^* lies between $\sum_{i=1}^{j} X_i$ and ν_j. Since by our assumption, $\frac{\sum_{i=1}^{j} X_i}{\nu_j} \overset{a.s.}{\longrightarrow} 1$, implying

$$\frac{Y_j^*}{\nu_j} \overset{a.s.}{\longrightarrow} 1 \text{ and } \frac{\psi'(Y_j^*)}{\psi'(\nu_j)} = \frac{\psi'(\nu_j.\frac{Y_j^*}{\nu_j})}{\psi'(\nu_j)} \overset{a.s.}{\longrightarrow} 1.$$

Let

$$A_k = \{\omega : |\frac{\psi(Y_j^*)}{\psi'(\nu_j)}| \leq 2,\ \forall\ j \geq k\}.$$

The almost sure convergence of the quantity within bracket implies $\mathbb{P}(\cup_{k=1}^{\infty} A_k) = 1$. Therefore given $\epsilon_0 > 0$, there exists a positive integer N_1 such that $\mathbb{P}(\cup_{i=1}^{N_1} A_k) > 1 - \epsilon_0$. Since $A_k' s$ are increasing, this implies $\mathbb{P}(A_{N_1}) > 1 - \epsilon_0$.

We will need the following inequality (where we use the lower bound of $\mathbb{P}(A_{N_1})$): Fix $\theta > 0$.

$$\begin{aligned}
\mathbb{P}[|\sum_{j=N_1}^{[n\epsilon]} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{s_n\psi'(\nu_n)}| > \theta] \quad &\leq \quad \mathbb{P}[\sum_{j=N_1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{s_n\psi'(\nu_n)}|I_{A_{N_1}} > \frac{\theta}{2}] \\
&+\mathbb{P}[\sum_{j=N_1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{s_n\psi'(\nu_n)}|I_{A_{N_1}^c} > \frac{\theta}{2}]
\end{aligned}$$

$$\leq \sum_{j=N_1}^{[n\epsilon]} \frac{2\mathbb{E}|\sum_{i=1}^{j} X_i - \nu_j||\psi'(\nu_j)|}{\frac{\theta}{2} s_n |\psi'(\nu_n)|} + \epsilon_0. \tag{3.14}$$

Now observe that,

$$\mathbb{E}\Big(\frac{|\sum_{i=1}^{j} X_i - \nu_j|}{s_j}\Big) \leq \Big[\mathbb{E}\Big(\frac{|\sum_{i=1}^{j} X_i - \nu_j|}{s_j}\Big)^2\Big]^{1/2} = 1.$$

Now choose a $\delta > 0$ sufficiently small so that $\alpha + \beta(\rho - 1) - 2\delta > -1$. Since $s_n \in RV_\alpha$ and $\psi'(\nu_n) \in RV_{\beta(\rho-1)}$, there exists $N_2 \geq 1$ such that $s_j/s_n \leq C(j/n)^{\alpha-\delta}$ and $\psi'(\nu_j)/\psi'(\nu_n) \leq C(j/n)^{\beta(\rho-1)}$ for all $n \geq N_2$, where C denotes a generic constant. Let $N_0 = \max\{N_1, N_2\}$. Then, by (3.14),

$$\begin{aligned}
\mathbb{P}\Big[\Big|\sum_{j=N_0}^{[n\epsilon]} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{n s_n \psi'(\nu_n)}\Big| > \theta\Big] &\leq \frac{\sum_{j=N_0}^{[n\epsilon]} 2 s_j |\psi'(\nu_j)|}{n s_n |\psi'(\nu_n)|} + \epsilon_0 \\
&\leq (C/n) \sum_{j=N_0}^{[n\epsilon]} (j/n)^{\alpha-\delta} (j/n)^{\beta(\rho-1)-\delta} + \epsilon_0. \\
&\longrightarrow C \int_0^{\epsilon} y^{\alpha+\beta(\rho-1)-2\delta} dy + \epsilon_0 \\
&\to C \frac{\epsilon^{1+\alpha+\beta(\rho-1)-2\delta}}{1 + \alpha + \beta(\rho - 1) - 2\delta} + \epsilon_0. \qquad (3.15)
\end{aligned}$$

So $\mathbb{P}\big[|\sum_{j=N_0}^{[n\epsilon]} \frac{\psi(\sum_{i=1}^{j} X_i)-\psi(\nu_j)}{n s_n \psi'(\nu_n)}| > \theta\big]$ can be made arbitrarily small by choosing $\epsilon > 0$ and $\epsilon_0 > 0$ small enough. Now we decompose the sum on the left-hand side of (2.2) in three parts as follows:

For simplicity of notation, we denote $\frac{[\psi(\sum_{i=1}^{j} X_i)-\psi(\nu_j)]}{n s_n \psi'(\nu_n)}$ by $\psi(j, n)$ in the following decomposition.

