HALF INDEPENDENCE AND HALF CUMULANTS

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The notion of half independence arises in random matrices and quantum groups. We define the half independent convolution of probability measures and show that appropriately defined half cumulants are additive across half independent convolutions. We also establish the central limit theorem for half independent convolutions and the limit is the symmetrized Rayleigh distribution. Cramer's theorem is also established in this set up.

1. Introduction. Along with classical independence, another well known notion of independence is *free independence* in the noncommutative set up. A third notion of independence in the noncommutative set up, is that of *half independence*. This has been described in Banica, Curran and Speicher (2010)[2] and Bose, Hazra and Saha (2010)[7]. In Section 2, we provide a quick description of these three notions and two examples of half independence. The goal of this article is to study half independence in details.

To motivate our results, let us recall the results from classical and free independence that are relevant to us. In classical independence, the two natural transforms, which are measure determining and convergence determining, are the characteristic function $\phi(\cdot)$ and the cumulant generating function $\chi(\cdot)$. For any probability measure μ on $\mathbb{R}$, these are defined as

$$\phi_\mu(t) = \int_{\mathbb{R}} e^{itx} d\mu(x) \text{ and } \chi_\mu(t) = \log \phi_\mu(t) \text{ (for } t \text{ in a neighbourhood of } 0).$$

If μ and ν are two probability measures and $\mu * \nu$ is their independent (additive) convolution, then

$$\chi_{\mu*\nu}(t) = \chi_\mu(t) + \chi_\nu(t). \tag{1}$$

Well known results for this convolution are the CLT and the Cramer's theorem. In free independence, the natural transforms are the *Cauchy transform* and the *R transform* (or Voiculescu transform) (see Nica and Speicher (2006)[10], Anderson et al. (2009)[1]). The R transform satisfies equation (1) when classical convolution is replaced by the *free convolution*. The *free CLT* holds: the n fold free convolution of identical laws with finite second moment, when scaled by $\sqrt{n}$ and centered, converges to the *semicircle law*. See Nica and Speicher (2006)[10]. However, the Cramer's

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[*] Research supported by J.C.Bose National Fellowship, Dept. of Science and Technology, Govt. of India.

AMS 2000 subject classifications: Primary 60B20; secondary 46L53, 46L54, 60B10.

Keywords and phrases: Cramer's theorem, Central limit theorem, cumulant, C^*probability space, free algebras, free independence, half commutativity, half independence, Mellin transform, moment generating function, noncommutative probability spaces, reverse circulant matrix, Rayleigh distribution, semicircular law, symmetric partition.

theorem fails–free convolution of two laws may be semicircular without the individual laws being semicircular. See Bercovici and Voiculescu (1995)[5].

It is well known that the *noncrossing partitions* play a crucial role in free independence. In half independence, *symmetric partitions* are the most natural objects. In Section 2.3 we use the symmetric partitions to develop the notions of *half cumulants* and *half cumulant generating function* under suitable restrictions on the growth of the moments. In Section 3.1 we relate the moments to the half cumulants via appropriate generating functions.

This relationship helps us to go beyond the restricted set up and we develop the notion of an appropriate transform $T_\mu(\cdot)$ for any symmetric (about zero) probability measure μ. This transform plays the same role in half independence as do the characteristic function in classical independence and the Cauchy transform in free independence. It turns out that T has a surprisingly simple description in terms of the characteristic function. Let $\circledast$ denote the product convolution. Then

$$T_\mu(t) = \phi_{\mu\circledast\rho}(2t) \text{ for all } t \in \mathbb{R}$$

where ρ is the probability measure on $(0, 1)$ with density

$$\frac{2}{\pi}\frac{1}{\sqrt{1-\alpha^2}}, \; 0 < \alpha < 1.$$

T is both measure determining (Lemma 3) and convergence determining (Theorem 3).

We define the *half independent convolution* $\mu \oslash \nu$, of symmetric probability measures μ and ν, as the (unique) probability measure which satisfies

$$T_{\mu\oslash\nu}(t) = T_\mu(t)T_\nu(t) \text{ for all } t \in \mathbb{R}.$$

and hence the half cumulant generating function, $H = \log T$, satisfies

$$H_{\mu\oslash\nu}(t) = H_\mu(t) + H_\nu(t) \text{ (for } t \text{ in a neighbourhood of } 0).$$

This is compatible with addition of half independent variables: if a and b are half independent in some C^* probability space with laws μ and ν, then the law of $(a + b)$ exists and is given by $\mu \oslash \nu$.

In half independence, the Rayleigh law takes the place of the Gaussian law in independence and the semicircular law in free independence. The *symmetrized Rayleigh law* R_σ has the density

$$f(x) = \frac{|x|}{\sigma^2}\exp(-x^2/\sigma^2), \; -\infty < x < \infty$$

with moments

$$\beta_{2k+1} = 0 \text{ and } \beta_{2k} = \sigma^{2k}k! \text{ for all } k \geq 0.$$

If $\sigma = 1$, then it is known as standard symmetrized Rayleigh law and is denoted by R.

In Section 4, we establish Cramer's theorem: $\mu \oslash \nu$ is symmetrized Rayleigh, if and only if both μ and ν are symmetrized Rayleigh. In Section 5 we prove the *half independent CLT*: the n fold half independent convolution of symmetric laws satisfying suitable moment condition, when scaled by $\sqrt{n}$, converges to the symmetrized Rayleigh distribution.

2. The three notions of independence.

2.1. *Preliminaries of noncommutative probability space.* Let $(\mathcal{A}, \tau)$ be a noncommutative $*$-probability space where $\mathcal{A}$ is a $*-$ unital complex algebra (with unity 1) and $\tau : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\tau(1) = 1$. At times we shall assume τ is a state, that is $\tau(a) \geq 0$ if $a \geq 0$ ($a \geq 0$ means $a^* = a$ and its spectrum $sp(a)$ is nonnegative). The elements of the algebra will be referred to as random variables and we shall concentrate only on self adjoint random variables. By the law of self adjoint $a \in \mathcal{A}$, we mean the collection $\{m_k(a)\}$ where

$$m_k(a) = \tau(a^k),\ k \geq 1 \tag{2}$$

are the *moments* of a. Suppose τ is a state and a is a self adjoint random variable. Then given any n and any collection of complex numbers $\{c_i, 0 \leq i \leq n\}$

$$\sum_{k=0}^{n}\sum_{l=0}^{n} c_k \tau(a^{k+l})\bar{c}_l = \tau\left((\sum_{k=0}^{n} c_k a^k)(\sum_{l=0}^{n} c_l a^l)^*\right) \geq 0.$$

Then $\{m_k(a), k \geq 0\}$ is a moment sequence, and there exists a measure μ_a on the real line such that

$$m_k(a) = \tau(a^k) = \int x^k d\mu_a(x). \tag{3}$$

μ_a may not be unique. A sufficient condition for μ_a to be unique is the Carleman's condition

$$\sum [\tau(a^{2k})]^{-1/(2k)} = \infty. \tag{4}$$

In that case we also call μ_a to be the law of a.

When $(\mathcal{A}, \tau)$ is a C^* algebra, the law μ_a of a exists and is compactly supported. More generally, suppose that the algebra is not necessarily C^* and τ is a state and for the self adjoint random variable a, $\tau(a^{2k}) \leq C^k k!$ for all k. Then $\{\tau(a^{2k})\}$ satisfies condition (4), and hence μ_a is the law of a but it is not necessarily compactly supported.

