

Large dimensional random k circulants

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December 29, 2008

Abstract

Circulant matrices with general shift by k places have been studied in the literature and formula for their eigenvalues is known. We first reestablish this formula and some further properties of these eigenvalues in a manner suitable for our use.

We then consider random $k = k(n)$ circulants $A_{k,n}$ with $n \rightarrow \infty$ and whose input sequence $\{a_i\}$ is independent with mean zero and variance one and $\sup_n n^{-1} \sum_{i=1}^n \mathbb{E}|a_i|^{2+\delta} < \infty$ for some $\delta > 0$. Under suitable restrictions on $\{k(n)\}$, we show that the limiting spectral distribution (LSD) of the empirical distribution of suitably scaled eigenvalues exists and identify the limits.

As examples, (i) if $k^g = -1 + sn$ where $g \geq 1$ fixed and $s = o(n^{1/3})$, then the LSD is $U_1(\prod_{i=1}^g E_i)^{1/2g}$ where E_i are i.i.d. Exp(1) and U_1 is uniformly distributed over the $(2g)$ th roots of unity, independent of the $\{E_i\}$, and (ii) if $k^g = 1 + sn$ where $g \geq 2$ is fixed and $s = o(n^{\frac{g+1}{g-1}})$ or $s = o(n)$ according as $g \geq 2$ is odd or even, then the LSD is $U_2(\prod_{i=1}^g E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. Exp(1) and U_2 is uniformly distributed over the unit circle, independent of the $\{E_i\}$.

We then consider the limit distribution of the spectral norm of $A_{k,n}$. We show that when $n = k^2 + 1 \rightarrow \infty$, the spectral norm, with appropriate scaling and centering, which we provide explicitly, converges to the Gumbel distribution.

Keywords: eigenvalue, circulant, k -circulant, empirical spectral distribution, limiting spectral distribution, central limit theorem, normal approximation, spectral norm, Gumbel distribution.

AMS Subject Classification: Primary 60F99, Secondary 62H99.

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1 Introduction

Suppose $\underline{a} = \{a_i, i = 0, 1, 2, \dots\}$ is a sequence of real numbers (called the *input* sequence). For positive integers k and n , define the $n \times n$ square matrix

$$A_{k,n}(\underline{a}) = \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-k} & a_{n-k+1} & \dots & a_{n-k-1} \\ a_{n-2k} & a_{n-2k+1} & \dots & a_{n-2k-1} \\ & & \vdots & \end{bmatrix}_{n \times n}.$$

We emphasize that all subscripts appearing in the matrix entries above are calculated modulo n . Our convention will be to start the row and column indices from zero. Thus, 0th row of $A_{k,n}(\underline{a})$ is

$$\underline{a}^T = (a_0, a_1, a_2, \dots, a_{n-1}).$$

Moreover, for $0 \leq j < n - 1$, the $(j + 1)$ -th row of $A_{k,n}$ is obtained by giving its j -th row a right-circular shift by k positions (equivalently, $k \bmod n$ positions). We will denote $A_{k,n}(\underline{a})$ by $A_{k,n}$ or A_n . $A_{k,n}(\underline{a})$ is said to be a *k-circulant matrix*. It may be noted that the k -circulant with $k = n + 1$ reduces to the usual circulant matrix. In general, without loss of generality, the value of k may always be reduced modulo n .

The k -circulant matrix and its block versions arise in many contexts and have been considered in many works in mathematics, statistics and related areas. As examples, we mention the book by Davis [5] and the article by Pollock (2002)[11]. Zhou (1996)[16] Davis (1979) provides a formula description of the eigenvalues of the k -circulant matrix. In this article, we first give a more detailed development leading to that formula and explore various special cases in some details.

Circulant matrices play a crucial role in the study of large dimensional Toeplitz matrices with nonrandom input. See, for example, Grenander and Szegö (1984)[9]. The eigenvalues of circulant matrices are essentially the discrete Fourier transform of the input sequence. The periodogram in time series analysis is a function of the eigenvalues of an appropriate circulant matrix with the time series inputs (see Fan and Yao (2003)[8]).

The k -circulant matrices, with fixed or increasing k and with the dimension n tending to infinity and the input sequence $\{a_i\}$ being an appropriate random sequence are examples of patterned large dimensional random matrix (LDRM). Consider the random distribution which puts equal mass at each of the eigenvalues of an appropriately centered and scaled version of an LDRM. Then its weak limit (say, in probability), is said to be the limiting spectral distribution (LSD) of the sequence of matrices. Deriving LSD for general patterned matrices has drawn significant attention in the literature. See for example the review article by Bai (1999)[1] or the more recent paper Bose and Sen (2007)[4].

In this article, we investigate the existence of LSD for the k -circulants. With varying k , the LSD is widely varying. The following results are known. Suppose the input sequence $\{a_i\}$ is independent with mean zero, variance one and with uniformly bounded finite third absolute moment. Then (see for example Bose and Mitra (2002)[3]),

- (i) If $k = 1$ (or equivalently $k = n + 1$), the LSD of the k -circulant is bivariate complex normal.
- (ii) If $k = n - 1$, the LSD of the k -circulant is the symmetric version of the positive square root of E where E is standard exponential.

In general, we can not rule out the possibility that the values of k and n are such that there are too many zero eigenvalues. In fact suppose $k = n^{o(1)}$, $n \rightarrow \infty$ with $\gcd(n, k) = 1$, and the input sequence is i.i.d. normal. In Theorem 2, we establish that, in this case, the LSD is the point mass at zero.

Thus, to avoid trivialities, we need $k = n^{\Omega(1)}$ as $n \rightarrow \infty$. To characterize the possible LSD for all combinations of k and n is an interesting problem. While we do not have a complete solution, we prove a few general results under suitable relations on k and n .

Under Assumption I given below, we establish that (i) if $k^g = -1 + sn$ where $g \geq 1$ fixed and $s = o(n^{1/3})$, then the LSD is $U_1(\prod_{i=1}^g E_i)^{1/2g}$ where E_i are i.i.d. Exp(1) and U_1 is uniformly distributed over the $(2g)$ th roots of unity, independent of the $\{E_i\}$, and (ii) if $k^g = 1 + sn$ where $g \geq 2$ is fixed and $s = o(n^{\frac{g+1}{g-1}})$ or $s = o(n)$ according as $g \geq 2$ is odd or even, then the LSD is $U_2(\prod_{i=1}^g E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. Exp(1) and U_2 is uniformly distributed over the unit circle, independent of the $\{E_i\}$.

The above results can be extended to situations where the LSD has a positive mass at zero and the rest of the mass is distributed as above. The results for $k = 1$ and $k = n - 1$ mentioned earlier follow as special cases of the above.

Assumption I The sequence $\{a_i\}$ is independent with mean zero, variance one and for some $\delta > 0$,

$$\rho_{2+\delta} = \sup_n n^{-1} \sum_{i=1}^n E|a_i|^{2+\delta} < \infty.$$

Another very important quantity associated with matrices is its largest (in magnitude) singular value. For an $n \times n$ matrix $B \in \mathcal{M}_n(\mathbb{C})$, the spectral norm $\|B\|_2$ is defined as

$$\|B\|_2 := \sqrt{\lambda_{\max}(B^*B)} = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } B\}.$$

For classical random matrix models such as the Wigner matrix and the i.i.d. matrix with Gaussian entries, the limiting distribution of an appropriately normalized spectral norm is known. See for example Tracy and Widom (2000)[14] and Johnstone (2001)[10].

One may ask similar question for the k -circulant matrices. Since we have explicit description of the eigenvalues of k -circulant at our disposal, finding the limiting distribution of spectral norms turns out to be not too hard, at least for some special choices of k and n , when the entries are Gaussian. But the problem becomes more interesting when the entries are not necessarily Gaussian.

For the usual circulant ($k = 1$), and for the reverse circulant ($k = n - 1$), the results of Davis and Mikosch (1999)[6] essentially provides the limits. Complete answer for all possible cases seems to be a difficult problem.

Nevertheless, the problem appears to be tractable under suitable assumptions on (k, n) . For simplicity, we deal with a restricted case where $n = k^2 + 1$. Suppose that $\{a_i\}$ is i.i.d. with mean zero and variance 1 and $\mathbb{E}|a_i|^\gamma < \infty$ for some $\gamma > 2$ and $n = k^2 + 1$. In Theorem 5 we show that as $n \rightarrow \infty$ an appropriately normalised spectral norm of $A_{k,n}$ converges in distribution to the Gumbel distribution. We also provide explicit expressions for the centering and scaling.

2 Eigenvalues of the k -circulant

We first describe the eigenvalues of a k -circulant and prove some related auxiliary properties. The formula solution, in particular is already known, see for example Zhou (1996). We provide a more detailed analysis which is used in our study of the LSD and the spectral norm in the next sections.

Let

$$\omega = \omega_n := \cos(2\pi/n) + i \sin(2\pi/n), \quad i^2 = -1 \quad \text{and} \quad \lambda_j = \sum_{l=0}^{n-1} a_l \omega^{jl}, \quad 0 \leq j < n. \quad (1)$$

For any two positive integers k and n , let $p_1 < p_2 < \dots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^c p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^c p_q^{\alpha_q}.$$

Here $\alpha_q, \beta_q \geq 1$ and n', k', p_q are pairwise relatively prime. For any positive integer s , let $\mathbb{Z}_s = \{0, 1, 2, \dots, s-1\}$. Define the following sets

$$S(x) = \{xk^b \bmod n' : b \geq 0\}, \quad 0 \leq x < n'.$$

We observe the following about the sets $S(x)$.

1. $S(x) = \{xk^b \bmod n' : 0 \leq b < |S(x)|\}$.
2. For $x \neq u$, either $S(x) = S(u)$ or, $S(x) \cap S(u) = \emptyset$. To see this, suppose $S(x) \cap S(u) \neq \emptyset$. Then, $xk^{b_1} \equiv uk^{b_2} \bmod n'$ for some integers b_1 and b_2 . Multiplying both sides by $k^{n(x)-b_1}$ we see that, $x \in S(u)$ so that, $S(x) \subseteq S(u)$. Hence, reversing the roles, $S(x) = S(u)$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \leq x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \dots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}\}, \quad \text{and} \quad n_i = |\mathcal{P}_i|, \quad 0 \leq i < \ell.$$

Define

$$x_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \dots, \ell-1 \quad \text{where} \quad y = n/n'.$$

The following theorem provides the formula solution for the eigenvalues of A_n and is inspired by the result of Zhou (1996)[16]. For completeness, we have provided a proof in the Appendix.

Theorem 1 (Zhou (1996)[16]) *The characteristic polynomial of $A_{k,n}$ is given by*

$$\chi(A_{k,n}) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} (\lambda^{n_j} - x_j). \quad (2)$$

Remark 1 $\{\lambda_j, 0 \leq j < n\}$ are eigenvalues of the usual circulant matrix $A_{1,n}$.

2.1 Some properties of the eigenvalue partition $\{\mathcal{P}_i, i \geq 0\}$

With a view to using them for the study of the LSD, we give below some properties of the eigenvalues.

1. If $j \in S(x)$ and $n' - j \in S(u)$, then for every $l \in S(x)$, we have $n' - l \in S(u)$ in which case we call $S(x)$ and $S(u)$ to be *conjugate* of each other. This is easy to see: let $l = jk^h \bmod n'$ for some $m \geq 0$. $n' - j = uk^p \bmod n'$ for some $p \geq 0 \Rightarrow n' - l = (n' - j)k^h \bmod n' = uk^{p+h} \bmod n'$.