$$\sum_{j=1}^{[nt]} \psi(j, n) = \sum_{j=1}^{N_0} \psi(j, n) + \sum_{j=N_0+1}^{[n\epsilon]} \psi(j, n) + \sum_{j=[n\epsilon]+1}^{[nt]} \psi(j, n) =: S_1 + S_2 + S_3. \tag{3.16}$$

Observe that $n s_n \psi'(\nu_n) = O(n^{1+\alpha+\beta(\rho-1)-2\delta})$ and hence tends to infinity as by our assumption $1 + \alpha + \beta(\rho - 1) - 2\delta > 0$. So the first sum

$$S_1 \xrightarrow{P} 0, \tag{3.17}$$

the second sum

$$S_2 \xrightarrow{P} 0 \text{ as } \epsilon \to 0 \text{ by (3.15)}. \tag{3.18}$$

The third sum is $S_3 = \int_\epsilon^t \xi_n(s) ds$, where $\xi_n(s) := \frac{\psi(\sum_{i=1}^{[ns]} X_i)-\psi(\nu_{[ns]})}{s_n \psi'(\nu_n)}$. By Lemma 1 of Bose, Gangopadhyay and Sarkar [3],

$$S_3 \xrightarrow{\mathcal{D}} \int_\epsilon^t s^{\alpha-1/2+\beta(\rho-1)} B(s) ds \text{ on } C[\epsilon,\ 1]. \tag{3.19}$$

Therefore by Slutsky's Theorem, (3.17), (3.18) and (3.19) together imply

$$\sum_{j=1}^{[nt]} \frac{\left[\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)\right]}{ns_n\psi'(\nu_n)} \xrightarrow{\mathcal{D}} \int_0^t s^{\alpha-1/2+\beta(\rho-1)}B(s)ds.$$

□

3.3 Proof of Theorem 2

(i) Using the mean value theorem we get

$$\frac{e^{f(\sum_{i=1}^{[nt]} X_i)} - e^{f(\nu_{[nt]})}}{s_n f'(\nu_n)e^{f(\nu_{[nt]})}} = \frac{\sum_{i=1}^{[nt]} X_i - \nu_{[nt]}}{s_n} \cdot \frac{e^{f(Y_{nt}^*)}}{e^{f(\nu_{[nt]})}} \cdot \frac{f'(Y_{nt}^*)}{f'(\nu_n)},$$

where Y_{nt}^* lies between $\sum_{i=1}^{[nt]} X_i$ and $\nu_{[nt]}$. Note that the first factor on the right-hand side goes to $t^{\alpha-1/2}B(t)$. Arguing as in Theorem 1, we have $\frac{f'(Y_{nt}^*)}{f'(\nu_n)} \xrightarrow{\mathbb{P}} t^{\beta(\rho-1)}$ in $D[\epsilon,\ 1]$. Now, again using the mean value theorem we get,

$$\frac{e^{f(Y_{nt}^*)}}{e^{f(\nu_{[nt]})}} = e^{f(Y_{nt}^*)-f(\nu_{[nt]})} = e^{\frac{Y_{nt}^*-\nu_{[nt]}}{s_n}\cdot s_n f'(\nu_n)\cdot \frac{f'(Y_{nt}^{**})}{f'(\nu_n)}}. \tag{3.20}$$

where Y_{nt}^{**} lies between Y_{nt}^* and ν_{nt}. Observe that

(a) $0 \leq |\frac{(Y_{nt}^*-\nu_{[nt]})}{s_n}|\cdot \leq |\frac{(\sum_{i=1}^{[nt]} X_i-\nu_{[nt]})}{s_n}| \xrightarrow{\mathcal{D}} t^{\alpha-1/2}B(t)$ and

(b) $\frac{f'(Y_{nt}^{**})}{f'(\nu_n)} \xrightarrow{\mathbb{P}} t^{\beta(\rho-1)}$ in $D[\epsilon,\ 1]$.

Using (a) and (b) and our assumption that $s_n \cdot f'(\nu_n) \longrightarrow 0$ we complete the proof of (i).

(ii) Denoting $ns_nf'(\nu_n)$ by a_n, we decompose the sum $(1/a_n)\sum_{j=1}^{[nt]} \frac{\psi(\sum_{i=1}^{j} X_i)-\psi(\nu_j)}{\psi(\nu_j)}$ in three parts as follows:

$$\begin{aligned} S_1 &= (1/a_n)\sum_{j=1}^{N_0} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{\psi(\nu_j)}, \\ S_2 &= (1/a_n)\sum_{j=N_0+1}^{[n\epsilon]} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{\psi(\nu_j)} \quad \text{and} \\ S_3 &= (1/a_n)\sum_{j=[n\epsilon]+1}^{[nt]} \frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{\psi(\nu_j)}. \end{aligned}$$

