

LIMITING SPECTRAL DISTRIBUTION OF BLOCK MATRICES WITH TOEPLITZ BLOCK STRUCTURE

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ABSTRACT. We study two specific symmetric random block Toeplitz (of dimension $k \times k$) matrices: where the blocks (of size $n \times n$) are (i) matrices with i.i.d. entries and (ii) asymmetric Toeplitz matrices. Under suitable assumptions on the entries, their limiting spectral distributions (LSDs) exist (after scaling by \sqrt{nk}) when (a) k is fixed and $n \rightarrow \infty$ (b) n is fixed and $k \rightarrow \infty$ (c) n and k go to ∞ simultaneously. Further the LSD's obtained in (a) and (b) coincide with those in (c) when n or respectively k tends to infinity. This limit in (c) is the semicircle law in case (i). In Case (ii) the limit is related to the limit of the random symmetric Toeplitz matrix as obtained by Bryc et al. (2006) and Hammond and Miller (2005).

1. INTRODUCTION

Limiting Spectral Distribution (LSD) of block matrices with large random blocks has been studied in the literature. Under certain moment assumptions, Oraby (2007) proved a general existence theorem for LSD of such matrices with large Wigner blocks and finite symmetric block structure. Banerjee and Bose (2011) later extended these results to relax the moment assumptions and also to include Wigner type blocks. They also showed the limit to be the semicircular law when both the size of a block and the size of the block structure is increased in a suitable way.

Our focus in this article is on matrices with the Toeplitz block structure. Matrices of the form $B_k = ((x_{i-j}))_{1 \leq i, j \leq k}$ where x_i is some sequence of numbers are called *Toeplitz matrices*. If the elements x_i are themselves $n \times n$ matrices, say $A_{n,i} \equiv A_i$, with $A_i = A_{-i}^T$ for all i then the corresponding symmetric matrices (say $B_k \star \{A_{n,i}\}$) are *block matrices with asymmetric Toeplitz block structure*. Visually,

$$B_k \star \{A_{n,i}\} = \begin{bmatrix} A_0 & A_{-1} & A_{-2} & \dots & A_{-(k-2)} & A_{-(k-1)} \\ A_1 & A_0 & A_{-1} & \dots & A_{-(k-3)} & A_{-(k-2)} \\ A_2 & A_1 & A_0 & \dots & A_{-(k-4)} & A_{-(k-3)} \\ & & & \vdots & & \\ A_{k-1} & A_{k-2} & A_{k-3} & \dots & A_1 & A_0 \end{bmatrix}. \quad (1.1)$$

Block Toeplitz matrices arise in many aspects of mathematics, physics and technology. Gazzah et al. (2001) researched the asymptotic behaviour of eigenvalue distribution for deterministic block Toeplitz matrices. Rashidi Far et al. (2008) proved

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the existence of the LSD for random block Toeplitz matrices with self-adjoint $\{A_{n,i}\}$ as $n \rightarrow \infty$. They assumed that the entries are complex Gaussian. The proofs use an operator-valued free probability approach and Wick's formula for moments of Gaussian variables. Li et al. (2011) study the LSD of these block Toeplitz matrices ($B_k \star \{A_{n,i}\}$) when $A_{n,i}$ have i.i.d. entries (with uniformly bounded moments of all order) as $k \rightarrow \infty$ and show the existence of the LSD. They also show that when in turn $n \rightarrow \infty$, the above limit converges to the semicircle law. They use the moment method and explicit calculations based on the eigenvalue trace formula.

We deal with two specific symmetric block matrices where the blocks are random Toeplitz matrices with $A_i = A_{-i}^T$ for all i and are otherwise independent:

Case (i) The entries of A_i are i.i.d. (as in Li et al. (2011)).

Case (ii) A_i are (asymmetric) Toeplitz matrices. This symmetric matrix with asymmetric Toeplitz block structure and individual Toeplitz blocks does not seem to have been studied in literature.

We show that in both cases, their LSD's exist (under finiteness of second moments) when

- (a) k is fixed and $n \rightarrow \infty$
- (b) n is fixed and $k \rightarrow \infty$
- (c) n and k go to ∞ simultaneously.