We now define for $0 \leq x < n'$,

$$O_x = \{t > 0 : t \text{ is integer and } xk^t \equiv x \bmod n'\} \quad \text{and} \quad g_x = \min\{t : t \in O_x\}.$$

We call g_x the *order* of x . Note that $g_x = |S(x)|$.

2. g_x divides g_1 for all $x \in \mathbb{Z}_{n'}$. To see this, note that $k^{g_1} = 1 \bmod n'$, implying that $xk^{g_1} = x \bmod n'$ for all x . Therefore $g_1 \in O_x$ for all x . But every element of O_x is divisible by g_x , and the result follows.

3. Suppose g divides g_1 . Let $X(g)$ be the set of all $x \in \mathbb{Z}_{n'}$ whose order is g . Let $m = \gcd(k^g - 1, n')$. Then

$$X(g) \subseteq Y(g) := \{cn'/m : 0 \leq c < m\} \text{ and } |Y(g)| = m.$$

Moreover,

$$\bigcup_{h:hg} X(h) = Y(g).$$

To prove this, note that clearly, $|Y(g)| = m$. Fix $x \in X(g)$. Then, $xk^g \equiv x \pmod{n'}$. Therefore n' divides $x(k^g - 1)$. So, n'/m divides $x(k^g - 1)/m$. But n'/m is relatively prime to $(k^g - 1)/m$ and hence n'/m divides x . So, $x = cn'/m$ for some integer $c \geq 0$. Since $0 \leq x < n'$, we have $0 \leq c < m$, and $x \in Y(g)$. It is now trivial to see $\bigcup_{h:hg} X(h) \subseteq Y(g)$.

On the other hand, take $0 \leq c < g'$. Then $(cn'/m)k^g = (cn'/m) \pmod{n'}$. Hence, $g \in \mathcal{O}_{cn'/m}$ and $gcn'/m \mid g$ so that $Y(g) \subseteq \bigcup_{h:hg} X(h)$.

The following lemma gives an expression for the number of elements in $\mathbb{Z}_{n'}$ with order less than g_1 .

Lemma 1 Let $g_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_t^{\alpha_t}$ where $p_1 < p_2 < \dots < p_t$ are primes. Then,

$$|\{x \in \mathbb{Z}_{n'} : g_x < g_1\}| = G_1 - G_2 + G_3 - G_4 \dots$$

where, for $j \geq 1$,

$$G_j = \sum_{\ell_j \in L_j} |Y(g_1/\ell_j)| = \sum_{\ell_j \in L_j} \gcd(k^{g_1/\ell_j} - 1, n')$$

with

$$L_j = \{p_{i_1} p_{i_2} \dots p_{i_j} : 1 \leq i_1 < \dots < i_j \leq t\}.$$

Proof Fix $x \in \mathbb{Z}_{n'}$. Let $g_x = p_1^{\beta_1} \dots p_t^{\beta_t}$ where, $0 \leq \beta_i \leq \alpha_i$ for $1 \leq i \leq t$. Suppose that exactly m -many β 's are equal to the corresponding α 's. To keep notation simple, we will assume that, $\beta_i = \alpha_i$, $1 \leq i \leq h$ and $\beta_i < \alpha_i$, $h + 1 \leq i \leq t$. Then $x \in Y(g_1/p_i)$ for $m + 1 \leq i \leq t$ and $x \notin Y(g_1/p_i)$ for $1 \leq i \leq h$. So, x is counted $(t - h)$ times in G_1 . Similarly, x is counted $\binom{t-h}{2}$ times in G_2 , $\binom{t-h}{3}$ times in G_3 , and so on. Hence, total number of times x is counted in $(G_1 - G_2 + G_3 - \dots)$ is

$$\binom{t-h}{1} - \binom{t-h}{2} + \binom{t-h}{3} - \dots = 1$$

completing the proof. ■

Define

$$\nu_{k,n'} := (n')^{-1} |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}|.$$

To derive the LSD in the special cases we have in mind, the asymptotic negligibility of $\nu_{k,n'}$ turns out to be important. The following two Lemmas establish upper bounds on $\nu_{k,n'}$ and will be crucially used later.

Lemma 2 (i) Suppose $g_1 = 2$. Then $\nu_{k,n'} = 1/n'$.

(ii) Suppose $g_1 > 2$ is even, and $k^{g_1/2} = -1 \pmod{n}$. Then $\nu_{k,n'} \leq \frac{g_1 k^{g_1/3}}{3n'}$.

(iii) Suppose p_1 is the smallest prime divisor of g_1 . Then $\nu_{k,n'} < \frac{2k^{g_1/p_1}}{n'}$.

Proof Proof of (i) is immediate from Lemma 1.

(ii) Suppose $g_x = g$. Then $x = xk^g = -x \pmod{n'}$. Therefore, $2x = 0 \pmod{n'}$ and x can be either 0 or $n'/2$, provided, of course, $n'/2$ is an integer. But $g_0 = 1 < g$. So, $|X(g)| \leq 1$. We now have,

$$\begin{aligned} |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}| &= |X(g)| + \sum_{1 \leq j \leq g_1/3} |\{x \in \mathbb{Z}_{n'} : g_x = j\}| \\ &\leq 1 + \sum_{1 \leq j \leq g_1/3} |Y(g_x)| \\ &\leq 1 + \sum_{1 \leq j \leq g_1/3} (k^{g_x} - 1) \\ &\leq 1 + \sum_{1 \leq j \leq g_1/3} (k^{g_1/3} - 1) \\ &\leq (g_1/3)k^{g_1/3}. \end{aligned}$$

(iii) As in Lemma 1, let $g_1 = p_1^{\alpha_1} \dots p_t^{\alpha_t}$ where $p_1 < p_2 < \dots < p_t$ are primes. Then

$$\begin{aligned} n' \times v_{k,n'} &= G_1 - G_2 + G_3 - G_4 + \dots \\ &\leq G_1 = \sum_{i=1}^t \gcd(k^{g_1/p_i} - 1, n') \\ &< \sum_{i=1}^t k^{g_1/p_i} \text{ where } g_1/p_1 > \dots > g_1/p_t \\ &\leq 2k^{g_1/p_1}. \end{aligned}$$

■

The next Lemma is a consequence of Lemma 2.

Lemma 3 Suppose $n, k = k(n) \rightarrow \infty$. Then in each of the following situations, $v_{k,n} \rightarrow 0$.

(i) $k^g = -1 + sn$ where $g \geq 1$ fixed and $s = o(n^{1/3})$.

(ii) $k^g = 1 + sn$, where $g > 1$ fixed, p_1 is the smallest prime divisor of g and $s = o(n^{p_1-1})$.

Proof (i) First of all, note that in either cases $\gcd(n, k) = 1$ and therefore $n' = n$. Since $k^{2g} = (sn - 1)^2 = 1 \pmod{n}$, therefore $g_1 | 2g$. Note $g_1 \neq g$ because $k^g = -1 \pmod{n}$.

If $g_1 \leq 2g/3$, then $k^{g_1} = (sn - 1)^{2/3} < n^{8/9}$. Hence, $g_1 = 2g$.

Now by Lemma 2(i),

$$v_{k,n} \leq \frac{2gk^{2g/3}}{3n} \leq \frac{2g(sn)^{2/3}}{3n} = o(1).$$

(ii) Clearly, $g_1 | g$ and so, p_1 is smaller than or equal to smallest prime in the prime factor decomposition of g_1 . Now Lemma 2(iii) immediately yields,

$$v_{k,n} < \frac{2k^{g_1/p_1}}{n} \leq \frac{2(1 + sn)^{1/p_1}}{n} = o(1).$$

■

3 Limiting Spectral distribution

Study of the distributional properties of large dimensional random matrices has arisen in many areas of science and has received considerable attention. For an excellent review, see Bai (1999)[1]. For a random $n \times n$ matrix B_n , let μ_1, \dots, μ_n denote its eigenvalues. The empirical spectral distribution (ESD) of B_n is defined to be the random distribution function on \mathbb{R}^2 given by

$$F_{B_n}(x, y) = n^{-1} |\{\mu_k : \Re(\mu_k) \leq x, \Im(\mu_k) \leq y\}|,$$

where for any $z \in \mathbb{C}$, $\Re(z)$ and $\Im(z)$ denote the real part and imaginary part of z respectively. As $n \rightarrow \infty$, if F_{A_n} converges weakly (either almost surely or in probability) to a (nonrandom) distribution F , then F is called the limiting spectral distribution (LSD) of $\{B_n\}$. See Bai (1999)[1] and Bose and Sen (2007)[4] for description of several interesting situations where the LSD exists and can be explicitly specified.

For the k -circulant, in many cases of course there are a host of zero eigenvalues and the limit distribution is the point mass at zero. Here is a general result in that direction.

Theorem 2 *Suppose $\{a_i\}$ is an i.i.d. sequence of $N(0, 1)$ random variables. Let $k \geq 2$ be such that $k = n^{o(1)}$ and $n \rightarrow \infty$ with $\gcd(n, k) = 1$. Then the ESD $F_{n^{-1/2}A_{k,n}}$ of $n \times n$ k -circulant matrices $\{A_{k,n}\}$ converges in probability to the distribution which is degenerate at 0.*

Proof Let λ_j be as given in (1). Then each $n^{-1}|\lambda_j|^2 \sim Y/2$ where Y is a chi-square random variable with 2 degrees of freedom. We also have that λ_j is independent of λ_l if $l \neq n - j$ and $|\lambda_j| = |\lambda_l|$ otherwise. Recall $g_1 = |S(1)|$. Let $m \in \{1, 2, \dots, n-1\}$ be an element whose order g_m divides $c > 1$. Then $mk^c = m \pmod n$, or equivalently, $m(k^c - 1) = sn$ for some $s \geq 1$. The number of such elements is clearly bounded by k^c . It easily follows that for a fixed $C > 1$,

$$n^{-1} |\{j : g_j \leq C\}| \leq n^{-1} \sum_{c=2}^C k^c = Cn^{-1}k^C \text{ as } n \rightarrow \infty.$$

Let r_j be the absolute value of j -th eigenvalue of k -circulant matrix $A_{k,n}$. It is enough to prove the corresponding empirical distribution $G_n(\cdot) = n^{-1} \sum_{j=1}^n \mathbb{I}(n^{-1/2}r_j \leq \cdot)$ converges in probability to δ_0 , the degenerate distribution at zero.

Fix $r > 0$. Suppose $Z_1, Z_2, \dots \stackrel{i.i.d.}{\sim} Y/2$ and define

$$\psi(m) := \mathbb{P}((Z_1 Z_2 \dots Z_m)^{1/2m} > r) = \mathbb{P}((2m)^{-1} \sum_{j=1}^m \log Z_j > \log r).$$

Fix $j \geq 1$. Then $j \in \mathcal{P}_l = S(x)$ for some l and $x \in \{1, 2, \dots, n-1\}$. Observe that if $S(x)$ is conjugate to itself, then $\mathbb{P}(n^{-1/2}r_j > r) = \psi(m/2)$. On the other hand, if $S(x)$ is not a conjugate to itself, then $\mathbb{P}(n^{-1/2}r_j > r) = \psi(m)$.