Observe that by our assumption, $ns_nf'(\nu_n) \longrightarrow \infty$. Therefore, $|S_1| \xrightarrow{\mathbb{P}} 0$. For S_2 consider,

$$\frac{\psi(\sum_{i=1}^{j} X_i) - \psi(\nu_j)}{\psi(\nu_j)s_jf'(\nu_j)} = \frac{\sum_{i=1}^{j} X_i - \nu_j}{s_j} \cdot \frac{f'(Y_j^*)}{f'(\nu_j)} \cdot \frac{e^{f(Y_j^*)}}{e^{f(\nu_j)}}. \tag{3.21}$$

As before Y_j^* lies between $\sum_{i=1}^j X_i$ and ν_j. Observe that by our assumption $\nu_n f'(\nu_n) \longrightarrow \infty$. Hence $(\sum_{i=1}^n X_i - \nu_n)f'(\nu_n) \xrightarrow{a.s.} 0$ implies $\frac{\sum_{i=1}^n X_i}{\nu_n} \xrightarrow{a.s.} 1$. So as in Theorem 1, we get N_1 for a given ϵ_0 such that

$$\mathbb{P}\big[\omega : |\frac{f'(\sum_{i=1}^j X_i)}{f'(\nu_j)}| \leq 2,\ j \geq N_1\big] > 1 - \epsilon_0. \tag{3.22}$$

Now,

$$\frac{e^{f(Y_j^*)}}{e^{f(\nu_j)}} = e^{f(Y_j^*) - f(\nu_j)} = e^{(Y_j^* - \nu_j)f'(Y_j^{**})} = e^{(Y_j^* - \nu_j)f'(\nu_j)\cdot\frac{f'(Y_j^{**})}{f'(\nu_j)}}. \tag{3.23}$$

Again as we did before, given $\epsilon_0 > 0$, we get a positive integer N_2 such that

$$\mathbb{P}\big[\omega : |(\sum_{i=1}^j X_i - \nu_j)f'(\nu_j)| \leq 1\ \ j \geq N_2\big] > 1 - \epsilon_0. \tag{3.24}$$

Let $N_0 = N_1 \vee N_2$. Denote the events considered in (3.22) and (3.24) by A_{N_1} and B_{N_2} respectively. Let $C_{N_0} = A_{N_0} \cap B_{N_0}$. Fix a $\theta > 0$. Then

$$\begin{aligned}
\mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)}{\psi(\nu_j)}| > \theta\Big] \quad &\leq \quad \mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)}{\psi(\nu_j)}| I_{C_{N_0}} > \frac{\theta}{2}\Big] \\
&+ \quad \mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)}{\psi(\nu_j)}| I_{C_{N_0}^c} > \frac{\theta}{2}\Big] \\
&\leq \quad \mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)}{\psi(\nu_j)}| I_{C_{N_0}} > \frac{\theta}{2}\Big] \\
&\quad +\epsilon_0.
\end{aligned} \tag{3.25}$$

Now,

$$\begin{aligned}
\mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} |\frac{\psi(\sum_{i=1}^j X_i) - \psi(\nu_j)}{\psi(\nu_j)}| I_{C_{N_0}} > \frac{\theta}{2}\Big] \quad &\leq \quad \mathbb{P}\Big[\sum_{j=N_0+1}^{[n\epsilon]} 2|\frac{\sum_{i=1}^j X_i - \nu_j}{s_j}| e^2 I_{C_{N_0}} \cdot |s_j f'(\nu_j)| > \frac{\theta}{2}\Big] \\
&\leq \quad \frac{\sum_{j=N_0+1}^{[n\epsilon]} 2|s_j f'(\nu_j)| e^2 \mathbb{E}|\frac{\sum_{i=1}^j X_i - \nu_j}{s_j}|}{\theta/2}.
\end{aligned} \tag{3.26}$$

Note that by Cauchy-Schwarz inequality $\mathbb{E}|\frac{\sum_{i=1}^j X_i - \nu_j}{s_j}| \leq 1$ for all $j \geq 1$. Therefore, arguing as in Theorem 1 we have $|S_2| \xrightarrow{\mathbb{P}} 0$, as $\epsilon \longrightarrow 0$. Using (i) above, we see that the third sum $S_3 \xrightarrow{D} \int_\epsilon^t s^\gamma B(s)ds$, where $\gamma = \alpha - 1/2 + \beta(\rho - 1)$. This completes the proof of (ii).

(iii) Using the mean value theorem we write

$$f(\sum_{i=1}^{[nt]}) - f(\nu[nt]) = \sum_{i=1}^{[nt]} X_i - \nu_{[nt]} f'(Y_{nt}^*) = \frac{\sum_{i=1}^{[nt]} X_i - \nu_{[nt]}}{s_n} \cdot s_n f'(\nu_n) \cdot \frac{f'(Y_{nt}^*)}{f'(\nu_n)}. \tag{3.27}$$

Then arguing as before we get the result. $\square$

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