For random variables $\{a_i\}_{i\in J}$, their joint moments is the collection $\{\tau(a_{i_1}a_{i_2}\ldots a_{i_k}),\ k \geq 1\}$, where each $a_{i_j} \in \{a_i\}_{i\in J}$.

Random variables $\{a_{i,n}\}_{i\in J} \in (\mathcal{A}_n, \tau_n)$ is said to converge in law to $\{a_i\}_{i\in J} \in (\mathcal{A}', \tau')$ if each joint moment of $\{a_{i,n}\}_{i\in J}$ converges to the corresponding joint moment of $\{a_{i,n}\}_{i\in J}$.

2.2. *Independence: free, classical, and half.* Unital subalgebras $\{\mathcal{A}_i\}_{i\in J} \subset \mathcal{A}$ are called *freely independent* or simply *free* if

$$\tau(a_j) = 0,\ a_j \in \mathcal{A}_{i_j} \text{ and } i_j \neq i_{j+1} \text{ for all } j \Rightarrow \tau(a_1 \cdots a_n) = 0.$$

The random variables (or elements of an algebra) $(a_1, a_2, \cdots)$ will be called free if the subalgebras generated by them are free.

Unital subalgebras $\{\mathcal{A}_j\}_{j\in J} \subset \mathcal{A}$ are said to be *independent* if they commute and

$$\tau(a_1 \cdots a_n) = \tau(a_1)\cdots\tau(a_n) \text{ for all } a_j \in \mathcal{A}_{i_j} \text{ where } k \neq l \Rightarrow i_k \neq i_l.$$

Two elements a and b of any algebra are said to be independent if the two unital algebras generated by them are independent.

Let $\{a_i\}_{i\in J}$ be noncommutative elements of $(\mathcal{A}, \tau)$. We say that they *half commute* if

$$a_i a_j a_k = a_k a_j a_i,$$

for all $i, j, k \in J$. Clearly, if $\{a_i\}_{i\in J}$ half commute then a_i^2 commutes with a_j and a_j^2 for all $i, j \in J$.

The random variable $a = a_{i_1} a_{i_2} \ldots a_{i_n}$ where each $a_{i_j} \in \{a_i\}_{i\in J}$, is said to be *balanced* (with respect to $\{a_i\}$), if each random variable a_i appears same number of times in odd and even positions of a. If a is not balanced, we say it is *unbalanced*. So if n is odd then a is automatically unbalanced.

Definition 1. *Half commuting elements $\{a_i\}_{i\in J}$ are said to be half independent if*

1. *The variables $\{a_i^2\}_{i\in J}$ are independent.*
2. *If $a = a_{i_1} a_{i_2} \cdots a_{i_n}$ is unbalanced with respect to $\{a_i\}_{i\in J}$, then $\tau(a) = 0$.*

This definition of half independent elements is equivalent to that given in Banica, Curran and Speicher (2009)[2]. Note that the second condition automatically implies $\tau(a^{2k+1}) = 0$ for all non-negative integers k. Whenever this happens, we say that a is *symmetric*.

The three notions of independence are different from each other. See Bose, Hazra and Saha (2010)[7] for details. Here are two examples of half independence.

Example 1. *(Banica and Speicher (2009) [2]) Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $\{\psi_i\}$ be a family of independent complex Gaussian random variables. Define $a_i \in (M_2(L(\mu)), \mathrm{E}[\mathrm{tr}(\cdot)])$by*

$$a_i = \begin{bmatrix} 0 & \psi_i \\ \overline{\psi_i} & 0 \end{bmatrix}.$$

where tr(·) *is normalized trace. Then $\{a_i\}$ are half independent. The law of each a_i is a symmetrized Rayleigh distribution.*

Example 2. *(Bose, Hazra and Saha (2010)[7]) A reverse circulant matrix is an $n\times n$ symmetric matrix whose (i, j)-th entry is given by $x_{(i+j) \bmod n}$. The sequence $\{x_i\}$ is called the input sequence. Let $\{RC_{i,n}\}_{1\leq i\leq p}$ be an independent (across i) sequence of $n\times n$ reverse circulant matrices each with an independent input sequence with mean 0, variance 1 and for all $k \geq 1$,*

$$\sup_{n\in\mathbb{N}} \sup_{1\leq i\leq p} \sup_{1\leq m\leq l\leq n} \mathrm{E}[|RC_{i,n}(m,l)|^k] \leq c_k < \infty.$$

Then $\{n^{-1/2}RC_{i,n}\}_{1\leq i\leq p} \in (M_n(L(\mu)), \mathrm{E}[\mathrm{tr}(\cdot)])$, converges to half independent $(a_1, a_2, \ldots, a_p)$ where a_i is as in Example 1 with $\mathrm{E}\,|\psi_i|^2 = 1$.

2.3. *Symmetric partition and half independence.* The proof of the above matricial limit given in Bose, Hazra and Saha (2010)[7] suggests that there is a suitable class of partitions which is tied to the notion of half independence just as the noncrossing partitions are tied to free independence. We now develop this notion.

Any set K of integers will be called *symmetric* if it has an equal number of odd and even integers. Any partition π of K will be called *symmetric* if each partition block is symmetric. The set of all symmetric partitions of $\{1, 2, \ldots, 2n\}$ will be denoted by $E(2n)$. The trivial partition of $\{1, 2, \ldots, 2n\}$ with one single block shall be denoted by I_{2n}. Given a sequence $\{l_n : \mathcal{A}^n \to \mathbb{C}\}$ of maps and partition π of $\{1, 2, \ldots, n\}$ define

$$l_\pi(a_1, a_2, \ldots, a_n) = \prod_{V\in\pi} l_{|V|}(a_{j_1}, \ldots, a_{j_{|V|}}), \tag{5}$$

where $V = (j_1, j_2, \cdots, j_{|V|})$ is a block of the partition π.

Definition 2. *Let $(\mathcal{A}, \tau)$ be a noncommutative probability space. Let $\{a_i\}_{i\in J}$ be random variables in $\mathcal{A}$ and suppose that $\tau(a_{i_1} \cdots a_{i_k}) = 0$ when k is odd and $i_i, \cdots, i_k \in J$. Then define half cumulants $\{r_k\}$ of $\{a_i\}_{i\in J}$, recursively by the following moment cumulant relation*

$$\tau(a_{i_1} a_{i_2} \ldots a_{i_k}) = \sum_{\pi\in E(k)} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_k}). \tag{6}$$

It is easily checked that $\{r_n\}$ are multilinear maps. For any self adjoint $a \in (\mathcal{A}, \tau)$, we let

$$r_n(a) = r_n(a, a, \ldots, a), \quad n \geq 1 \tag{7}$$

denote the half cumulants of a. Note that $r_{2n+1}(a) = 0$ for all $n \geq 0$. The *half cumulant generating function* of a is defined as the formal power series

$$H_a(t) = \sum_{n\geq 1} (-1)^n \frac{r_{2n}(a)}{(n!)^2} t^{2n}. \tag{8}$$

The reason for the extra $n!$ in the denominator shall be clear as we proceed.