Note that,

$$\mathbb{E}\bar{G}_n(r) = n^{-1} \sum_j \mathbb{P}(n^{-1/2}r_j > r) \tag{3}$$

$$= n^{-1} \sum_{l: |\mathcal{P}_l| \leq C} \sum_{j \in \mathcal{P}_l} \mathbb{P}(n^{-1/2}r_j > r) + n^{-1} \sum_{l: |\mathcal{P}_l| > C} \sum_{j \in \mathcal{P}_l} \mathbb{P}(n^{-1/2}r_j > r). \tag{4}$$

Note that $\mathbb{E}((\log Z_1)^+) < \infty$ but $\mathbb{E}(\log Z_1) = -\infty$. Hence, by law of large numbers, as $m \rightarrow \infty$,

$$(2m)^{-1} \sum_{j=1}^m \log Z_j \xrightarrow{P} -\infty, \text{ implying } \psi(m) \rightarrow 0.$$

Given $\epsilon > 0$, take C large enough so that $\psi(m) \leq \epsilon$. Then

$$\limsup \mathbb{E} \bar{G}_n(r) \leq 0 + \limsup n^{-1} \sum_{l: |\mathcal{P}_l| > C} \sum_{j \in \mathcal{P}_l} \mathbb{P}(n^{-1/2} r_j > r) \leq \epsilon.$$

Therefore $\bar{G}_n(r) \xrightarrow{L_1} 0$ and hence $\bar{G}_n(r) \xrightarrow{P} 0 \quad \forall r > 0$. ■

We now investigate the case of possible nondegenerate LSD. If the input sequence $\{a_i\}$ is i.i.d. with finite third moment, then the limit distribution of the (one-) circulant is bivariate complex normal (Bose and Mitra (2002)[3]). For the *symmetric* circulant with i.i.d. input having finite second moment, the LSD is real normal, (Bose and Sen (2007)[4]). For the k -circulant with $k = n - 1$, the LSD is the symmetric version of the positive square root of $Y/2$ where Y is chi square with two degrees of freedom (Bose and Mitra (2002)[3]). Establishing the LSD for general k -circulants appears to be a difficult problem. Below we consider a few cases. In particular, the results on one-circulant and k -circulant with $k = n - 1$ follow as special cases.

The following observation will be useful to us. Suppose $\{a_i\}$ are independent, mean zero and variance one random variables. Then for all $j \geq 1$,

$$\bar{\lambda}_j = \lambda_{n-j}, \quad \mathbb{E}(\lambda_j \lambda_l) = n \mathbb{1}(l = n - j), \quad E(\Re(\lambda_j) \Im(\lambda_j)) = 0, \quad \text{and} \quad \mathbb{E} \Re(\lambda_j)^2 = \mathbb{E} \Im(\lambda_j)^2 = n/2.$$

Theorem 3 *Suppose $\{a_i\}$ satisfies Assumption I. Suppose $k^g = -1 + sn$ where $g \geq 1$ fixed and $s = o(n^{1/3})$. Then $F_{n^{-1/2} A_{k,n}}$ converges weakly in probability to the LSD $U_1(\prod_{i=1}^g E_i)^{1/2g}$ as $n \rightarrow \infty$ where $\{E_i\}$ are i.i.d. $\text{Exp}(1)$ and U_1 is uniformly distributed over the $(2g)$ th roots of unity, independent of the $\{E_i\}$.*

Remark 2 *In view of Theorem 1, the above theorem can easily extended to yield an LSD has some positive mass at the origin. For example, suppose the sequences k and n satisfy $k^g = -1 + sn$ where $g \geq 1$ fixed and $s = o(n^{1/3})$. Fix primes p_1, p_2, \dots, p_t and positive integers $\beta_1, \beta_2, \dots, \beta_t$. Define*

$$\tilde{n} = p_1^{\beta_1} p_2^{\beta_2} \dots p_t^{\beta_t} n.$$

Suppose $k = p_1 p_2 \dots p_t m \rightarrow \infty$. Then the ESD of $\tilde{n}^{-1/2} A_{k, \tilde{n}}$ converges weakly in probability to the LSD which has $1 - \left(\prod_{s=1}^t p_s^{\beta_s}\right)^{-1}$ mass at zero, and rest of the probability mass is distributed as $U_1(\prod_{i=1}^g E_i)^{1/2g}$ where U_1 and $\{E_i\}$ are as above.

Proof First note that $\gcd(k, n) = 1$, and hence in this case, $n' = n$ in Theorem 1. Thus, the index of each eigenvalue belongs to one of the sets \mathcal{P}_l in the eigenvalue partition of $\{0, 1, \dots, n - 1\}$. Recall that $v_{k,n}$ is the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{P}_l$ and $|\mathcal{P}_l| < g_1$. In view of Lemma 3 (i), we have $v_{k,n}/n \rightarrow 0$ and hence they do not contribute to the LSD. Hence, it remains to consider only the eigenvalues corresponding to the sets \mathcal{P}_l which have size *exactly* equal to g_1 . Also it follows from the argument in the proof of Lemma 3 (i) that $g_1 = 2g$.

Recall the quantities $n_j = |\mathcal{P}_j|$, $x_j = \prod_{t \in \mathcal{P}_j} \lambda_t$, where $\lambda_j = \sum_{l=0}^{n-1} a_l \omega^{jl}$, $0 \leq j < n$. Also, for every integer $t \geq 0$, $tk^g = (-1 + sn)t = -t \pmod n$, so that, λ_t and λ_{n-t} belong to same partition block $S(t) = S(n - t)$. Thus each x_j is real. Let us define

$$I_n = \{l : |\mathcal{P}_l| = 2g\}.$$

It is clear that $n/|I_n| \rightarrow 2g$. Without any loss, we denote the index set of such j as $I_n = \{1, 2, \dots, |I_n|\}$.

Let $1, \omega, \omega^2, \dots, \omega^{2g-1}$ be all the $(2g)$ th roots of unity. Note that for every j , the eigenvalues of A_n corresponding to the set \mathcal{P}_j are: $x_j^{1/2g}, x_j^{1/2g} \omega, \dots, x_j^{1/2g} \omega^{2g-1}$. Hence, it suffices to consider only the modulus of eigenvalues $x_j^{1/2g}$ as j varies: if these have an LSD F , say, then the LSD of the whole sequence will be

(r, θ) in polar coordinates where r is distributed according to F and θ is distributed uniformly across all the $2g$ roots of unity and r and θ are independent. With this in mind, and remembering the scaling \sqrt{n} , we consider

$$F_n(x) = |I_n|^{-1} \sum_{i=1}^{|I_n|} \mathbb{I} \left(\left[\frac{x_j}{\sqrt{n}} \right]^{\frac{1}{2g}} \leq x \right).$$

Since the set of λ values corresponding to any \mathcal{P}_j is closed under conjugation, there exists a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size g such that

$$\mathcal{P}_i = \{x : x \in \mathcal{A}_i \text{ or } n - x \in \mathcal{A}_i\}.$$

Combining each λ_j with its conjugate, we may write x_j in the form,

$$x_j = \prod_{t \in \mathcal{A}_j} (a_{t,n}^2 + b_{t,n}^2)$$

where $\{a_{t,n}\}$ and $\{b_{t,n}\}$ are of the form

$$a_{t,n} = \sum_{l=0}^{n-1} a_l \cos\left(\frac{2\pi tl}{n}\right), \quad b_{t,n} = \sum_{l=0}^{n-1} a_l \sin\left(\frac{2\pi tl}{n}\right), \quad t \in \mathcal{A}_j, \quad 1 \leq j \leq |I_n|.$$

It may be noted that the sine and cosine terms above have the useful orthogonality properties:

$$\begin{aligned} \sum_{l=0}^{n-1} \cos\left(\frac{2\pi tl}{n}\right) \sin\left(\frac{2\pi t'l}{n}\right) &= 0, \quad \text{and} \quad \sum_{l=0}^{n-1} \cos^2\left(\frac{2\pi tl}{n}\right) = \sum_{l=0}^{n-1} \sin^2\left(\frac{2\pi tl}{n}\right) = n/2 \quad \forall t, t'. \\ \sum_{l=0}^{n-1} \cos\left(\frac{2\pi tl}{n}\right) \cos\left(\frac{2\pi t'l}{n}\right) &= 0, \quad \text{and} \quad \sum_{l=0}^{n-1} \sin\left(\frac{2\pi tl}{n}\right) \sin\left(\frac{2\pi t'l}{n}\right) = 0 \quad \forall t \neq t'. \end{aligned}$$

Claim. Suppose that the variables $\{a_i\}$ are i.i.d. standard normal. Then for every n , $n^{-1/2}x_j$ are i.i.d. and each is distributed as g -fold product of i.i.d. random variables each of which is in turn distributed as $Y/2$ where Y is Chi-square with two degrees of freedom. Note that $Y/2$ has the same distribution as that of $\text{Exp}(1)$ random variable.

This claim follows easily from the above orthogonality property which implies that $\{a_{t,n}, b_{t,n}\}$ are i.i.d. normal with mean zero and variance $n/2$.

Continuing with the proof, first assume that $\{a_i\}$ are i.i.d. standard normal. In this case, F_n is the usual empirical distribution of $|I_n|$ observations on $(\prod_{i=1}^g E_i)^{1/2g}$ where E_i are i.i.d. $\text{Exp}(1)$. Thus by Glivenko-Cantelli Lemma, this converges to the distribution of $(\prod_{i=1}^g E_i)^{1/2g}$. Note that though the variables involved in the empirical distribution form a triangular sequence here, the convergence is still almost sure due to the specific bounded nature of the indicator functions involved. This may be proved easily by considering higher moments, and using Borel-Cantelli lemma.

As mentioned earlier, all eigenvalues corresponding to any partition block \mathcal{P}_j are all the $(2g)$ th roots of the product x_j . Thus, the limit claimed in the statement of the theorem holds. So we have proved the result when the $\{a_j\}$ are i.i.d. standard normal.

Now suppose that the variables $\{a_j\}$ are not necessarily normal. This case is tackled by normal approximation arguments similar to Bose and Mitra (2002)[3] who deal with the case $k = n - 1$ (and hence $g = 1$). We now sketch some of the steps. The basic idea remains the same but with some added notational and technical complications. Before we use the normal approximation, we establish the following lemma.

Lemma 4 Define for some for some $r > 0$, the set

$$A = \{(x_1, x_2, \dots, x_{2p}) : (x_1^2 + x_2^2)^{1/2}(x_3^2 + x_4^2)^{1/2} \dots (x_{2p-1}^2 + x_{2p}^2)^{1/2} \leq r\}.$$

Let $(\partial A)^\eta$ be the η -boundary of A and let $\Phi_{2p}(\cdot)$ denote the cumulative distribution function of $2p$ -dimensional standard normal vector. Then there exists a constant C such that for all small $\eta > 0$, for any $\epsilon > 0$,

$$\sup_{y \in \mathbb{R}^p} \Phi_{2p}((\partial A)^\eta - y) \leq C\eta^{1-\epsilon}.$$

Proof Note that for convex sets A results of above kind are well known. Here the main difficulty comes from the fact that the given set A is not convex. To overcome that we will work towards converting this problem into one where we only have to deal with a convex set.