Further the LSD's obtained in (a) and (b) coincide with those in (c) when n or respectively k tends to infinity. This limit in (c) is the semicircle law in Case (i). In particular this establishes results of Li et al. (2011) under weaker moment conditions when the blocks are i.i.d. In Case (ii) the limit is related to the limit of the random symmetric Toeplitz matrix as obtained by Bryc et al. (2006) and Hammond and Miller (2005). Our study shows that one can also accommodate other kinds of patterned matrices in place of the i.i.d. or Toeplitz matrices A_i .

Li et al. (2011) also studied the Hankel block structure where the individual blocks are i.i.d. matrices and the symmetry of the whole matrix was assumed. It would be interesting to study Hankel block structures where individual blocks are either Toeplitz or Hankel. Note that the Hankel matrix is a symmetric matrix and hence the block structure is symmetric. This is an additional feature which is absent in the two block matrices that we deal with in this article. We shall deal with such symmetric block matrices separately in Basu et al. (2011). This symmetry will be exploited to establish additional results and for the more general situation where there is a general patterned block structure and where each individual block is another patterned matrix.

Incidentally, block matrices with asymmetric Toeplitz block structures are specific block versions of patterned matrices. The problem of showing existence of LSD of random patterned matrices using the moment method was studied in the general framework by Bose and Sen (2008) who built upon the so called volume method ideas that were first propounded by Bryc et al. (2006) in the context of Toeplitz and Hankel matrices. Our proofs are based on these ideas and results and we also draw generously from the developments in Banerjee and Bose (2011).

In Section 2 we recollect a few useful concepts from Bose and Sen (2008) and introduce our model. In Section 3 we state and prove our results.

2. PRELIMINARIES

2.1. Background material. We first present in brief some of the material from Bose and Sen (2008) that we need. A patterned matrix is defined through a *link function*. Let d be a positive integer. Let L_n be functions $L_n : \{1, 2, \dots, n\}^2 \rightarrow Z_{\geq}^d$, $n \geq 1$. Here Z_{\geq}^d denotes all d -tuples of non-negative integers ($d = 1$ or 2). We write L for L_n and write \mathbb{N}^2 as the common domain of $\{L_n\}$.

Then the sequence of patterned matrices $\{A_n\}$ of order $n \times n$ with link function L is defined as

$$A_n \equiv ((a_{i,j})) = ((x_{L_n(i,j)}))$$

where $\{x_{i,j}\}$ or $\{x_i\}$ as the case may be, is called the *input sequence*. L_W and L_T will denote the link functions respectively for the Wigner and symmetric Toeplitz matrices, so that

$$L_W(i, j) = (\min(i, j), \max(i, j)), \quad L_T(i, j) = |i - j|.$$

The h -th moment of the *empirical spectral distribution* (ESD) of $n^{-1/2}A_n$ equals

$$\frac{1}{n} \sum_i \left(\frac{\lambda_i}{\sqrt{n}} \right)^h = \frac{1}{n} \text{Tr} \left(\frac{A_n}{\sqrt{n}} \right)^h = \frac{1}{n^{1+h/2}} \sum_{1 \leq i_1, i_2, \dots, i_h \leq n} x_{L(i_1, i_2)} x_{L(i_2, i_3)} \cdots x_{L(i_{h-1}, i_h)} x_{L(i_h, i_1)}.$$

Any function $\pi : \{0, 1, 2, \dots, h\} \rightarrow \{1, 2, \dots, n\}$ is a *circuit of length h* if $\pi(0) = \pi(h)$. A circuit depends on h and n but we suppress this dependence. A circuit π is said to be *matched*. if given any i , there is at least one $j \neq i$ such that $L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))$.

Two circuits π_1 and π_2 (of same length) are *equivalent* if $\{L(\pi_1(i-1), \pi_1(i)) = L(\pi_1(j-1), \pi_1(j))\} \Leftrightarrow \{L(\pi_2(i-1), \pi_2(i)) = L(\pi_2(j-1), \pi_2(j))\}$. This defines an equivalence relation. *Equivalence classes* are identified with partitions of $\{1, 2, \dots, h\}$. Partitions will be labelled by *words* w of length $l(w) = h$ of letters where the first occurrence of each letter is in alphabetical order. For example, if $h = 5$, then the partition $\{\{1, 3, 5\}, \{2, 4\}\}$ is represented by the word *ababa*. Let $w[i]$ denote the i -th entry of w . The equivalence class corresponding to w will be denoted by

$$\Pi_{L,n}(w) = \{\pi : w[i] = w[j] \Leftrightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\}.$$

A word is *pair-matched* if every letter occurs exactly two times. The set of pair-matched words of length k will be denoted by $\mathcal{W}_k(2)$. For any set G , let $\#G$ denote the number of elements in G .