Example 3. *If the law of a is R, then it is easy to see that*

$$r_2(a) = 1, \quad r_{2n}(a) = 0 \text{ (for } n > 1\text{) and } H_a(t) = -t^2. \tag{9}$$

Suppose $\{a_i\}$ are half independent. Using (6) recursively, it is easy to see that for any even $k > 1$,

$$r_k(a_{i_1}, a_{i_2}, \ldots, a_{i_k}) = 0 \text{ if } a_{i_1} a_{i_2} \ldots a_{i_k} \text{ is unbalanced.} \tag{10}$$

For example, let $\{a_1, a_2\} \in (\mathcal{A}, \tau)$, be half independent. Then

$$r_2(a_1, a_2) = \tau(a_1 a_2) = 0 \text{ and } r_2(a_2, a_1) = \tau(a_2 a_1) = 0.$$

Again using (6),

$$r_4(a_1,a_1,a_1,a_2) = \tau(a_1a_1a_1a_2) - r_2(a_1,a_1)r_2(a_1,a_2) - r_2(a_1,a_1)r_2(a_1,a_2) = 0,$$

$$r_4(a_1,a_2,a_1,a_2) = \tau(a_1a_2a_1a_2) - r_2(a_1,a_2)r_2(a_1,a_2) - r_2(a_1,a_2)r_2(a_1,a_2) = 0.$$

It is well known that classical independence and freeness can be characterized via the classical cumulants and free cumulants respectively (see Theorem 5.3.15 of Anderson et. al. (2009)[1] or Theorem 11.16 of Nica and Speicher (2006)[10] for the free cumulant results). A similar characterization holds for half independent random variables via half cumulants:

THEOREM 1. *Let* $\{a_i\}_{1\le i\le d}$ *be a sequence of self adjoint half commuting random variables and suppose for all* $1 \le j \le d$, a_j *occurs* k_j *times in* $a = a_{i_1}a_{i_2}\dots a_{i_{2n}}$.
(i) Suppose $\{a_i\}$ *are half independent and* a *is balanced with respect to* $\{a_i\}_{1\le i\le d}$. *If* $k_i, k_j \ge 2$ *for some* $1 \le i, j \le d$, *then* $r_{2n}(a_{i_1}, a_{i_2}, \dots, a_{i_{2n}}) = 0$.
(ii) If $k_i, k_j \ge 2$ *for some* $1 \le i, j \le d$ *implies* $r_{2n}(a_{i_1}, a_{i_2}, \dots, a_{i_{2n}}) = 0$ *and moments of all monomials of odd length vanish then* $\{a_i\}_{1\le i\le d}$ *are half independent.*

Before we prove the Theorem, we state an important Corollary.

COROLLARY 1. *If* a *and* b *are self adjoint, half commuting and half independent then*

$$r_{2n}(a+b) = r_{2n}(a) + r_{2n}(b) \text{ for all } n \ge 1, \text{ and } H_{a+b}(t) = H_a(t) + H_b(t),$$

where the second equality holds between formal power series.

PROOF. Note that $r_2(a,b) = \tau(ab) = 0$. Hence the result is true for $n = 2$ since

$$\begin{aligned} r_2(a+b) &= r_2(a+b, a+b) \\ &= r_2(a, a+b) + r_2(b, a+b) \\ &= r_2(a,a) + r_2(a,b) + r_2(b,a) + r_2(b,b) \end{aligned}$$

For $n > 2$,

$$\begin{aligned} r_{2n}(a+b) &= r_{2n}(a+b, .., a+b) \\ &= r_{2n}(a, a+b, ..., a+b) + r_{2n}(b, a+b, ..., a+b) \\ &= r_{2n}(a, a..., a) + r_{2n}(b, b, ..., b) + \sum_{(a_1,a_2,...a_{2n}):\ a_i\in\{a,b\},\ a_i\ne a\forall i} r_{2n}(a_1, a_2, \dots, a_{2n}). \end{aligned}$$

Now if $(a_1, a_2, \dots, a_{2n})$ is unbalanced then by (10), $r_{2n}(a_1, a_2, \dots, a_{2n}) = 0$. If $(a_1, a_2, \dots, a_{2n})$ is balanced but a, b both appear in the tuple, then by Theorem 1 (i), $r_{2n}(a_1, a_2, \dots, a_{2n}) = 0$. So only the first two terms survive in the last expression and hence

$$r_{2n}(a+b) = r_{2n}(a) + r_{2n}(b).$$

□

Proof of Theorem 1. (i) We prove this through induction on d and n. For each fixed values of d we use induction on n. We use notation D_1 for induction on d and D_2 for induction on n.

Now from relation (6),

$$\tau(a_{i_1}a_{i_2}\ldots a_{i_{2n}}) = \sum_{\pi\in E(2n)} r_\pi(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}). \tag{11}$$

If $\pi \in E(2n)$ has a block V such that $(a_{j_1}a_{j_2}\ldots a_{j_{|V|}})$ is unbalanced, then by observation (10)

$$r_\pi(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}) = 0.$$

Therefore relation (11) reduces to

$$\tau(a_{i_1}a_{i_2}\ldots a_{i_{2n}}) = \sum_{\pi\in E'(2n)} r_\pi(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}), \tag{12}$$

where

$$E'(2n) = \{\pi \in E_{2n} : \text{for each block } V \text{ of } \pi,\ a_{j_1}a_{j_2}\ldots a_{j_{|V|}} \text{ is balanced}\}.$$

Note that $E'(2n)$ depends on the considered tuple $(a_{i_1},a_{i_2},\ldots,a_{i_{2n}})$. We shall use relation (12) repeatedly in the proof.

First assume $d = 2$. Consider $(a_{i_1},a_{i_2},\ldots,a_{i_{2n}})$ where $a_{i_j} \in \{a_1,a_2\}$ and $k_1,k_2 \geq 2$. So the minimum possible value of n is 2. For $n = 2$, using (12), half commutativity and half independence,

$$\begin{aligned} r_4(a_{i_1},a_{i_2},a_{i_3},a_{i_4}) &= \tau(a_{i_1}a_{i_2}a_{i_3}a_{i_4}) - \sum_{\pi\in E'(4)-I_4} r_\pi(a_{i_1},a_{i_2},a_{i_3},a_{i_4}) \\ &= \tau(a_1^2)\tau(a_2^2) - r_2(a_1,a_1)r_2(a_2,a_2) \\ &= \tau(a_1^2)\tau(a_2^2) - \tau(a_1^2)\tau(a_2^2) = 0. \end{aligned}$$

Now assuming that the result is true upto $(n-1)$, we prove it for any n (induction D_2). Again by (12), half commutativity and half independence,

$$\begin{aligned} r_{2n}(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}) &= \tau(a_{i_1}a_{i_2}\ldots a_{i_{2n}}) - \sum_{\pi\in E'(2n)-I_{2n}} r_\pi(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}) \\ &= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\pi\in E'(2n)-I_{2n}} r_\pi(a_{i_1},a_{i_2},\ldots,a_{i_{2n}}). \end{aligned}$$

Let

$$J_j = \{j_1,j_2,\ldots,j_{2k_j}\}$$

be the positions of a_j for $j = 1,2$ in $(a_{i_1},a_{i_2},\ldots,a_{i_{2n}})$. Define

$$\Pi = \{\pi \in E'(2n) - I_{2n} :\ \text{if } V \text{ is a block in } \pi \text{ then either } V \subset J_1 \text{ or } V \subset J_2\}. \tag{13}$$