Recall that η -boundary of A is given by

$$(\partial A)^\eta = \{(x_1, x_2, \dots, x_{2p}) : r - \eta \leq (x_1^2 + x_2^2)^{1/2}(x_3^2 + x_4^2)^{1/2} \dots (x_{2p-1}^2 + x_{2p}^2)^{1/2} \leq r + \eta\}.$$

Let $g(y) = \Phi_{2p}((\partial A)^\eta - y)$. Then it is easy to see that g is continuous. We claim that as $\max_i |y_i| \rightarrow \infty$, $g(y) \rightarrow 0$. To see this, without loss, assume that $\max_i |y_i| = |y_1| \rightarrow \infty$. Let

$$S = \{(x_3, x_4, \dots, x_{2p}) : (x_3^2 + x_4^2)^{1/2} \dots (x_{2p-1}^2 + x_{2p}^2)^{1/2} > 0\}.$$

On the set S , define the following two functions,

$$r_1(x_3, x_4, \dots, x_{2p}) = (r - \eta) / [(x_3^2 + x_4^2)^{1/2} \dots (x_{2p-1}^2 + x_{2p}^2)^{1/2}]$$

and

$$r_2(x_3, x_4, \dots, x_{2p}) = (r + \eta) / [(x_3^2 + x_4^2)^{1/2} \dots (x_{2p-1}^2 + x_{2p}^2)^{1/2}].$$

Observe that

$$\begin{aligned} \Phi_{2p}((\partial A)^\eta + y) &= \int_{(\partial A)^\eta} \phi(x_1 - y_1)\phi(x_2 - y_2) \dots \phi(x_{2p} - y_{2p}) dx_1 dx_2 \dots dx_{2p} \\ &= \int_{(x_3, \dots, x_{2p}) \in S} \left[\int_{r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2} \phi(x_1 - y_1)\phi(x_2 - y_2) dx_1 dx_2 \right] \dots \phi(x_{2p} - y_{2p}) \dots dx_{2p}. \end{aligned}$$

A simple application of DCT shows that $\Phi_{2p}((\partial A)^\eta + y) \rightarrow 0$ as $|y_1| \rightarrow \infty$. Hence there exists a point $y^0 = (y_1^0, y_2^0, \dots, y_{2p}^0) \in \mathbb{R}^{2k}$ such that $g(y^0) = \sup_{y \in \mathbb{R}^{2k}} g(y)$. Let $L := \max_i |y_i^0|$.

Now suppose, X_i is distributed as $N(y_i^0, 1)$. Then we have the very crude tail bound, $\mathbb{P}(|X_i| > M) \leq e^{-(M-L)}$. For sufficiently small η , let $M = -2 \log \eta$. Then $\mathbb{P}(|X_i| > M) \leq \eta$. Now,

$$\begin{aligned} \Phi_{2p}((\partial A)^\eta + y^0) &= \int_{(\partial A)^\eta} \phi(x_1 - y_1^0)\phi(x_2 - y_2^0) \dots \phi(x_{2p} - y_{2p}^0) dx_1 dx_2 \dots dx_{2p} \\ &\leq 2k\eta + \int_{(\partial A)^\eta, |x_i| \leq M \forall i} \phi(x_1 - y_1^0)\phi(x_2 - y_2^0) \dots \phi(x_{2p} - y_{2p}^0) dx_1 dx_2 \dots dx_{2p}. \end{aligned}$$

If $(x_1, x_2, \dots, x_{2p}) \in (\partial A)^\eta$ and $\max_i |x_i| \leq M$, then

$$(x_{2i-1}^2 + x_{2i}^2)^{1/2} \geq \frac{r - \eta}{(2M)^{k-1}} \quad \forall 1 \leq i \leq k.$$

Thus after fixing $(2k - 2)$ coordinates x_3, x_4, \dots, x_{2p} , (x_1, x_2) can vary atmost within the following range.

$$(r - \eta) \left(\frac{(2M)^{k-1}}{r - \eta} \right)^{k-1} \leq (x_1^2 + x_2^2)^{1/2} \leq (r + \eta) \left(\frac{(2M)^{k-1}}{r - \eta} \right)^{k-1}.$$

This set is contained in a set which can be given in a compact form as follows

$$R - c\eta(-\log \eta)^{(k-1)^2} \leq (x_1^2 + x_2^2)^{1/2} \leq R + c\eta(-\log \eta)^{(k-1)^2}, \text{ for some suitable constants } R \text{ and } c.$$

Let $D((a, b); s)$ denote the closed disc in \mathbb{R}^2 with centre at (a, b) and radius s . Let $\delta = c\eta(-\log \eta)^{(k-1)^2}$. Then from the above calculations,

$$\begin{aligned} \int_{(\partial A)^n, |x_i| \leq M \quad \forall i} \phi(x_1 - y_1^0) \dots \phi(x_{2p} - y_{2k}^0) dx_1 \dots dx_{2p} &\leq \int_{(\partial D((0,0); R); \delta)} \phi(x_1 - y_1^0) \phi(x_2 - y_2^0) dx_1 dx_2 \\ &= \int_{(\partial D((y_1^0, y_2^0); R); \delta)} \phi(x_1) \phi(x_2) dx_1 dx_2 \\ &\leq c_1 \delta \quad \text{for some constant } c_1 > 0. \end{aligned}$$

Since $D((y_1^0, y_2^0); R)$ is convex, the last step above follows from Corollary 3.2 of Bhattacharya and Ranga Rao (1976)[2] which implies that for all convex sets E ,

$$\int_{(\partial E)^\delta} \phi(x_1) \phi(x_2) dx_1 dx_2 \leq K\delta$$

where $(\partial E)^\delta$ is the δ -boundary of E and K is an absolute constant (independent of E).

Combining all the estimates, the proof is complete. ■

Continuing with the proof of the theorem, let us define for $\{E_i\}$ are i.i.d. $\text{Exp}(1)$,

$$F(x) = \mathbb{P} \left(\left(\prod_{i=1}^g E_i \right)^{1/2g} \leq x \right), \quad x > 0.$$

We show that for every x ,

$$\mathbb{E}[F_n(x)] \rightarrow F(x) \text{ and } \text{Var}[F_n(x)] \rightarrow 0.$$

This will prove that the ESD converges to the required LSD in probability.

Note that for $x > 0$,

$$\mathbb{E}[F_n(x)] = |I_n|^{-1} \sum_{j=1}^{|I_n|} \mathbb{P} \left(\frac{x_j}{\sqrt{n}} \leq x^{2g} \right).$$

Using the orthogonality properties mentioned earlier, and the central limit theorem for independent random variables, it is clear that for any finite sets K and J of indices, of k and j , as $n \rightarrow \infty$,

$$(a_{t,n}, b_{t',n}, t \in K, t' \in J) \rightarrow N(0, D) \text{ in distribution}$$

where D is the diagonal matrix with diagonal entries $1/2$. This motivates using normal approximations.

Towards using an appropriate Berry-Esséen bound, define

$$X_{l,j} = 2^{1/2} \left(a_l \cos \left(\frac{2\pi t l}{n} \right), a_l \sin \left(\frac{2\pi t l}{n} \right), t \in \mathcal{A}_j \right) \quad 0 \leq l < n, 1 \leq j \leq |I_n|.$$

Note that

$$\mathbb{E}(X_{l,j}) = 0 \text{ and } n^{-1} \sum_{l=1}^{n-1} \text{Cov}(X_{l,j}) = I_{2g} \quad \forall j, n.$$

Define the set $A \subseteq \mathbb{R}^{2g}$ as

$$A = \left\{ (a_i, b_i; i = 1, 2, \dots, g) : \prod_{i=1}^g [2^{-1}(a_i^2 + b_i^2)] \leq x^{2g} \right\}.$$

Note that

$$\left\{ \frac{x_j}{\sqrt{n}} \leq x^{2g} \right\} = \left\{ \sum_{l=1}^{n-1} X_{l,j} \in A \right\}.$$

The proof of the following Lemma follows easily from Theorem 18.1, page 181 of Bhattacharya and Ranga Rao [2]. We omit the proof.

Lemma 5 *Let X_1, \dots, X_k be independent random vectors with values in \mathbb{R}^d , having zero means and an average positive-definite covariance matrix $V_k = k^{-1} \sum_{j=1}^k \text{Cov}(X_j)$. Let G_k denote the distribution of $k^{-1/2} T_k(X_1 + \dots + X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \geq 1$. If for some $\delta > 0$, $\mathbb{E} \|X_j\|^{(2+\delta)} < \infty$, then there exists $C_1, C_2 > 0$ (depending only on d), such that for any Borel set A ,*

$$\begin{aligned} |G_k(A) - \Phi_d(A)| &\leq C_1 k^{-\delta/2} \left[k^{-1} \sum_{j=1}^k \mathbb{E} \|T_k X_j\|^{(2+\delta)} \right] + 2 \sup_{y \in \mathbb{R}^d} \Phi_d((\partial A)^\eta - y), \\ &\leq C_1 k^{-\delta/2} (\lambda_{\min}(V_k))^{-(2+\delta)} \rho_{2+\delta} + 2 \sup_{y \in \mathbb{R}^d} \Phi_d((\partial A)^\eta - y), \end{aligned}$$

where $\rho_{2+\delta} = k^{-1} \sum_{j=1}^k \mathbb{E} \|X_j\|^{(2+\delta)}$ and $\eta = C_2 \rho_{2+\delta} n^{-\delta/2}$.

Using this result and Lemma 4, we get for any small $\epsilon > 0$, and some constant C ,

$$\left| \mathbb{P}\left(\frac{x_j}{\sqrt{n}} \leq x^{2g}\right) - \Phi_{2g}(A) \right| \leq C n^{-\delta/2} \rho_{2+\delta} + C (\rho_{2+\delta} n^{-\delta/2})^{1-\epsilon}$$

where

$$\rho_{2+\delta} = n^{-1} \sum_{l=0}^{n-1} \mathbb{E} \|X_{l,j}\|^{2+\delta}.$$

Now using Assumption I, it is easy to see that

$$\mathbb{E}[F_n(x)] - \Phi_{2g}(A) = |I_n|^{-1} \sum_{l=1}^{|I_n|} \left(\mathbb{P}\left(\frac{x_j}{\sqrt{n}} \leq x^{2g}\right) - \Phi_{2g}(A) \right) \rightarrow 0.$$

This proves that for $x > 0$, $\mathbb{E}[F_n(x)] \rightarrow F(x)$.

To show that $\text{Var}[F_n(x)] \rightarrow 0$, since the variables involved are all bounded, it is enough to show that

$$n^{-2} \sum_{j \neq j'} \text{Cov} \left(\mathbb{I}\left(\frac{x_j}{\sqrt{n}} \leq x^{2g}\right), \mathbb{I}\left(\frac{x_{j'}}{\sqrt{n}} \leq x^{2g}\right) \right) \rightarrow 0.$$

Along the lines of the proof used to show $\mathbb{E}[F_n(x)] \rightarrow F(x)$, one may now extend the vectors with $2g$ coordinates defined above to ones with $4g$ coordinates and proceed exactly as above to verify this. We omit the routine details. This completes the proof of Theorem 3. \blacksquare

We now state another result which also describes the LSD for k -circulant matrices but where k, n satisfy a different relation.

Theorem 4 *Suppose $\{a_i\}$ satisfies Assumption I. Suppose $k^g = 1 + sn$ where $g \geq 2$ is fixed and*

$$s = \begin{cases} o(n) & \text{if } g \text{ is even} \\ o\left(n^{\frac{g+1}{g-1}}\right) & \text{if } g \text{ is odd.} \end{cases}$$

Then $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to the LSD $U_2(\prod_{i=1}^g E_i)^{1/2g}$ as $n \rightarrow \infty$ where E_i are i.i.d. $\text{Exp}(1)$ and U_2 is uniformly distributed over the circle of unit radius, independent of the $\{E_i\}$.