Note that $\#\mathcal{W}_k(2) = \frac{(2k)!}{2^k k!}$. A word $w \in \mathcal{W}_k(2)$ is *Catalan* if it has a double letter and removing double letters successively leads to the empty word.

Define for any (matched) word w ,

$$\Pi_{L,n}^*(w) = \{\pi : w[i] = w[j] \Rightarrow L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j))\} \supseteq \Pi_{L,n}(w).$$

$\Pi_{L,n}^*(w)$ is often equivalent to $\Pi_{L,n}(w)$ for asymptotic considerations, but is easier to work with.

Any $\pi(i)$ is a *vertex* and it is *generating* if either $i = 0$ or $w[i]$ is the position of the *first* occurrence of a letter. For example, if $w = abbcab$ then $\pi(0), \pi(1), \pi(2), \pi(4)$ are generating vertices.

For any word of length $2t$, let (whenever the limit exists),

$$p(w) = \lim_{n \rightarrow \infty} \frac{\#\Pi_{L,n}^*(w)}{n^{1+t}}.$$

It is known that with appropriate input sequences, $p(w)$ exists for many symmetric patterned matrices, including the Wigner and the symmetric Toeplitz matrices and $\sum_{w \in \mathcal{W}_k(2)} p(w)$ is the $2k$ -th moment of the LSD. We shall denote by $p_W(w)$ and $p_T(w)$ the value of $p(w)$ for these two matrices respectively.

2.2. Our model. A *finite block structure* $B_k = B_k(a_0, a_1, \dots)$ is a $k \times k$ patterned matrix with link function L_1 and input sequence $\{a_i\}_{i \in \mathbb{Z}}$. B_k is called symmetric if $L_1(i, j) = L_1(j, i)$. Let $\{A_{n,0}, A_{n,1}, \dots\}_{n \times n}$ be a sequence of independent patterned matrices with link functions $L_{2,0}, L_{2,1}, \dots$ and respective independent input sequences $\{x_{0,k}, x_{1,k}, \dots\}_{k \geq 1}$. Then the (i, j) -th entry of $A_{n,k}$ is given by $x_{k, L_{2,k}(i,j)}$. The $kn \times kn$ block matrix $B_k \star A_{n,i}$ is defined by replacing (i, j) -th entry of B_k by $A_{n, L_1(i,j)}$, i.e. $B_k \star A_{n,i} = B_k(A_{n,0}, A_{n,1}, \dots)$. Let

$$m(i, j) = L_1 \left(\left[\frac{i-1}{n} \right] + 1, \left[\frac{j-1}{n} \right] + 1 \right).$$

Thus $B_k \star A_{n,i}$ is a $nk \times nk$ matrix with link function L defined as

$$L(i, j) = (m(i, j), L_{2, m(i,j)}((i-1) \bmod n + 1, (j-1) \bmod n + 1)).$$

Hence all the concepts introduced in the previous subsection remain valid. In this article we shall deal with the following two sequences of block matrices. It may be noted that all of these are symmetric matrices, even though most of the blocks are asymmetric.

Toeplitz block matrix with i.i.d. blocks ($TBI_{k,n}$): Here $L_1(i, j) = i - j$ is the link function for the block structure B_k , $A_{0,n}$ is a Wigner matrix with link function L_W and for $i > 0$, $A_{n,i}$ are i.i.d. matrices with the common link function $L_2(k, l) = (k, l)$ for every k, l . For $i < 0$, $A_{n,i} = A_{n,-i}^T$. We denote this block matrix by $TBI_{k,n}$.

Toeplitz block matrix with Toeplitz blocks ($TBT_{k,n}$): Here B_k is as above, $A_{0,n}$ is a symmetric Toeplitz matrix with $L_{2,0}(k, l) = L_T(k, l)$ and for $i > 0$, $A_{n,i}$ is an asymmetric Toeplitz matrix with link function $L_2(k, l) = k - l$. For $i < 0$, $A_{n,i} = A_{n,-i}^T$. We denote this block matrix by $TBT_{k,n}$.

We shall make the following assumption on the input sequence:

Assumption A: The input sequence is independent with mean 0 and variance 1 and either uniformly bounded or identically distributed or with uniformly bounded moments of all order.