Note that if $\pi \in E'(2n) - \Pi$, then using (2) and induction D_2, $r_\pi(a_{i_1}, \ldots, a_{i_{2n}}) = 0$. Hence

$$r_{2n}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) = \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\pi \in \Pi} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}).$$

Since $a_{i_1}a_{i_2}\ldots a_{i_{2n}}$ is balanced, $J_j,\ j = 1, 2$ are symmetric. Hence if Π_j denotes all symmetric partitions of $\{j_1, \ldots, j_{2k_j}\}$ then it is in bijection with $E(2k_j)$. That is

$$\Pi_j := E\{j_1, \ldots, j_{2k_j}\} \simeq E(2k_j) \text{ for } j = 1, 2. \tag{14}$$

Now observe that for any $\pi \in \Pi$ there exist unique $\sigma = \{V_1, V_2, \ldots V_l\} \in \Pi_1$ and unique $\eta = \{U_1, U_2, \ldots U_s\} \in \Pi_2$ where V_i, U_i are blocks such that $\pi = \{V_1, V_2, \ldots V_l, U_1, U_2, \ldots U_s\} := \sigma \sqcup \eta$. Thus there is a bijection between Π and $\Pi_1 \star \Pi_2 := \{\sigma \sqcup \eta : \sigma \in \Pi_1 \text{ and } \eta \in \Pi_2\}$. Therefore

$$\begin{aligned}
r_{2n}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) &= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\pi \in \Pi} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) \\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\sigma \in \Pi_1,\ \eta \in \Pi_2} r_{\sigma \sqcup \eta}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) \\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\sigma \in \Pi_1,\ \eta \in \Pi_2} r_\sigma(a_{1_1}, a_{1_2}, \ldots, a_{1_{2k_1}}) r_\eta(a_{2_1}, a_{2_2}, \ldots, a_{2_{2k_2}}) \\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \sum_{\sigma \in E(2k_1),\ \eta \in E(2k_2)} r_\sigma(a_1, a_1, \ldots, a_1) r_\eta(a_2, a_2, \ldots, a_2) \\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - [\sum_{\sigma \in E(2k_1)} r_\sigma(a_1, a_1, \ldots, a_1)][\sum_{\eta \in E(2k_2)} r_\eta(a_2, a_2, \ldots, a_2)] \\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2}) - \tau(a_1^{2k_1})\tau(a_2^{2k_2}) = 0 \ \text{ (using (6))}.
\end{aligned}$$

Hence the result is true for $d = 2$.

Now assuming that the result is true for l many random variables where $l \leq d - 1$, we shall prove the result for $l = d$ (induction D_1). For $l = d$, the minimum possible value of n is d and each $k_j = 2$ for $1 \leq j \leq d$. So for $n = d$,

$$\begin{aligned}
r_{2d}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2d}}) &= \tau(a_{i_1}a_{i_2}\ldots a_{i_{2d}}) - \sum_{\pi \in E'(2d) - I_{2d}} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2d}}) \\
&= \tau(a_1^2)\tau(a_2^2)\cdots\tau(a_d^2) - \sum_{\pi \in E'(2d) - I_{2d}} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2d}}). \qquad (15)
\end{aligned}$$

Now observe that if $\pi \in E'(2d) - I_{2d}$ then either π has a block V such that $4 \leq \#V \leq 2(d-1)$ or each block of π has size 2. For the first case, by definition of cumulant (2) and induction D_1,

$$r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2d}}) = 0.$$

So in (15), we are left with partitions $\pi \in E'(2d) - I_{2d}$ whose each partition block is of size 2. Hence

$$r_{2d}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2d}}) = \tau(a_1^2)\cdots\tau(a_d^2) - \sum_{\substack{\pi \in E'(2d) \\ \pi = \{V_1, \ldots, V_d\}, \#V_i = 2\ \forall\ i}} r_\pi(a_{i_1}, \ldots, a_{i_{2d}})$$

$$\begin{aligned}
&= \tau(a_1^2)\cdots\tau(a_d^2) - r_2(a_1,a_1)\cdots r_2(a_d,a_d)\\
&= \tau(a_1^2)\cdots\tau(a_d^2) - \tau(a_1^2)\cdots\tau(a_d^2) = 0.
\end{aligned}$$

So we have established the result for $n = d$.

Still holding $l = d$ and assuming that the result is true upto $(n-1)$, we now prove it for n. Let $a_{i_1}a_{i_2}\ldots a_{i_{2n}}$ be balanced. Since we have d distinct random variables, each $k_j \geq 2$ for $1 \leq j \leq d$. As before, let $J_j = \{j_1, j_2, \ldots, j_{2k_j}\}$ be the positions of a_j in $(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}})$ for $1 \leq j \leq d$. By similar argument as given in $d = 2$ case, each J_j is symmetric. Now

$$\begin{aligned}
r_{2n}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) &= \tau(a_{i_1}a_{i_2}\ldots a_{i_{2n}}) - \sum_{\pi\in E'(2n)-I_{2n}} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}})\\
&= \tau(a_1^{2k_1})\tau(a_2^{2k_2})\cdots\tau(a_d^{2k_d}) - \sum_{\pi\in E'(2n)-I_{2n}} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}).
\end{aligned}$$

Suppose $\pi \in E'(2n) - I_{2n}$ has t (≥ 2) many blocks, $V_1, \ldots, V_t$. Suppose for some s, $1 \leq s \leq t$; $V_s \cap J_m \neq \emptyset$ for at least two values of m, where $1 \leq m \leq d$. If number of such m is strictly less than d, then by induction D_1 and Definition (2), $r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) = 0$. If number of such m is equal to d, then by induction D_2 (since, $\#V < 2(n-1)$) and (2), $r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) = 0$. So we are left with only $\pi \in \Pi$ where Π is as defined below. Let

$$\begin{aligned}
\Pi &= \{\pi \in E'(2n) - I_{2n} : \pi = \{V_1, \ldots, V_l\} \text{ and for all } i,\ V_i \subset J_r \text{ for some } 1 \leq r \leq d\},\\
\Pi_j &= E(j_1, j_2, \ldots, j_{2k_j}) \simeq E(2k_j),\ 1 \leq j \leq d.
\end{aligned}$$

Then following the argument given for $d = 2$ case, we have

$$\begin{aligned}
r_{2n}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}}) &= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \sum_{\pi\in\Pi} r_\pi(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}})\\
&= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \sum_{\pi_1\in\Pi_1,\ldots,\pi_d\in\Pi_d} r_{\pi_1\sqcup\cdots\sqcup\pi_d}(a_{i_1}, a_{i_2}, \ldots, a_{i_{2n}})\\
&= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \sum_{\pi_1\in\Pi_1,\ldots,\pi_d\in\Pi_d} \prod_{j=1}^{d} r_{\pi_j}(a_{j_1}, a_{j_2}, \ldots, a_{j_{2k_j}})\\
&= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \prod_{j=1}^{d}\left[\sum_{\pi_j\in\Pi_j} r_{\pi_j}(a_{j_1}, a_{j_2}, \ldots, a_{j_{2k_j}})\right]\\
&= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \prod_{j=1}^{d}\left[\sum_{\pi\in E(2k_j)} r_\pi(a_j, a_j, \ldots, a_j)\right]\\
&= \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) - \tau(a_1^{2k_1})\cdots\tau(a_d^{2k_d}) = 0.
\end{aligned}$$

Hence the first part of the Theorem is proved.