Remark 3 *Note that while the radial coordinates of the LSD described in Theorem 3 and 4 are same, their angular coordinates differ. While one puts its mass only at discrete places $e^{i2\pi t/2g}$, $1 \leq t \leq 2g$ on the unit circle, the other spreads its mass uniformly over the entire unit circle. See Figure 1.*

We will not attempt to give a detailed proof of Theorem 4 here but let us sketch the main idea. First of all, note that $\gcd(k, n) = 1$. Since $k^g = 1 + sn = 1 \pmod{n}$, we have $g_1 | g$. If $g_1 < g$, then $g_1 \leq g/\alpha$ where $\alpha = 2$ if g is even and $\alpha = 3$ if g is odd. In either case, it is easy to check that

$$k^{g_1} \leq k^{g/\alpha} \leq (1 + sn)^{1/\alpha} = o(n).$$

Hence, $g = g_1 \cdot$ By Lemma 3 (ii) the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{P}_l$ and $|\mathcal{P}_l| < g$ is asymptotically negligible.

Unlike the previous theorem, here the partition sets \mathcal{P}_l are not necessarily self-conjugate. In this case one can show that the number of indices l such that \mathcal{P}_l is self-conjugate is asymptotically negligible compared to n . For that, we need to bound the cardinality of the following set for $1 \leq l < g$:

$$\{t \in \{1, 2, \dots, n\} : tk^l = -t \pmod{n}\} = \{t \in \{1, 2, \dots, n\} : n | t(k^l + 1)\}.$$

Note that $t_0 = n / \gcd(n, k^l + 1)$ is the minimum element of the above set and every other element of the set is a multiple of t_0 . Let $t_0(k^l + 1) = nm$ for some positive integer m . Note that $(k^l + 1) | nm$ implies $m \geq \frac{k^l + 1}{\gcd(n, k^l + 1)}$. Combining, we have

$$t_0(k^l + 1) \geq \frac{k^l + 1}{\gcd(n, k^l + 1)} n.$$

Thus the size of the above set is bounded by

$$\frac{n}{t_0} \leq \gcd(n, k^l + 1).$$

Let us now estimate $\gcd(n, k^l + 1)$. For $l > [g/2]$,

$$\gcd(n, k^l + 1) \leq \gcd(k^g - 1, k^l + 1) = \gcd(k^{g-l}(k^l + 1) - (k^{g-l} - 1), k^l + 1) \leq k^{g-l},$$

which implies $\gcd(n, k^l + 1) \leq k^{[g/2]}$ for all $1 \leq l < g$. Therefore,

$$\frac{\gcd(n, k^l + 1)}{n} \leq \frac{k^{[g/2]}}{(k^g - 1)/s} \leq \frac{2s}{k^{[(g+1)/2]}} \leq \frac{2s}{((sn)^{1/g})^{[(g+1)/2]}} = o(1),$$

since, by hypothesis, $s = o\left(n^{\frac{[(g+1)/2]}{[g/2]}}\right)$. So, we can ignore the partition sets which are self-conjugate.

For other \mathcal{P}_j ,

$$x_j = \prod_{t \in \mathcal{P}_j} (a_{t,n} + ib_{t,n})$$

will be complex. Hence the empirical distribution of x_j for those j for which $|\mathcal{P}_j| = g$ should converge to $\prod_{t=1}^g N_t$ where $N_t, 1 \leq t \leq g$ are i.i.d. complex normals, the components being independent with mean zero and variance $1/2$. Observe that such a complex normal can be alternatively expressed as

$$N_t \stackrel{\mathcal{D}}{=} (E_t)^{1/2} U_t,$$

where E_t is $\text{Exp}(1)$ and U_t is distributed uniformly over the unit circle in the plane and they are independent. Hence the LSD of $n^{-1/2}A_{k,n}$ should be as described in the theorem. This can be rigorously proved using the same line of argument as given in the proof of Theorem 3. We omit the details.

4 Spectral Norm

In this section we seek the limiting distribution of the spectral norm of k -circulant matrices for the simple subcase of Theorem 3 where $n = k^2 + 1$. Consider, as in the proof of Theorem 3, that the input sequence is i.i.d. standard normal. Then, as we have seen in that proof, the modulus of the nonzero eigenvalues are essentially independent and distributed according to $(E_1 E_2)^{1/4}$, where $E_i, i = 1, 2$ are i.i.d. standard exponential random variables. Thus, the behavior of the spectral norm automatically translates to the problem of studying the maxima of $(E_1 E_2)^{1/4}$, which given the extreme value theory, is not too difficult. In addition, as suggested by the results of Davis and Mikosch (1999)[6], even when the input sequence is not i.i.d. normal but just mean zero and variance 1 with suitable moment condition, some kind of invariance principle holds and the same limit persists. The next theorem affirms our belief. But before that, let us recall the following definition.

Definition 1 A probability distribution is said to be Gumbel with parameter $\theta > 0$ if its cumulative distribution function is given by

$$\Lambda_\theta(x) = \exp\{-\theta \exp(-x)\}, \quad x \in \mathbb{R}.$$

The special case when $\theta = 1$ is known as standard Gumbel distribution and its cumulative distribution function is simply denoted by $\Lambda(\cdot)$.

Theorem 5 Suppose $\{a_i\}_{i=0}^\infty$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathbb{E}|a_i|^\gamma < \infty$ for some $\gamma > 2$. If $n = k^2 + 1$ then

$$\frac{\|n^{-1/2}A_{k,n}\|_2 - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

as $n \rightarrow \infty$ where $q = q(n) = \lceil \frac{n}{4} \rceil$ and c_n and d_n can be taken as given in (7).

Notation. For the subsequent development, we are going to need the following notations some of which have been already used in the proof of Theorem 3. Set $q = \lceil \frac{n}{4} \rceil$. By Lemma 7, in the present case, the eigenvalue partition of $\{0, 1, 2, \dots, n-1\}$ contains exactly q sets of size 4, which we denote by $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_q$. Since each \mathcal{P}_i is self-conjugate, we can find a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size 2 such that

$$S_i = \{x : x \in \mathcal{A}_i \text{ or } n - x \in \mathcal{A}_i\}. \quad (5)$$

For any sequence of numbers $(b_l)_{l \in \mathbb{Z}}$ define

$$\beta_{b,n}(t) = n^{-2} \prod_{j \in \mathcal{A}_t} \left| \sum_{l=0}^{n-1} b_l \exp(i\omega t l) \right|^2, \quad \omega = \frac{2\pi}{n}, \quad 1 \leq t \leq q. \quad (6)$$

Proof of Theorem 5 Note that with the above notation,

$$\|n^{-1/2} A_{k,n}\|_2 = \max_{1 \leq t \leq q} (\beta_{a,n}(t))^{1/4}.$$

As in the proof of Theorem 3, first assume that $\{a_i\}$ are i.i.d. standard normal. Let E_1, E_2, \dots, E_{2q} be i.i.d. standard Exponential. Recalling the claim made in the proof of Theorem 3, it easily follows that

$$\mathbb{P}\left(\max_{1 \leq t \leq q} (\beta_{a,n}(t))^{1/4} > c_q x + d_q\right) = \mathbb{P}\left((E_{2i-1} E_{2i})^{1/4} > c_q x + d_q \text{ for some } 1 \leq i \leq q\right).$$

The Theorem then follows in this special case from the Lemma below. This result may be known in the literature. Since we could not find a reference, we provide a sketch of the proof.

Lemma 6 Suppose G is the distribution of $(E_1 E_2)^{1/4}$ where E_i are i.i.d. standard exponential random variables. If G_i are i.i.d. random variables with the distribution G , and $G^{(n)} = \max\{G_i : 1 \leq i \leq n\}$, then

$$\frac{G^{(n)} - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda.$$

where c_n and d_n are normalising constants which can be taken as follows

$$c_n = (8 \log n)^{-1/2} \quad \text{and} \quad d_n = \frac{(\log n)^{1/2}}{\sqrt{2}} \left(1 + \frac{1}{4} \frac{\log \log n}{\log n}\right) + \frac{1}{2(8 \log n)^{1/2}} \log \frac{\pi}{2}. \quad (7)$$

Proof Define

$$\bar{K}(x) := P\{E_1 E_2 > x\} = \int_0^\infty \exp(-y) \exp(-xy^{-1}) dy, \quad x > 0. \quad (8)$$

Differentiating (8) twice, we get

$$\frac{d^2}{dx^2} \bar{K}(x) = \int_0^\infty y^{-2} \exp(-y) \exp(-xy^{-1}) dy, \quad (9)$$

which implies that \bar{K} satisfies the differential equation

$$\begin{aligned} x \frac{d^2}{dx^2} \bar{K}(x) - \bar{K}(x) &= - \int_0^\infty (1 - xy^{-2}) \exp(-(y + xy^{-1})) dy \\ &= \exp(-(y + xy^{-1})) \Big|_0^\infty = 0, \quad \text{for } x > 0, \end{aligned} \quad (10)$$

with the boundary conditions $\bar{K}(0) = 1$ and $\bar{K}(\infty) = 0$.

The theory of second order differential equations now tells us that the only solution to (10) is given by

$$\bar{K}(x) = \pi x^{1/2} H_1^1(2ix^{1/2}), \quad x > 0$$

where $i^2 = -1$ and H_1^1 is related to Bessel function, as for example given in Watson (1944)[15]. More precisely,

$$H_1^1(x) = J_1(x) + iY_1(x)$$

and J_1 and Y_1 are order one Bessel functions of the first and second kind respectively.

It also follows from the well-known theory of the asymptotic properties of the Bessel functions J_1 and Y_1 , that

$$\overline{K}(x) \asymp \pi^{1/2} x^{1/4} \exp(-2x^{1/2}) \text{ as } x \rightarrow \infty. \quad (11)$$

Now let

$$\overline{G}(x) = P\{(E_1 E_2)^{1/4} > x\}. \quad (12)$$

From (11)

$$\overline{G}(x) \asymp \pi^{1/2} x \exp(-2x^2) \text{ as } x \rightarrow \infty \quad (13)$$

We now use Proposition 1.1 and the development on pages 43 and 44 of Resnick (1996)[13] to establish the Lemma. In particular, we need to show that,

$$\overline{G}(x) = \theta(x)(1 - F_{\#}(x))$$

where

$$\lim_{x \rightarrow \infty} \theta(x) = \theta > 0$$

and, there exists some x_0 and a function f such that $f(y) > 0$ for $y > x_0$ such that f has an absolute continuous density with $f'(x) \rightarrow 0$ as $x \rightarrow \infty$ so that

$$1 - F_{\#}(x) = \exp\left(-\int_{x_0}^x (1/f(y))dy\right), \quad x > x_0. \quad (14)$$

Moreover, a choice for the normalizing constants c_n and d_n is then given by

$$d_n^* = \left(1/(1 - F_{\#})\right)^{-1}(n), \quad c_n^* = f(d_n^*). \quad (15)$$

Then

$$\frac{G^{(n)} - d_n^*}{c_n^*} \xrightarrow{\mathcal{D}} \Lambda_{\theta}.$$

Towards this end, define for $x \geq 1$,

$$\theta(x) \asymp \pi^{1/2} e^{-2}, \quad 1 - F_{\#}(x) = x \exp\left(-2(x^2 - 1)\right), \quad x \geq 1 = x_0. \quad (16)$$