2.3. Address functions. The following definitions are taken from Banerjee and Bose (2011) and modified for our set-up.

Block Address Function: Let us consider any of the above two block matrices. Let $U_i = \{(i-1)n+1, \dots, in\}$, $i = 1, 2, \dots, k$, be a partition of $1, 2, \dots, nk$.

For any $w \in \mathcal{W}_{2t}(2)$, the **block address** π_b of $\pi \in \Pi_{L,nk}^*(w)$ is defined as:

$$\pi_b(i) = j \text{ if } \pi(i) \in U_j.$$

It denotes which of the k blocks of rows (or columns) an index belongs to. Note that from the definition of the block structure, if two elements of $B_k \star A_{n,i}$ are same, then they must belong to blocks which are either same or transpose of each other. Hence,

$$w[i] = w[j] \Rightarrow L_1(\pi_b(i-1), \pi_b(i)) = \pm L_1(\pi_b(j-1), \pi_b(j)) \quad (2.1)$$

$$\Rightarrow L_T(\pi_b(i-1), \pi_b(i)) = L_T(\pi_b(j-1), \pi_b(j)). \quad (2.2)$$

Also $\pi(0) = \pi(2t) \Rightarrow \pi_b(0) = \pi_b(2t)$. Hence, $\pi_b \in \Pi_{L_T,k}^*(w)$.

This leads to the definition of the **block address function** as: $\phi_B : \Pi_{L,nk}^*(w) \rightarrow \Pi_{L_T,k}^*(w)$ by:

$$(\pi(0), \dots, \pi(2t)) \mapsto (\pi_b(0), \dots, \pi_b(2t)).$$

Entry Address function: Analogous to the block address we define entry address as follows. Let $w \in \mathcal{W}_{2t}(2)$. The **entry address** π_e of $\pi \in \Pi_{L,nk}^*(w)$ is defined as

$$\pi_e(i) = \pi(i) - (\pi_b(i) - 1)n.$$

Clearly then, $1 \leq \pi_e(i) \leq n$. This denotes the address of an entry inside a block. Now, from the definitions it follows that

$$w[i] = w[j] \Rightarrow L_{2,m_1}(\pi_e(i-1), \pi_e(i)) = L_{2,m_2}(\pi_e(j-1), \pi_e(j))$$

where

$$m_1 = L_1(\pi_b(i-1), \pi_b(i)) \text{ and } m_2 = L_1(\pi_b(j-1), \pi_b(j)).$$

Conversely,

$$L_{2,m_1}(\pi_e(i-1), \pi_e(i)) = L_{2,m_2}(\pi_e(j-1), \pi_e(j)) \Rightarrow m_1 = \pm m_2$$

and hence

(i) $(\pi_e(i-1), \pi_e(i)) = (\pi_e(j-1), \pi_e(j))$ or $(\pi_e(i-1), \pi_e(i)) = (\pi_e(j), \pi_e(j-1))$ (when we have iid blocks)

(ii) $\pi_e(i) - \pi_e(i-1) = \pm(\pi_e(j) - \pi_e(j-1))$ (when we have asymmetric Toeplitz blocks).

Also, as before $\pi_e(0) = \pi_e(2t)$ and hence $\pi_e \in \Pi_{L_W,n}^*(w)$ (when we have iid blocks) and $\pi_e \in \Pi_{L_T,n}^*(w)$ (when we have asymmetric Toeplitz blocks).

This leads to the definition of the **entry address function** $\phi_A : \Pi_{L,nk}^*(w) \rightarrow \Pi_{L_W,n}^*(w)$ (when we have iid blocks) or $\phi_A : \Pi_{L,nk}^*(w) \rightarrow \Pi_{L_T,n}^*(w)$ (when we have asymmetric Toeplitz blocks) as:

$$(\pi(0), \dots, \pi(2t)) \mapsto (\pi_e(0), \dots, \pi_e(2t)).$$

Note that (π_b, π_e) determines π uniquely but for any $\pi_b \in \Pi_{L_T,k}^*(w)$ and $\pi_e \in \Pi_{L_W,n}^*(w)$, the π determined by π_b and π_e need not be in $\Pi_{L,nk}^*(w)$.

Now let $w \in \mathcal{W}_{2t}(2)$. Let S be the set of all non-zero generating vertices of w . For every $i \in S$, let us denote by j_i the index such that $w[j_i]$ is the second occurrence of the letter $w[i]$. Let i_1, i_2, \dots, i_t be all the non-zero generating vertices. Let $l = (l_{i_1}, \dots, l_{i_t}) \in \{-1, 1\}^t$.