(ii) For the second part observe that the definition of half cumulant reduces to the definition of *half-liberated cumulant* given in Banica, Curren and Speicher (2010)[2]. They showed in Theorem 2.11 that half-liberated cumulant characterizes half independence for half commuting random variables $\{x_i\} \in (\mathcal{A}, \tau)$. In particular if $\tau(x_{i_1} \cdots x_{i_k}) = 0$ when k is odd and $r_\pi(x_{i_1}, \cdots, x_{i_k}) = 0$ whenever $\pi \in E(2k)$ is not symmetric with respect to $\boldsymbol{i} = (i_1, i_2, \cdots, i_k)$ (that is, it is not the case that s and t are in the same block of π and $i_s = i_t$) then $\{x_i\}$ are half independent. Now these two conditions follow from our assumptions. □

3. *T* transform.

Neither the Cauchy transform nor the characteristic function seem to be appropriate for the study of half independence. In this section we introduce a new transform, which we call the T transform, appropriate for the study of half independence.

3.1. *Moment and cumulant generating functions.*

From Definition 2, $m_{2n}(a) = \sum_{\pi \in E(2n)} r_\pi(a)$. The relation between $\{m_{2k}\}$ and $\{r_{2k}\}$ is given in the next theorem.

Theorem 2.

$$m_{2n}(a) = \sum_{s=0}^{n-1} \binom{n}{s} \binom{n-1}{s} r_{2(n-s)}(a) m_{2s}(a).$$

Proof. We adapt the proof of Proposition 7.7 of Banica, Curran and Speicher (2009)[3] for the recursion formula for count of $E(2n)$. Fix any $\pi \in E(2n)$. Then each partition block has same number of odd and even members. Now place the sequence of odd numbers in one row and place the even members in the next row. In this way, $E(2n)$ is viewed as the set of partitions between the upper and lower rows of n points such that each partition block has same number of upper and lower members. Now to form any such partition, we perform the following steps:

- First pick a number $s \in \{1, 2, \cdots, n\}$. This shall be total number of odd elements in the partition block containing 1.
- Connect 1 in the upper row to some *other* $(s-1)$ points in the upper row. So we have chosen all the *odd* members of the partition block that contains 1.
- Now choose s points in the lower row and connect them to the already connected s points in the upper row, thus completing the symmetric partition block of size $2s$ containing 1.
- Now partition the remaining $(n-s)$ odd points of the upper row and the remaining $(n-s)$ even points in the lower row in any symmetric way to essentially partition a set of size $2(n-s)$ in a symmetric way.

The number of such partitions is clearly $\binom{n-1}{s-1}\binom{n}{s} \# E_{2(n-s)}$. Now observe that for any fixed s, the total contribution to $m_{2n}(a)$ from all these partitions equals $r_{2(n-s)}(a) m_{2s}(a)$. Hence

$$m_{2n}(a) = \sum_{s=1}^{n} \binom{n-1}{s-1} \binom{n}{s} r_{2(n-s)}(a) m_{2s}(a).$$

□

We now convert the above relation to a relation between appropriate generating functions of $\{m_{2k}\}$ and $\{r_{2k}\}$. We drop the argument a in these expressions for ease of notation. Consider the following two formal power series:

$$\tilde{T}(t) = \sum_{n=0}^{\infty} \frac{m_{2n}t^{2n}}{(n!)^2} \text{ and } \tilde{H}(t) = \sum_{n=0}^{\infty} \frac{r_{2n}t^{2n}}{(n!)^2}. \tag{16}$$

Their formal derivatives are:

$$\tilde{T}'(t) = 2\sum_{n=1}^{\infty} \frac{m_{2n}t^{2n-1}}{n!(n-1)!} \text{ and } \tilde{H}'(t) = 2\sum_{n=1}^{\infty} \frac{r_{2n}t^{2n-1}}{n!(n-1)!}.$$

Lemma 1. *The above formal power series are related by the following relation,* $\tilde{T}'(t) = \tilde{T}(t)\tilde{H}'(t)$ *where* $\tilde{T}(0) = 1$ *and* $\tilde{H}(0) = 1$. *When considered as appropriate functions, this differential equation has a solution* $\tilde{T}(t) = \exp(\tilde{H}(t) - 1)$. *Hence, formally* $\tilde{H}(t) = 1 + \log \tilde{T}(t)$.

Proof. Using Theorem 2,

$$\begin{aligned}
\tilde{T}'(t) &= 2\sum_{n=1}^{\infty} \frac{m_{2n}t^{2n-1}}{n!(n-1)!} = 2\sum_{n=1}^{\infty}\sum_{s=0}^{n-1} \frac{\binom{n}{s}\binom{(n-1)}{s}}{(n-s)!(n-s-1)!} r_{2(n-s)}m_{2s}t^{2n-1}. \\
&= 2\sum_{n=1}^{\infty}\sum_{s=0}^{n-1} \frac{1}{(s!)^2 n!(n-1)!} r_{2(n-s)}m_{2s}t^{2n-1} = 2\sum_{s=0}^{\infty}\sum_{n=s+1}^{\infty} \frac{r_{2(n-s)}t^{2(n-s)-1}}{(n-s)!(n-s-1)!}\frac{m_{2s}t^{2s}}{(s!)^2} \\
&= 2\sum_{s=0}^{\infty}\sum_{l=1}^{\infty} \frac{r_{2l}t^{2l-1}}{l!(l-1)!}\frac{m_{2s}t^{2s}}{(s!)^2} = \tilde{T}(t)\tilde{H}'(t).
\end{aligned}$$

□

3.2. *T transform for general probability measure.* The finiteness of the two power series in (16) impose growth conditions on the moments, thereby restricting the class of variables (or measures) for which the transforms are defined. We now remedy this.

When $a \in (\mathcal{A}, \tau)$ is a self adjoint random variable with a compactly supported law μ_a, using Fubini's theorem,

$$\tilde{T}(t) = \sum_{n=0}^{\infty} \frac{m_{2n}(a)t^{2n}}{(n!)^2} = \int_{\mathbb{R}} \sum_{n=0}^{\infty} \frac{(xt)^{2n}}{(n!)^2} d\mu_a(x) = \int_{\mathbb{R}} I_0(2xt) d\mu_a(x) \tag{17}$$

where

$$I_0(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n}}{(n!)^2} = \frac{1}{\pi}\int_0^{\pi} \cosh(t\cos\theta)d\theta$$

is the modified Bessel function of order 0 (see Section 3.7 of Watson (1995)[11]).