To solve for f , taking log on both sides,

$$\log x - 2(x^2 - 1) = -\int_1^x \frac{1}{f(y)} dy. \quad (17)$$

Taking derivative,

$$\frac{1}{x} - 2(2x) = -\frac{1}{f(x)}$$

or

$$f(x) = \frac{x}{4x^2 - 1}$$

and hence as $x \rightarrow \infty$,

$$f(x) \asymp \frac{1}{4x}.$$

Note that d_n^* (to be obtained) will tend to ∞ as $n \rightarrow \infty$. Hence

$$c_n^* = f(d_n^*) \asymp (4d_n^*)^{-1}.$$

We now proceed to obtain (the asymptotic form of) d_n^* . Using the defining equation (15),

$$d_n^* \exp^{-2((d_n^*)^2-1)} = n^{-1}. \quad (18)$$

Clearly, from the above, we may write

$$d_n^* = \left(\frac{\log n}{2}\right)^{1/2}(1 + \delta_n)$$

where $\delta_n \rightarrow 0$ is a *positive* sequence to be appropriately chosen. Thus, again using (18), we obtain

$$(\log n)(\delta_n^2 + 2\delta_n) - \left(\frac{1}{2} \log \log + \xi_n\right) = 0$$

where

$$\xi_n = 2 - \frac{1}{2} \log 2 + \log(1 + \delta_n).$$

“Solving” the quadratic,

$$\delta_n = \frac{-2 + \sqrt{4 + 4\left(\frac{1}{2} \log \log + \xi_n\right)/\log n}}{2}.$$

Using expansion $\sqrt{1+x} = 1 + \frac{1}{2}x + O(x^2)$ as $x \rightarrow 0$, we easily see that

$$\delta_n = \frac{1}{2} \left(\frac{\frac{1}{2} \log \log n + \xi_n}{\log n} \right) + O\left(\frac{(\log \log n)^2}{(\log n)} \right).$$

Hence

$$d_n^* = \left(\frac{\log n}{2}\right)^{1/2} \left(1 + \frac{\frac{1}{2} \log \log n + \xi_n}{2 \log n} \right) + O\left(\frac{(\log \log n)^2}{(\log n)^{3/2}} \right).$$

Simplifying, and dropping appropriate small order terms, we see that

$$\frac{G^{(n)} - \hat{d}_n}{\hat{c}_n} \xrightarrow{\mathcal{D}} \Lambda_{\pi^{1/2}e^{-2}}.$$

where

$$\hat{d}_n = \frac{(\log n)^{1/2}}{\sqrt{2}} \left(1 + \frac{1}{4} \frac{\log \log n}{\log n} \right) + \frac{1}{(8 \log n)^{1/2}} \left(2 - \frac{1}{2} \log 2 \right)$$

and

$$\hat{c}_n = (8 \log n)^{-1/2}.$$

To convert the above convergence to standard Gumbel distribution, we can borrow the following result from de Haan and Ferreira (2006) [7][Theorem 1.1.2] which says that the following two statements are equivalent for any sequence of $a_n > 0, b_n$ of constants and any nondegenerate distribution function H .

1.

$$\lim_{n \rightarrow \infty} G^n(c_n x + d_n) = H(x),$$

for each continuity point x of H .

2.

$$\lim_{t \rightarrow \infty} \frac{(1/(1-G))^{-1}(tx) - d_{[t]}}{c_{[t]}} = H^{-1}(e^{-1/x}),$$

for each $x > 0$ continuity point of $H^{-1}(e^{-1/x})$.

Now the relation $\Lambda_\theta^{-1}(e^{-1/x}) - \Lambda^{-1}(e^{-1/x}) = \log \theta$ and a simple calculation yield that

$$c_n = \hat{c}_n, \quad d_n = \hat{d}_n + \hat{c}_n \log(\pi^{1/2} e^{-2}).$$

■

We now tackle the general case by using truncation of $\{a_i\}$, Bonferroni's inequality and an appropriately sharp normal approximation result. We start with a simple Lemma about the structure of the eigenvalue partition of $\{0, 1, 2, \dots, n-1\}$.

Lemma 7 *Let $n = k^2 + 1$. Then*

$$v_{k,n} \leq \frac{2}{n}.$$

Proof Clearly, $S(1) = \{1, k, k^2, k^2 - k + 1\}$ and thus $g_1 = 4$. By Lemma 1,

$$|\{x : 0 \leq x < n, g_x < g_1\}| = \gcd(k^2 - 1, k^2 + 1) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}.$$

■

For each $n \geq 1$, define a triangular array of centered random variables $(\bar{a}_t^{(n)})_{0 \leq t < n}$ by

$$\bar{a}_t = \bar{a}_t^{(n)} = a_t I_{|a_t| \leq n^{1/\gamma}} - \mathbb{E} a_t I_{|a_t| \leq n^{1/\gamma}}.$$

Lemma 8 (Truncation) *Assume $\mathbb{E}|a_t|^\gamma$ for some $\gamma > 2$. Then*

$$\max_{1 \leq t \leq q} (\beta_{a,n}(t))^{1/4} - \max_{1 \leq t \leq q} (\beta_{\bar{a},n}(t))^{1/4} = o(1) \quad \text{a.s.},$$

where $\beta_{\bar{a},n}(t)$ is as given in (6) based on $\bar{a}_0^{(n)}, \bar{a}_1^{(n)}, \dots, \bar{a}_{n-1}^{(n)}$.

Proof Since $\sum_{l=0}^{n-1} \exp(il\omega j) = 0$ for $j \neq 0$, it follows that $\beta_{\bar{a},n}(t) = \beta_{a,n}(t)$ where

$$\bar{a}_t = \tilde{a}_t^{(n)} = \bar{a}_t + \mathbb{E} a_t I_{|a_t| \leq n^{1/\gamma}} = a_t I_{|a_t| \leq n^{1/\gamma}}$$

By Borel-Cantelli lemma, with probability one, $\sum_{t=0}^{\infty} |a_t| I_{|a_t| > t^{1/\gamma}}$ is finite and has only finitely many non-zero terms. Thus there exists an integer $N(\omega) \geq 0$ such that

$$\sum_{t=m}^{n-1} |\tilde{a}_t^{(n)} - a_t| = \sum_{t=m}^{n-1} |a_t| I_{|a_t| > n^{1/\gamma}} \leq \sum_{t=m}^{\infty} |a_t| I_{|a_t| > t^{1/\gamma}} = \sum_{t=m}^{N(\omega)} |a_t| I_{|a_t| > t^{1/\gamma}}, \quad (19)$$

Consequently, if $m > N(\omega)$, the left side of (19) is zero. Therefore, the sequences $(a_t)_{m \leq t < n}$ and $(\tilde{a}_t^{(n)})_{m \leq t < n}$ are identical a.s. for all sufficiently large n and the assertion follows immediately. ■

Lemma 9 (Bonferroni inequality) *Let B_1, B_2, \dots, B_n be events from a σ -field \mathcal{F} and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Then for every integer $k \geq 1$,*

$$\sum_{j=1}^{2m} (-1)^{j-1} S_{j,n} \leq \mathbb{P}\left(\bigcup_{i=1}^n B_i\right) \leq \sum_{j=1}^{2m-1} (-1)^{j-1} S_{j,n}, \quad (20)$$

where

$$S_{j,n} = \sum_{1 \leq i_1 < i_2 < \dots < i_j \leq n} \mathbb{P}\left(\bigcap_{l=1}^j B_{i_l}\right).$$

For $d \geq 1$, and any distinct integers i_1, i_2, \dots, i_d , from $\{1, 2, \dots, \lceil \frac{n-1}{2} \rceil\}$, define

$$v_d(l) = \left(\cos(\omega_{i_1} l), \sin(\omega_{i_1} l), \dots, \cos(\omega_{i_d} l), \sin(\omega_{i_d} l) \right)', \quad l = 0, 1, 2, \dots, n-1.$$

Let $\varphi_{\Sigma}(\cdot)$ denote the density of the $2d$ -dimensional Gaussian vector having mean zero and covariance matrix Σ and let I_{2d} be the identity matrix of order $2d$.

Lemma 10 (Normal approximation, Davis and Mikosch (1999)[6]) Fix $d \geq 1$ and $\gamma > 2$ and let \tilde{p}_n be the density function of

$$2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_d(l),$$

where (N_l) is a sequence of i.i.d. $N(0, 1)$ random variables, independent of (a_l) and $\sigma_n^2 = \text{Var}(\bar{a}_0) s_n^2$. If $n^{-2c} \ln n \leq s_n^2 \leq 1$ with $c = 1/2 - (1 - \delta)/\gamma$ for arbitrarily small $\delta > 0$, then the relation

$$\tilde{p}_n(x) = \varphi_{(1+\sigma_n^2)I_{2d}}(x)(1 + \varepsilon_n) \quad \text{with } \varepsilon_n \rightarrow 0$$

holds uniformly for $\|x\|^3 = o_d(n^{1/2-1/\gamma})$, $x \in \mathbb{R}^{2d}$.

Corollary 1 Let $\gamma > 2$ and $\sigma_n^2 = n^{-c}$ where c is as given in Lemma 10. Let E be a measurable set in \mathbb{R}^{2d} . Then

$$\left| \int_E \tilde{p}_n(x) dx - \int_E \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx \right| \leq \varepsilon_n \int_E \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx + O_d(\exp(-n^\eta)),$$

for some $\eta > 0$ and uniformly over all the d -tuples of distinct integers $1 \leq i_1, i_2, \dots, i_d \leq \lceil \frac{n-1}{2} \rceil$.

Proof Set $r = n^\alpha$ where $0 < \alpha < 1/2 - 1/\gamma$. Using Lemma 10, we have,

$$\begin{aligned} & \left| \int_E \tilde{p}_n(x) dx - \int_E \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx \right| \\ & \leq \left| \int_{E \cap \{\|x\| \leq r\}} \tilde{p}_n(x) dx - \int_{E \cap \{\|x\| \leq r\}} \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx \right| + \int_{E \cap \{\|x\| > r\}} \tilde{p}_n(x) dx + \int_{E \cap \{\|x\| > r\}} \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx \\ & \leq \varepsilon_n \int_{E \cap \{\|x\| \leq r\}} \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx + \int_{\{\|x\| > r\}} \tilde{p}_n(x) dx + \int_{\{\|x\| > r\}} \varphi_{(1+\sigma_n^2)I_{2d}}(x) dx, \end{aligned}$$

where $\varepsilon_n \rightarrow 0$. First let us bound the second term above. Denoting the j -th coordinate of the vector $v_d(l)$ by $v_d^{(j)}(l)$, $1 \leq j \leq 2d$,

$$\begin{aligned} \int_{\{\|x\| > r\}} \tilde{p}_n(x) dx & = \mathbb{P} \left(\left\| 2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_d(l) \right\| > r \right) \\ & \leq 2d \max_{1 \leq j \leq 2d} \mathbb{P} \left(\left| 2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_d^{(j)}(l) \right| > r/(2d) \right) \\ & \leq 2d \max_{1 \leq j \leq 2d} \mathbb{P} \left(\left| n^{-1/2} \sum_{l=0}^{n-1} \bar{a}_l v_d^{(j)}(l) \right| > r/(4\sqrt{2}d) \right) + 4d \exp(-rn^{c/2}/(4\sqrt{2}d)) \end{aligned}$$

where we have used the normal tail bound, $\mathbb{P}(|N(0, \sigma^2)| > x) \leq 2e^{-x/\sigma}$ for $x > 0$.