Let $\Pi_{L_T, k, l}^*(w)$ be the subset of $\Pi_{L_T, k}^*(w)$ such that,

$$\pi_e(i-1) - \pi_e(i) = l_i(\pi_e(j_i-1) - \pi_e(j_i)) \quad \forall i \in S.$$

Now clearly,

$$\Pi_{L_T, k}^*(w) = \bigcup_l \Pi_{L_T, k, l}^*(w)$$

which is *not* a disjoint union.

Also, define $\Pi_{L_W, n, l}^*(w) = \{\pi_e \in \Pi_{L_W, n}^*(w) : w[i] = w[j] \Rightarrow (\pi_e(i-1), \pi_e(i)) = (\pi_e(j-1), \pi_e(j)) \text{ if } l_i = 1 \text{ and } \pi_e(i-1), \pi_e(i) = (\pi_e(j), \pi_e(j-1)) \text{ if } l_i = -1\}$

Clearly,

$$\Pi_{L_W, n}^*(w) = \bigcup_l \Pi_{L_W, n, l}^*(w).$$

It is to be noted that $\Pi_{L_W, n, l}^*(w)$'s are closely related with R_1 and R_2 constraints as described in Bose and Sen (2008). In fact, it is easy to see that the pair (i, j_i) satisfies an R_1 constraint if $l_i = 1$ and it satisfies an R_2 constraint if $l_i = -1$.

3. RESULTS AND PROOFS

3.1. Results. We are now ready to state the main results.

Theorem 3.1. *Consider the block matrix $TBI_{k, n}$ where $A_{n, i}$ satisfy Assumption A. Then,*

(i) *for fixed k , as $n \rightarrow \infty$ LSD of $\frac{1}{\sqrt{nk}}TBI_{k, n}$ exists w.p. 1, and has all moments finite, all odd moments 0, and $\forall t > 0$, the $2t - th$ moment β_{2t} satisfy*

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2), w \text{ catalan}} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} \quad \text{where } l_0 = (-1, -1, \dots, -1).$$

(ii) *for fixed n , as $k \rightarrow \infty$, LSD of $\frac{1}{\sqrt{nk}}TBI_{k, n}$ exists w.p. 1, and has all moments finite, all odd moments 0, and $2t - th$ moments β_{2t} satisfy*

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_W, n, l_0}^*(w)}{n^{t+1}} p_T(w) \quad \text{where } l_0 = (-1, -1, \dots, -1).$$

(iii) *as n and k both go to ∞ , LSD of $\frac{1}{\sqrt{nk}}TBI_{k, n}$ exists w.p. 1 and is the semicircular law.*

(iv) *in (i) and (ii) if we let $k \rightarrow \infty$ or $n \rightarrow \infty$ respectively, the LSD converge to that in (iii).*

Theorem 3.2. *Consider the block matrix $TBT_{k, n}$ where the input sequence satisfies Assumption B. Then,*

(i) for fixed k , as $n \rightarrow \infty$ LSD of $\frac{1}{\sqrt{nk}}TBT_{k,n}$ exists w.p. 1, has all moments finite, all odd moments 0, and $(2t) - th$ moments β_{2t} satisfy

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} p_T(w) \quad \text{where } l_0 = (-1, -1, \dots, -1).$$

(ii) for fixed n , as $k \rightarrow \infty$, LSD of $\frac{1}{\sqrt{nk}}TBT_{k,n}$ exists w.p. 1, has all moments finite, all odd moments 0, and $(2t) - th$ moments β_{2t} satisfy

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_T, n, l_0}^*(w)}{n^{t+1}} p_T(w) \quad \text{where } l_0 = (-1, -1, \dots, -1)$$

(iii) as n and k both go to ∞ , LSD of $\frac{1}{\sqrt{nk}}TBT_{k,n}$ exists w.p. 1, is symmetric, has all moments finite and is determined by the even moments

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} (p_T(w))^2.$$

(iv) in (i) and (ii) if we let $k \rightarrow \infty$ or $n \rightarrow \infty$ respectively, the LSD converge to that in (iii).

3.2. Proofs. We first state a Proposition which follows from the results of Bose and Sen (2008) and which helps to reduce many computational aspects.