Suppose μ_a is symmetric about 0. Let ϕ_{μ_a} denote the characteristic function of μ_a. Then we know that $\phi_{\mu_a}(t)$ is real. Using this and (17),

$$\begin{aligned} \tilde{T}(it) &= \frac{1}{\pi}\int_{\mathbb{R}}\int_0^{\pi} \cosh(2xit\cos\theta)d\theta d\mu_a(x) \\ &= \frac{1}{\pi}\int_0^{\pi}\int_{\mathbb{R}} \cos(2xt\cos\theta)d\mu_a(x)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\int_{\mathbb{R}} \exp(i2xt\cos\theta)d\mu_a(x)d\theta \quad \text{(since } \mu_a \text{ is symmetric about 0)} \\ &= \frac{1}{\pi}\int_0^{\pi} \phi_{\mu_a}(2t\cos\theta)d\theta. \end{aligned}$$

We have thus arrived at the following definition:

Definition 3. *For any law μ which is symmetric about zero (but does not necessarily have compact support), define the transform,*

$$T_\mu(t) = \tilde{T}(it) = \frac{1}{\pi}\int_0^{\pi} \phi_\mu(2t\cos\theta)d\theta, \ t \in \mathbb{R}.$$

We shall also use T_a to denote the transform T_{μ_a}. If μ is compactly supported or more generally if $m_{2n} = \int_{\mathbb{R}} x^{2n}d\mu(x) \leq C^n n!$ for all n, then it easily follows from the above discussions that

$$T_\mu(t) = \sum_{n=0}^{\infty} \frac{m_{2n}(-1)^n t^{2n}}{(n!)^2} \text{ for all } t \in \mathbb{R}. \tag{18}$$

Example 4. *If μ is symmetrized Rayleigh R_σ, then $m_{2n} = \sigma^{2n}n!$ for all n and hence*

$$T_\mu(t) = \sum_{n=0}^{\infty} \frac{m_{2n}(-1)^n t^{2n}}{(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n(\sigma t)^{2n}}{n!} = \exp(-\sigma^2 t^2), \quad t \in \mathbb{R}.$$

We now connect T and ϕ through product convolution. For probability measure μ on $\mathbb{R}$ and ρ on $(0, \infty)$, we use the product convolution $\circledast$ where

$$(\mu \circledast \rho)(B) = \int_0^{\infty} \mu(x^{-1}B)\rho(dx) = (\mu \otimes \rho)f^{-1}(B) \text{ with } f(x,y) = xy, \tag{19}$$

for any Borel set B. This gives a probability measure on $\mathbb{R}$ since $\mu(x^{-1}\mathbb{R}) = 1$ for every $x \neq 0$.

Lemma 2. *Let ρ be the probability measure on $(0, 1)$ with density $\frac{2}{\pi}\frac{1}{\sqrt{1-\alpha^2}}$ for $0 < \alpha < 1$. If μ is a probability measure on $\mathbb{R}$ which is symmetric around 0, then*

$$T_\mu(t) = \phi_{\mu\circledast\rho}(2t).$$

Proof.

$$T_\mu(t) = \frac{1}{\pi}\int_0^\pi \phi_\mu(2t\cos\theta)d\theta = \frac{1}{\pi}\int_{-1}^1 \frac{\phi_\mu(2t\alpha)}{\sqrt{1-\alpha^2}}d\alpha.$$

As μ is symmetric, $\phi_\mu(t) = \phi_\mu(-t)$ and $(\mu \circledast \rho)(B) = (\mu \circledast \rho)(-B)$ and hence

$$T_\mu(t) = \frac{2}{\pi}\int_0^1 \int_{\mathbb{R}} \frac{\cos(2t\alpha x)}{\sqrt{1-\alpha^2}}d\mu(x)d\rho(\alpha) = \phi_{\mu\circledast\rho}(2t).$$

□

3.3. *T is measure determining.* We show this by using Mellin transform. See Galambos and Simonelli (2004)[8] for the details of this transform. The main points we need are (i) the Mellin transform, $\mathcal{M}_\mu(s)$, is defined for any probability measure μ on $\mathbb{R}$ at least for all values of s in some complex strip which contains the imaginary axis, and possibly equals to this axis, (ii) $\mathcal{M}_{\mu\circledast\nu}(s) = \mathcal{M}_\mu(s)\mathcal{M}_\nu(s)$ for suitable values of s, (iii) Mellin transform uniquely determines the measure (Theorem 1.19 of Galambos and Simonelli (2004)[8]), and (iv) if μ is a probability measure μ on $[0,\infty)$ then

$$\mathcal{M}_\mu(s) = \int_0^\infty x^s d\mu(x) \text{ (for appropriate complex values } s).$$

Lemma 3. *Let μ and ν be two symmetric (about zero) probability measures. Then $T_\mu(t) = T_\nu(t)$ for all $t \in \mathbb{R}$ if and only if $\mu = \nu$.*

Proof. Let ρ be as in Lemma 2. Since $T_\mu(t) = T_\nu(t)$, Lemma 2 implies that $\phi_{\mu\circledast\rho}(2t) = \phi_{\nu\circledast\rho}(2t)$ for all t. Now by property (ii), (in an appropriate region),

$$\mathcal{M}_\mu(s)\mathcal{M}_\rho(s) = \mathcal{M}_\nu(s)\mathcal{M}_\rho(s).$$

Now let $s = it$ and observe that by property (iv),

$$\begin{aligned}\mathcal{M}_\rho(s) &= \frac{2}{\pi}\int_0^1 \frac{\alpha^s}{\sqrt{1-\alpha^2}}d\alpha \\ &= \frac{1}{\pi}\int_0^1 y^{(s-1)/2}(1-y)^{-1/2}dy = \frac{1}{\pi}\int_0^1 y^{s/2}y^{-1/2}(1-y)^{-1/2}dy \\ &= \frac{1}{B(1/2,1/2)}\int_0^1 y^{s/2}y^{-1/2}(1-y)^{-1/2}dy = \frac{\Gamma(\frac{s+1}{2})}{\Gamma(\frac{s+2}{2})}.\end{aligned}$$

The last equality follows from Lomnicki (1967)[9]. For $s = it$, $\Gamma(\frac{s+2}{2})$ does not have a pole and the numerator is also not equal to zero and hence $\mathcal{M}_\rho(it) \neq 0$ for all $t \in \mathbb{R}$. Hence $\mathcal{M}_\mu(s) = \mathcal{M}_\nu(s)$ in an appropriate region and then the result follows from property (iii). □

3.4. *T is convergence determining.*

Theorem 3. *Suppose $\{\mu_n\}$ and μ are symmetric probability measures. Then μ_n converges to μ weakly, if and only if $T_{\mu_n}(t)$ converges to $T_\mu(t)$ for all $t \in \mathbb{R}$.*

Proof. First suppose μ_n converges to μ. Hence $\phi_{\mu_n}(t)$ converges to $\phi_\mu(t)$ for every t. Since $T_{\mu_n}(t) = \frac{1}{\pi}\int_0^\pi \phi_{\mu_n}(2t\cos\theta)d\theta$, the result easily follows by dominated convergence theorem.

Conversely, suppose that $T_{\mu_n}(t)$ converges to $T_\mu(t)$. So from Lemma 2 we have that $\mu_n \circledast \rho$ weakly converge to $\mu \circledast \rho$ where ρ is as in Lemma 2. Since ρ does not put any mass at zero, this implies that $\{\mu_n\}$ is also tight and there exists a subsequence μ_{n_k} which converge weakly to μ_0. Moreover, μ_0 is symmetric. It then follows that $\mu_{n_k} \circledast \rho$ weakly converge to $\mu_0 \circledast \rho$.

On the other hand, we know that $\mu_n \circledast \rho$ converges to $\mu \circledast \rho$. Hence we have

$$\mu_0 \circledast \rho = \mu \circledast \rho.$$

Now from Lemma 3 we have that $\mu_0 = \mu$. So every subsequence has a unique limit point and hence the original sequence of measures converge. Moreover, μ_0 is symmetric. ◻

3.5. *Half convolution of two symmetric measures.* Since $T_\mu(t) = \phi_{\mu\circledast\rho}(t)$ and ϕ is continuous at 0 the logarithm of T is well defined in a neighbourhood of 0. This leads us to the following definition of half cumulant generating function which is more general than that given in (8).