Note that the random variables $\bar{a}_l v_d^{(j)}(l)$, $0 \leq l < n$ are independent, have mean zero, variance at most one and are bounded by $2n^{1/\gamma}$. Therefore, by applying Bernstein's inequality and subsequent simplifications,

$$\mathbb{P}\left(\left|n^{-1/2} \sum_{l=0}^{n-1} \bar{a}_l v_d^{(j)}(l)\right| > r/4 \sqrt{2d}\right) \leq \exp(-Kr^2)$$

for some constant $K > 0$. On the other hand, the third term can easily be bounded by

$$\int_{\{\|x\|>r\}} \varphi_{(1+\sigma_n^2)L_{2d}}(x) dx \leq 4d \exp(-r/4d).$$

■

Lemma 11 Fix $x \in \mathbb{R}$. Let $\sigma_n^2 = n^{-c}$ for some $c > 0$. With notation as in the previous lemma, we have

$$\mathbb{P}\left((E_1 E_2)^{1/4} > (1 + \sigma_n^2)^{-1/2}(c_n x + d_n)\right) \leq \frac{K}{n},$$

where $K = K(x)$ is some positive constant depending on x .

Proof Since $(1 + y)^{-1/2} \geq 1 - y/2$ for $y > 0$,

$$\mathbb{P}\left((E_1 E_2)^{1/4} > (1 + \sigma_n^2)^{-1/2}(c_n x + d_n)\right) \leq \mathbb{P}\left((E_1 E_2)^{1/4} > (1 - \sigma_n^2/2)(c_n x + d_n)\right).$$

Recall the representation

$$\mathbb{P}((E_1 E_2)^{1/4} > x) = \theta(x)(1 - F_{\#}(x)) \text{ as } x \rightarrow \infty.$$

Note that $(1 - \sigma_n^2/2)(c_n x + d_n) = d_n^* + (d_n - d_n^*) + c_n x - (c_n x + d_n)\sigma_n^2/2 = d_n^* + o_x(1)$ where we use the facts that $c_n \rightarrow 0$, $(d_n - d_n^*)/c_n = o(1)$ and $d_n \sim \sqrt{\log n}$. The lemma now easily follows once we note that $1 - F_{\#}(d_n^*) = n^{-1}$. ■

We now continue with the proof of Theorem 5. Fix $x \in \mathbb{R}$. For notational convenience, define

$$\begin{aligned} Q_1^{(n)} &:= \mathbb{P}\left(\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/4} > c_q x + d_q\right), \\ Q_2^{(n)} &:= \mathbb{P}\left(\max_{1 \leq t \leq q} (1 + \sigma_n^2)(E_{2t-1} E_{2t})^{1/4} > c_q x + d_q\right), \end{aligned}$$

where $\{N_i\}$ is a sequence of i.i.d. standard normals random variables and E_i , $1 \leq i < \infty$ are i.i.d. $\text{Exp}(1)$. Our goal is to approximate $Q_1^{(n)}$ by the simpler quantity $Q_2^{(n)}$. By Bonferroni's inequality, for all $m \geq 1$,

$$\sum_{j=1}^{2m} (-1)^{j-1} S_{j,n} \leq Q_1^{(n)} \leq \sum_{j=1}^{2m-1} (-1)^{j-1} S_{j,n}, \quad (21)$$

where

$$S_{j,n} = \sum_{1 \leq t_1 < t_2 < \dots < t_j \leq q} \mathbb{P}\left((\beta_{\bar{a} + \sigma_n N, n}(t_1))^{1/4} > c_q x + d_q, \dots, (\beta_{\bar{a} + \sigma_n N, n}(t_j))^{1/4} > c_q x + d_q\right).$$

Similarly, we have

$$\sum_{j=1}^{2m} (-1)^{j-1} T_{j,n} \leq Q_2^{(n)} \leq \sum_{j=1}^{2m-1} (-1)^{j-1} T_{j,n}, \quad (22)$$

where

$$T_{j,n} = \sum_{1 \leq t_1 < t_2 < \dots < t_j \leq q} \mathbb{P}\left((1 + \sigma_n^2)(E_{2t_1-1}E_{2t_1})^{1/4} > c_q x + d_q, \dots, (1 + \sigma_n^2)(E_{2t_j-1}E_{2t_j})^{1/4} > c_q x + d_q\right).$$

Therefore, the difference between $Q_1^{(n)}$ and $Q_2^{(n)}$ can be bounded as follows:

$$\sum_{j=1}^{2m} (-1)^{j-1} (S_{j,n} - T_{j,n}) - T_{2m+1,n} \leq Q_1^{(n)} - Q_2^{(n)} \leq \sum_{j=1}^{2m-1} (-1)^{j-1} (S_{j,n} - T_{j,n}) + T_{2m,n}, \quad (23)$$

for each $m \geq 1$. By independence and Lemma 11,

$$T_{j,n} \leq \binom{n}{j} \frac{K^j}{n^j} \leq \frac{K^j}{j!} \quad \text{for all } n, j \geq 1. \quad (24)$$

Consequently, $\lim_{j \rightarrow \infty} \limsup_n T_{j,n} = 0$. Now fix $j \geq 1$. Let us bound the difference between $S_{j,n}$ and $T_{j,n}$. Let $1 \leq t_1 < t_2 < \dots < t_j \leq q$. Define

$$v_{2j}(l) = (\cos(\omega e_{t_1} l), \sin(\omega e_{t_1} l), \cos(\omega e'_{t_1} l), \sin(\omega e'_{t_1} l), \dots, \cos(\omega e'_{t_j} l), \sin(\omega e'_{t_j} l)),$$

where \mathcal{A}_t is as defined in (5) and we represent \mathcal{A}_t as $\{e_t, e'_t\}$. Then,

$$\begin{aligned} & \mathbb{P}\left((\beta_{\bar{a} + \sigma_n N, n}(t_1))^{1/4} > c_q x + d_q, \dots, (\beta_{\bar{a} + \sigma_n N, n}(t_j))^{1/4} > c_q x + d_q\right) \\ &= \mathbb{P}\left(2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_{2j}(l) \in B_n^{(j)}\right), \end{aligned}$$

where

$$B_n^{(j)} = \{y \in \mathbb{R}^{4j} : (y_{4i+1}^2 + y_{4i+2}^2)^{1/4} (y_{4i+3}^2 + y_{4i+4}^2)^{1/4} > 2^{1/2} (c_q x + d_q), 0 \leq i < j\}.$$

By Corollary 1 and the fact $Z_1^2 + Z_2^2 \stackrel{D}{=} 2E_1$ where Z_1, Z_2 are i.i.d. standard normal, we deduce

$$\begin{aligned} & \left| \mathbb{P}\left(2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_{2j}(l) \in B_n^{(j)}\right) - \mathbb{P}\left((1 + \sigma_n^2)^{1/2} (E_{2t_1-1} E_{2t_1})^{1/4} > c_q x + d_q, 1 \leq i \leq j\right) \right| \\ & \leq \varepsilon_n \mathbb{P}\left((1 + \sigma_n^2)^{1/2} (E_{2t_1-1} E_{2t_1})^{1/4} > c_q x + d_q, 1 \leq i \leq j\right) + O(\exp(-n^\eta)) \end{aligned}$$

uniformly over all the d -tuples $1 \leq t_1 < t_2 < \dots < t_j \leq q$.

Therefore, as $n \rightarrow \infty$,

$$|S_{j,n} - T_{j,n}| \leq \varepsilon_n T_{j,n} + \binom{n}{j} O(\exp(-n^\eta)) \leq \varepsilon_n \frac{K^j}{j!} + o(1) \rightarrow 0, \quad (25)$$

Hence using (21), (22), (24) and (25), we have

$$\limsup_n |Q_1^{(n)} - Q_2^{(n)}| \leq \limsup_n T_{2m+1,n} + \limsup_n T_{2m,n} \quad \text{for each } m \geq 1.$$

Letting $m \rightarrow \infty$, we conclude $\lim_n Q_1^{(n)} - Q_2^{(n)} = 0$. Since by Lemma 6,

$$\max_{1 \leq t \leq q} (E_{2t-1} E_{2t})^{1/4} = O_p((\log n)^{1/2}) \quad \text{and} \quad \sigma_n^2 = n^{-c},$$

it follows that

$$\frac{(1 + \sigma_n^2)^{1/2} \max_{1 \leq t \leq q} (E_{2t-1} E_{2t})^{1/4} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda$$

and consequently,

$$\frac{\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/4} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda.$$

In view of Lemma 8, it now suffices to show that

$$\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/4} - \max_{1 \leq t \leq q} (\beta_{\bar{a}, n}(t))^{1/4} = o_p(c_q).$$

We can use the basic inequality

$$||z_1 z_2| - |w_1 w_2|| \leq (|z_1| + |w_2|) \max\{|z_1 - w_1|, |z_2 - w_2|\}, \quad z_i, w_i \in \mathbb{C}, 1 \leq i \leq 2$$

to obtain

$$\begin{aligned} |\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/2} - \max_{1 \leq t \leq q} (\beta_{\bar{a}, n}(t))^{1/2}| &\leq (M_n(\bar{a} + \sigma_n N) + M_n(\bar{a})) M_n(\sigma_n N) \\ &\leq (2M_n(\bar{a} + \sigma_n N) + M_n(\sigma_n N)) M_n(\sigma_n N) \end{aligned}$$

where, for any sequence of random variables $X = (X_i)_{0 \leq i < n}$,

$$M_n(X) = \max_{1 \leq t \leq n} \left| n^{-1/2} \sum_{l=0}^{n-1} X_l \exp(i\omega t l) \right|.$$

As a trivial consequence of Theorem 2.1 of Davis and Mikosch (1999) [6], we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n \log n) \quad \text{and} \quad M_n^2(\bar{a} + \sigma_n N) = O_p(\log n).$$

Together with $\sigma_n = n^{-c/2}$ they imply that

$$\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/2} - \max_{1 \leq t \leq q} (\beta_{\bar{a}, n}(t))^{1/2} = o_p(n^{-c/4}).$$

From the inequality

$$|\sqrt{y_1} - \sqrt{y_2}| \leq \frac{1}{\min\{\sqrt{y_1}, \sqrt{y_2}\}} |y_1 - y_2|, \quad y_1, y_2 > 0$$

it easily follows that

$$\max_{1 \leq t \leq q} (\beta_{\bar{a} + \sigma_n N, n}(t))^{1/4} - \max_{1 \leq t \leq q} (\beta_{\bar{a}, n}(t))^{1/4} = o_p(n^{-c/8}) = o_p(c_q).$$

This completes the proof of Theorem 5. ■

Acknowledgement

We are grateful to Z.D. Bai for help in accessing the article Zhou (1996). We also thank Rajat Hazra and Koushik Saha for some interesting discussions.

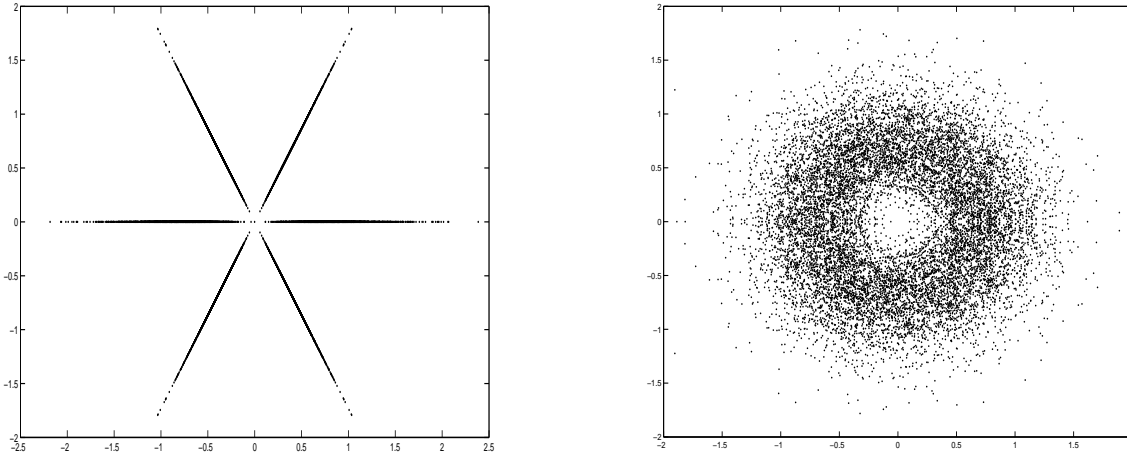


Figure 1: Eigenvalues of 20 realizations of $n^{-1/2}A_{k,n}$ with $a_i \sim \text{Exp}(1) - 1$ when (i) $k = 11, k^3 = -1 + 2n$ (left) and (ii) $k = 11, k^3 = 1 + 2n$ (right).

Appendix

For completeness, here we provide a proof of Theorem 1. Recall that for any two positive integers k and n , $p_1 < p_2 < \dots < p_c$ are all their common prime factors so that,

$$n = n' \prod_{q=1}^c p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^c p_q^{\alpha_q}.$$

Here $\alpha_q, \beta_q \geq 1$ and n', k', p_q are pairwise relatively prime.

Define

$$M = \max_{1 \leq q \leq c} [\beta_q / \alpha_q], \quad [t] = tk^M \pmod{n}.$$

Let $\underline{e}_{m,d}$ denote a $d \times 1$ vector with $(m \bmod d)$ -th element unity and other elements zero. If there is no scope for confusion, we shall write \underline{e}_m for $\underline{e}_{m,d}$. The matrix $E_{m,d}$ is the $d \times d$ matrix with j -th column equal to $\underline{e}_{jm,d}$, $0 \leq j < d$.

Define the diagonal matrix

$$\Lambda_{m,d} = \text{diag} [\lambda_{[0]}, \lambda_{[1]}, \dots, \lambda_{[j]}, \dots, \lambda_{[d-1]}],$$

where $[j] = jk^m \pmod{n}$ as defined above.

To prove Theorem 1, we need two lemmas. We omit the proof of the first lemma since it is easy.

Lemma 12 *Let $(\pi(0), \pi(1), \dots, \pi(n-1))$ be a permutation of $(0, 1, \dots, n-1)$. Define the $n \times n$ permutation matrix P_π as*

$$P_\pi = [\underline{e}_{\pi(0)} \ \underline{e}_{\pi(1)} \ \dots \ \underline{e}_{\pi(n-1)}]$$

Then,

$$(P_\pi^T E_{k,n} \Lambda_{0,n} P_\pi)_{i,j} = \begin{cases} \lambda_t & \text{if } (i, j) = (\pi^{-1}(kt \bmod n), \pi^{-1}(t)), \quad 0 \leq t < n \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 13 Let k and b be positive integers, $g = \gcd(k, b)$, $x = b/g$. Then the following equation holds.

$$\chi(A_{k,b}) = \chi(E_{k,b}\Lambda_{0,b}) = \lambda^{b-x}\chi(E_{k,x} \times \text{diag}(\lambda_{0 \bmod b}, \lambda_{k \bmod b}, \lambda_{2k \bmod b}, \dots, \lambda_{(x-1)k \bmod b})).$$

Proof of Lemma 13. Define the $b \times b$ permutation matrix

$$\Xi = \begin{bmatrix} 0 & I_{b-1} \\ 1 & \underline{0}^T \end{bmatrix}.$$

Observe that for $0 \leq j < b$, the j -th row of $A_{k,b}$ can be written as $a^T \Xi^{jk}$. From direct calculation, it is easy to verify that $\Xi = UDU^*$ is a spectral decomposition of Ξ where

$$D = \text{diag}(1, \omega, \dots, \omega^{b-1}), U = [u_0 : u_1 : \dots : u_{b-1}] \text{ with } u_j = b^{-1/2}(1, \omega^j, \omega^{2j}, \dots, \omega^{(b-1)j}), 0 \leq j < b.$$

Note that $\lambda_j = a^T u_j$, $0 \leq j < b$. From easy computations, it now follows that

$$U^* A_{k,b} U = E_{k,b} \Lambda_{0,b},$$

so that, $\chi(A_{k,b}) = \chi(E_{k,b} \Lambda_{0,b})$, proving the first equality.

To prove the second equality, define the following matrices

$$B_{b \times x} = [\underline{e}_{0,b} \ \underline{e}_{k,b} \ \underline{e}_{2k,b} \ \dots \ \underline{e}_{(x-1)k,b}] \text{ and } P = [B : B^c]$$

where B^c consists of those columns of I_b that are not in B which makes P a permutation matrix. Clearly, $E_{k,b} = [B : B : \dots : B]$ which is a $b \times b$ matrix of rank x , and we have

$$\chi(E_{k,b} \Lambda_{0,b}) = \chi(P^T E_{k,b} \Lambda_{0,b} P).$$

Note that,

$$\begin{aligned} P^T E_{k,b} \Lambda_{0,b} P &= \begin{bmatrix} I_x & I_x & \dots & I_x \\ 0_{(b-x) \times x} & 0_{(b-x) \times x} & \dots & 0_{(b-x) \times x} \end{bmatrix} \Lambda_{0,b} P \\ &= \begin{bmatrix} C \\ 0_{(b-x) \times b} \end{bmatrix} P \text{ for some } x \times b \text{ matrix } C \\ &= \begin{bmatrix} C \\ 0_{(b-x) \times b} \end{bmatrix} [B : B^c] = \begin{bmatrix} CB & CB^c \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Clearly, the characteristic polynomial of $P^T E_{k,b} \Lambda_{0,b} P$ does not depend on CB^c , explaining why we did not bother to specify the order of columns in B^c . Thus we have,

$$\chi(E_{k,b} \Lambda_{0,b}) = \chi(P^T E_{k,b} \Lambda_{0,b} P) = \lambda^{b-x} \chi(CB).$$

It now remains to show that $CB = E_{k,x} \Lambda_{1,x}$. Note that, the j -th column of B is $\underline{e}_{jk,b}$. So, j -th column of CB is actually the $(jk \bmod b)$ -th column of C , where,

$$\begin{aligned} C &= [I_x : I_x : \dots : I_x] \Lambda_{0,b} \\ &= [I_x : I_x : \dots : I_x] \times \text{diag}(\lambda_0, \lambda_1, \dots, \lambda_{b-1}). \end{aligned}$$

Hence, $(jk \bmod b)$ -th column of C is $\lambda_{jk \bmod b} \underline{e}_{jk \bmod x}$. So,

$$CB = E_{k,x} \times \text{diag}(\lambda_{0 \bmod b}, \lambda_{k \bmod b}, \lambda_{2k \bmod b}, \dots, \lambda_{(x-1)k \bmod b})$$

and the Lemma is proved completely. \blacksquare

Proof of Theorem 1. We first establish the Theorem for $A_{k,n'}$. The general case will be subsequently proved by using this case. Recall the notations given in Section 2. Since, k and n' are relatively prime, by using Lemma 13,

$$\chi(A_{k,n'}) = \chi(E_{k,n'} \Lambda_{0,n'}).$$

Get the sets S_0, S_1, \dots to form a partition of $\{0, 1, \dots, n' - 1\}$, as in Section 2.

Define the permutation π on the set $\mathbb{Z}_{n'}$ by setting $\pi(t) = s_t, 0 \leq t < n'$. This permutation π automatically gives rise to a permutation matrix P_π as in Lemma 12, and consider the positions of λ_b for $b \in S_j$ in the product $P_\pi^T E_{k,n'} \Lambda_{0,n'} P_\pi$. Let $N_{j-1} = \sum_{t=0}^{j-1} |S_t|$. We know, $S_j = \{r_j k^x \bmod n', x \geq 0\}$ for some integer r_j . Thus,

$$\pi^{-1}(r_j k^{t-1} \bmod n') = N_{j-1} + t, \quad 1 \leq t \leq n_j$$

so that, position of λ_b for $b = r_j k^{t-1} \bmod n', 1 \leq t \leq n_j$ in $P_\pi^T E_{k,n'} \Lambda_{0,n'} P_\pi$ is given by

$$(\pi^{-1}(r_j k^y \bmod n'), \pi^{-1}(r_j k^{y-1} \bmod n')) = \begin{cases} (N_{j-1} + t + 1, N_{j-1} + t) & \text{if, } 1 \leq t < n_j \\ (N_{j-1} + 1, N_{j-1} + n_j) & \text{if, } t = n_j \end{cases}$$

Hence,

$$P_\pi^T E_{k,n'} \Lambda_{0,n'} P_\pi = \text{diag}(L_0, L_1, \dots)$$

where, for $j \geq 0$, if $n_j = 1 \Rightarrow L_j = [\lambda_{r_j}]$, and if $n_j > 1$, then,

$$L_j = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \lambda_{r_j k^{n_j-1} \bmod n'} \\ \lambda_{r_j \bmod n'} & 0 & 0 & \dots & 0 & 0 \\ 0 & \lambda_{r_j k \bmod n'} & 0 & \dots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \dots & \lambda_{r_j k^{n_j-2} \bmod n'} & 0 \end{bmatrix}$$

Clearly, $\chi(L_j) = \lambda^{n_j} - x_j$. Now the result follows from the identity

$$\chi(E_{k,n'}) = \prod_{j \geq 0} \chi(L_j) = \prod_{j \geq 0} (\lambda^{n_j} - x_j).$$

Now let us prove the general case. Fix positive integers $\beta_q, 1 \leq q \leq c$. Define $n = n' \times \prod_{q=1}^c p_q^{\beta_q}$. Then, again using Lemma 13,

$$\chi(A_{k,n}) = \chi(E_{k,n} \Lambda_{0,n})$$

where $\Lambda_{0,n} = \text{diag}(\lambda_0, \dots, \lambda_{n-1})$.

Recalling that, $M = \max_{1 \leq q \leq c} \lceil \beta_q / \alpha_q \rceil$, $[t] = tk^M \bmod n$, and again, using Lemma 13 repeatedly,

$$\begin{aligned} \chi(A_{k,n}) &= \chi(E_{k,n} \Lambda_{0,n}) \\ &= \lambda^{n-n'} \chi(E_{k,n'} \Lambda_{M,n'}) \\ &= \lambda^{n-n'} \chi(E_{k,n'} \Lambda_{M+j,n'}) \text{ for all } j \geq 0 \\ &= \lambda^{n-n'} \chi(E_{k,n'} \times \text{diag}(\lambda_0 \bmod n, \lambda_y \bmod n, \lambda_{2y} \bmod n, \dots, \lambda_{(n'-1)y} \bmod n)) \\ &= \lambda^{n-n'} \chi(E_{k,n'} \times \Gamma_{0,n'}) \end{aligned}$$

Now replacing $\Lambda_{0,n'}$ by $\Gamma_{0,n'}$, we can mimic the rest of the proof given in case of $A_{k,n'}$, and complete the proof of the general case. \blacksquare

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