Definition 4. *The half cumulant generating function of any symmetric probability measure μ is defined as*

$$H_\mu(t) = \log T_\mu(t) \ \ \textit{(in an appropriate neighborhood of zero).}$$

The half cumulant generating function H_a of any a in $(\mathcal{A}, \tau)$ is H_{μ_a} whenever μ_a exists and is symmetric.

Example 5. *If μ is symmetrized Rayleigh R_σ, (see Example 4), then $T_\mu(t) = \exp(-\sigma^2 t^2)$ is non-zero for all $t \in \mathbb{R}$. So*

$$H_\mu(t) = -\sigma^2 t^2 \ \ \textit{for all } t \in \mathbb{R}.$$

Conversely, if $H_\mu(t) = -\sigma^2 t^2$, then μ is the symmetrized Rayleigh R_σ.

To define half convolution of probability measures, we proceed as follows: Suppose $(\mathcal{A}, \tau)$ is a C^*-probability space and $a, b \in (\mathcal{A}, \tau)$ are two self adjoint, half independent random variables with compactly supported measures μ_a and μ_b on $\mathbb{R}$ respectively. Then obviously, μ_a and μ_b are symmetric and

$$H_{a+b}(t) = H_a(t) + H_b(t) \ \ \text{and} \ \ T_{a+b}(t) = T_a(t)T_b(t). \tag{20}$$

In this case the measure μ_{a+b} is the measure corresponding to the random variable $(a + b) \in (\mathcal{A}, \tau)$ and is compactly supported and symmetric.

We now define the half convolution (notation: $\oslash$) of two arbitrary symmetric measures μ and ν. First assume that μ and ν are compactly supported measures. Now using similar argument as given in Proposition 2.8 of Banica, Curran and Speicher (2010)[2], we can construct two half independent random variables y_1, y_2 as follows:
Let X_1, X_2 be two independent random variables such that X_1 and X_2 have law μ and ν respectively. Let U_1, U_2 be two independent Haar unitary random variables which are independent from X_1, X_2 and let $\xi_i = U_i X_i$. Then ξ_1, ξ_2 are independent and

$$\begin{aligned} \mathrm{E}[\xi_i^n \bar{\xi}_i^m] = \mathrm{E}[X_i^{n+m}]\,\mathrm{E}[U_i^n \bar{U}_i^m] &= \delta_{nm}\,\mathrm{E}[X_i^{n+m}] \\ &= \begin{cases} \mathrm{E}[X_i^{2n}] & \text{if} \quad n = m \\ 0 & \text{if} \quad n \neq m. \end{cases} \end{aligned} \tag{21}$$

Now define the variables y_1, y_2 as

$$y_i = \begin{pmatrix} 0 & \xi_i \\ \bar{\xi}_i & 0 \end{pmatrix}.$$

Then it is easy to check using (21) that y_1, y_2 are half independent and have laws $\mu,\ \nu$ respectively. Then from Lemma 2

$$T_{y_1}(t) = T_\mu(t) = \phi_{\mu\circledast\rho}(2t) \ \text{ and } \ T_{y_2}(t) = T_\nu(t) = \phi_{\nu\circledast\rho}(2t).$$

Consider the product $T_\mu(t)T_\nu(t)$. Then due to relation (20)

$$T_\mu(t)T_\nu(t) = T_{y_1}(t)T_{y_2}(t) = T_{y_1+y_2}(t) = T_\beta(t) = \phi_{\beta\circledast\rho}(2t),$$

where β is a symmetric measure corresponding to random variable $(y_1 + y_2)$ and this is unique by Lemma 3. We define β as half convolution of μ and ν, that is,

$$\mu \oslash \nu := \beta.$$

Now suppose μ and ν are two arbitrary symmetric measure. Define for $n \in \mathbb{N}$, the two measures μ_n and ν_n by

$$\mu_n(B) = \mu(B \cap [-n, n]) \ \text{ and } \ \nu_n(B) = \nu(B \cap [-n, n]), \ \text{for any Borel set } \ B \subseteq \mathbb{R}.$$

Then μ_n and ν_n are compactly supported symmetric measure on $\mathbb{R}$ and μ_n, ν_n converges weakly to μ, ν respectively. Then as before we have measures β_n such that

$$T_{\mu_n}(t)T_{\nu_n}(t) = \phi_{\mu_n\circledast\rho}(2t)\phi_{\nu_n\circledast\rho}(2t) = \phi_{\beta_n\circledast\rho}(2t).$$

Since μ_n, ν_n converges weakly to μ, ν respectively, $\phi_{\mu_n\circledast\rho}(2t)\phi_{\nu_n\circledast\rho}(2t)$ converges and hence $\phi_{\beta_n\circledast\rho}(2t)$ converges for all t. Hence $\beta_n \circledast \rho$ converges weakly to some measure on $\mathbb{R}$. It easily follows that $\{\beta_n\}$ is tight and $\beta_n \circledast \rho$ converges weakly to $\beta \circledast \rho$ for some unique measure β. Uniqueness of β follows from Lemma 3. Hence we arrive at

Definition 5. *For any two symmetric probability measures μ and ν, their half independent convolution is defined as the unique measure β described above. We write*

$$\mu \oslash \nu := \beta.$$

Note that $\oslash$ is associative and commutative on the space of symmetric measures.

4. Cramer's theorem.

Theorem 4. *(i) Let a and b be self adjoint and half independent random variables of $*$-probability space $(\mathcal{A}, \tau)$ where τ is a state. Suppose $\mu_{\frac{a+b}{\sqrt{2}}}$ is R_σ. Then μ_a and μ_b exist and are symmetrized Rayleigh.*
(ii) If μ and ν are symmetric probability measures such that $\mu \oslash \nu$ is symmetrized Rayleigh, then both μ and ν are symmetrized Rayleigh.

Proof. (i) Without loss, assume that $\sigma = 1$. Then

$$\tau(((a+b)/\sqrt{2})^{2k}) = k!.$$

From this, and half independence, it is easy to see that

$$\tau(a^{2k+1}) = 0 \text{ and } \tau(a^{2k}) \leq 2^k k!. \tag{22}$$

Thus there exists a unique symmetric probability measure μ_a and μ_b such that

$$\tau(a^k) = \int x^k d\mu_a(x) \text{ and } \tau(b^k) = \int x^k d\mu_b(x).$$

It easily follows that the characteristic functions of μ_a and μ_b are real and hence they are symmetric about origin. By Corollary 1 and Example 5, we have for all $t \in \mathbb{R}$,

$$H_{\frac{a}{\sqrt{2}}}(t) + H_{\frac{b}{\sqrt{2}}}(t) = H_R(t) = -t^2 \ .$$

So using Definition 4 we have

$$\log T_{\frac{a}{\sqrt{2}}}(t) + \log T_{\frac{b}{\sqrt{2}}}(t) = -t^2 \ .$$

Hence

$$\left(T_{\frac{a}{\sqrt{2}}}(t) T_{\frac{b}{\sqrt{2}}}(t)\right) = \exp(-t^2) \ .$$

So we have

$$\left(\phi_{\mu_a \circledast \rho}(2t)\right)\left(\phi_{\mu_b \circledast \rho}(2t)\right) = \exp(-t^2).$$

Hence, by the classical Cramer's theorem it follows that each of the product on the left side must be the characteristic function of a normal distribution. So $T_{\frac{a}{\sqrt{2}}}(t) = \exp(-\frac{t^2}{\sigma^2})$ for some σ. As a consequence, by the uniqueness Lemma 3, a is symmetrized Rayleigh $R_{\sqrt{2}/\sigma}$. Likewise, b is symmetrized Rayleigh $R_{\sqrt{2}/\sigma'}$.

The proof of the second part follows easily from the above arguments. ☐

5. Central limit theorem for half independent convolution.

THEOREM 5. *Let* $\{\mu_n\}$ *be a sequence of symmetric probability measures with variance* 1. *If for every* $\epsilon > 0$,

$$\frac{1}{n}\sum_{k=1}^{n}\int_{\{x:\ |x|\geq\epsilon\sqrt{n}\}} x^2 d\mu_k(x) \to 0 \ \textit{ as } n\to\infty \textit{ (Lindeberg's condition)}, \tag{23}$$

then

$$\delta_n := D_{\frac{1}{\sqrt{n}}}\mu_1 \oslash D_{\frac{1}{\sqrt{n}}}\mu_2 \oslash\cdots\oslash D_{\frac{1}{\sqrt{n}}}\mu_n$$

converges weakly to R *where* $D_{c\mu_i}(B) = \mu_i(c^{-1}B)$ *for any Borel set* B *in* $\mathbb{R}$.

PROOF. By definition of half convolution

$$\begin{aligned} T_{\delta_n}(t) &= T_{D_{\frac{1}{\sqrt{n}}}\mu_1}(t)T_{D_{\frac{1}{\sqrt{n}}}\mu_2}(t)\cdots T_{D_{\frac{1}{\sqrt{n}}}\mu_n}(t)\\ &= \phi_{D_{\frac{1}{\sqrt{n}}}\mu_1\circledast\rho}(2t)\phi_{D_{\frac{1}{\sqrt{n}}}\mu_2\circledast\rho}(2t)\cdots\phi_{D_{\frac{1}{\sqrt{n}}}\mu_n\circledast\rho}(2t), \end{aligned}$$

where ρ is as defined in Lemma 2. Note that for any $\epsilon > 0$

$$\begin{aligned} \frac{1}{n}\sum_{k=1}^{n}\int_{\{x:\ |x|\geq\epsilon\sqrt{n}\}} x^2 d(\mu_k\circledast\rho)(x) &= \frac{1}{n}\sum_{k=1}^{n}\frac{2}{\pi}\int_0^1\int_{\{x:\ |x\alpha|\geq\epsilon\sqrt{n}\}}\frac{x^2\alpha^2}{\sqrt{1-\alpha^2}}\mu_k(x)d\alpha\\ &= \frac{2}{\pi}\int_0^1\frac{\alpha^2}{\sqrt{1-\alpha^2}}\left[\frac{1}{n}\sum_{k=1}^{n}\int_{\{x:\ |x\alpha|\geq\epsilon\sqrt{n}\}} x^2 d\mu_k(x)\right]d\alpha. \end{aligned} \tag{24}$$

Now as $n\to\infty$, last expression in (24) goes to 0 by dominated convergence theorem and condition (23). Hence the sequence of measure $\{\mu_n\circledast\rho\}$ satisfies Lindeberg's condition. Also

$$\int_0^1 x^2 d(\mu_k\circledast\rho)(x) = \frac{2}{\pi}\int_0^1\frac{1}{\sqrt{1-\alpha^2}}\int_{\mathbb{R}} x^2\alpha^2 d\mu_k(x)d\alpha = \frac{1}{2}.$$

Hence by classical central limit theorem (see Billingsley (1995)[6]) as $n\to\infty$,

$$T_{\delta_n}(t) \to \phi_{N_{0,\frac{1}{2}}}(t) = \exp(-t^2)$$

where $N_{0,\frac{1}{2}}$ is the Gaussian measure with mean zero and variance $1/2$. We know $T_R(t) = \exp(-t^2)$. Hence δ_n converges weakly to the standard Rayleigh measure R. □

The following result gives a central limit theorem for half independent random variables in a noncommutative probability space.

Theorem 6. *Let* $\{x_i\}$ *be a sequence of self adjoint half independent random variables in* $(\mathcal{A}, \tau)$ *with* $\tau(x_i) = 0$ *and* $\tau(x_i^2) = 1$ *and* $\sup_i \tau(x_i^k) < \infty$ *for every* k. *If* $S_n = x_1 + x_2 + \cdots + x_n$, *then* $S_n/\sqrt{n}$ *converges in law to the standard symmetrized Rayleigh distribution* R.

Proof. If we assume $(\mathcal{A}, \tau)$ is a C^*-probability space, then we have a sequence of symmetric compactly supported measure $\{\mu_i\}$ corresponding to the random variables $\{x_i\}$. Then the result follows immediately from Theorem 5.

In the general case, to show the required convergence it is enough to show that,

$$\frac{\tau(S_n^{2k})}{n^k} \to k! \text{ and } \frac{\tau(S_n^{2k+1})}{n^{k+1/2}} \to 0.$$

By applying half independence and unbalancedness of each term in the expansion of $(x_1 + \cdots + x_n)^{2k+1}$, it immediately follows that $\tau((x_1 + \cdots + x_n)^{2k+1}) = 0$. It is thus enough to consider the even moments.

Consider any $q = x_{i_1} \cdots x_{i_{2k}}$ in the expansion of $(x_1 + \cdots + x_n)^{2k}$ which is unbalanced. Then by half independence $\tau(q) = 0$. Thus we are left with only balanced monomials q. We divide such monomials into two sets:

$M_1 = \{x_{i_1} \cdots x_{i_{2k}}; i_j \in \{1, 2, \cdots, n\}$ balanced and every random variable appears exactly twice$\}$,
$M_2 = \{x_{i_1} \cdots x_{i_{2k}}; i_j \in \{1, 2, \cdots, n\}$ balanced and at least one random variable appears more than twice$\}$.

Observe that M_2 has at most $(k-1)$ many distinct random variables and hence

$$\#M_2 \leq C_k n(n-1)\cdots(n-k+2),$$

for some constant C_k that depends only on k. Now since all moments are finite,

$$\frac{1}{n^k} \sum_{M_2} \tau(x_{i_1} \cdots x_{i_{2k}}) \leq C'_k \frac{n(n-1)\cdots(n-k+2)}{n^k} \to 0 \text{ as } n \to \infty.$$

Now consider M_1. Pick a fixed set of k variables from $\{x_1, \ldots, x_n\}$. Then there are exactly $k!$ symmetric monomials in M_1 of these k variables. Hence

$$\#M_1 = k! \times n(n-1)\cdots(n-k+1).$$

Therefore as $n \to \infty$

$$\frac{1}{n^k} \sum_{M_1} \tau(x_{i_1} \cdots x_{i_{2k}}) = \frac{\#M_1}{n^k} \tau(x_1^2 \cdots x_k^2) = \frac{k! \times n(n-1)\cdots(n-k+1)}{n^k} \to k!.$$

Hence the result follows. □

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