A link function is said to satisfy *Property B*, if,

$$\Delta(L) = \sup_n \sup_t \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n, L(k, l) = t\} < \infty. \quad (3.1)$$

Assumption B: Let $k_n = \#\{L_n(i, j) : 1 \leq i, j \leq n\}$ and $\alpha_n = \max_k \#\{(i, j) : L_n(i, j) = k\}$. Then $k_n \rightarrow \infty$ and $k_n \alpha_n = O(n^2)$.

Proposition 3.1. *Suppose A_n is a sequence of $n \times n$ patterned random matrices with link function L satisfying Assumption B and Property B.*

- (1) *Suppose for every bounded, mean zero and variance one i.i.d. input sequence, the LSD exists almost surely and is non-random. Then the same limit continues to hold if the input sequence satisfies Assumption A.*
- (2) *if w is matched but not pair-matched then $p(w)$ exists and is equal to 0.*
- (3) *if for every $t > 0$ and for every $w \in \mathcal{W}_{2t}(2)$, $p(w) = \lim_{n \rightarrow \infty} \frac{\#\Pi_{L, n}^*(w)}{n^{1+t}}$ exists then the LSD of $\frac{A_n}{\sqrt{n}}$ exists and the $2t$ -th moment of the LSD is given by*

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} p(w).$$

- (4) *Assumption B and Property B are satisfied for the Toeplitz and the Wigner link functions. Both for the Toeplitz matrix and the Wigner matrix, $p(w)$ exists for every $t > 0$ and every $w \in \mathcal{W}_{2t}(2)$. For the Wigner matrix $p_W(w) = 0$ iff w is non-Catalan. For every Catalan word w , $p_W(w) = p_T(w) = 1$.*

Before we prove Theorems 3.1 and 3.2 we state a lemma which quantifies some of the properties of Toeplitz and Wigner limit and these were proved in Bryc et al. (2006) and Bose and Sen (2008). See also Theorem 5.1 of Bose et al. (2010).

Lemma 3.1. *Let $w \in \mathcal{W}_{2t}(2)$ and l_0 be the k -tuple $(-1, -1, \dots, -1)$.*

- (1) *If $l \neq l_0$, then $\lim_{k \rightarrow \infty} \frac{1}{k^{t+1}} \# \Pi_{L_T, k, l}^*(w) = 0$.*
- (2) *$\lim_{k \rightarrow \infty} \frac{1}{k^{t+1}} \# \Pi_{L_T, k, l_0}^*(w) = \lim_{k \rightarrow \infty} \frac{1}{k^{t+1}} \# \Pi_{L_T, k}^*(w) = p_T(w)$.*
- (3) *If w is Catalan, then $\forall l \neq l_0$,*

$$\lim_{n \rightarrow \infty} \frac{1}{n^{t+1}} \# \Pi_{L_W, n, l}^*(w) = 0 \text{ and } \lim_{n \rightarrow \infty} \frac{1}{n^{t+1}} \# \Pi_{L_W, n, l_0}^*(w) = 1.$$

3.3. Proof of Theorem 3.1. We begin by proving a few lemma involving the block and entry address functions.

Lemma 3.2. *Let $w \in \mathcal{W}_{2t}(2)$.*

- (1) *Let $\pi_b \in \Pi_{L_T, k}^*(w)$ and l^1, l^2, \dots, l^m be all the distinct values of l such that, $\pi_b \in \Pi_{L_T, k, l}^*(w)$. Then,*

$$\# \phi_B^{-1}(\pi_b) = \# \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w).$$

- (2) *Let $\pi_e \in \Pi_{L_W, n}^*(w)$ and l^1, l^2, \dots, l^m be all the distinct values of l such that, $\pi_e \in \Pi_{L_W, n, l}^*(w)$. Then,*

$$\# \phi_A^{-1}(\pi_e) = \# \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w).$$

Proof of Lemma 3.2. (1) Let $\pi \in \phi_B^{-1}(\pi_b)$. Then we claim that $\pi_e = \phi_A(\pi) \in \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w)$. To prove this, let i_1, i_2, \dots, i_t be all the non-zero generating vertices for w . Since $\pi_e \in \Pi_{L_W, n}^*(w)$, then either, $(\pi_e(i-1), \pi_e(i)) = (\pi_e(j_i-1), \pi_e(j_i))$ or $(\pi_e(i-1), \pi_e(i)) = (\pi_e(j_i), \pi_e(j_i-1))$. For $i \in \{i_1, i_2, \dots, i_t\}$, define

$$m_i = \begin{cases} 1 & \text{if } (\pi_e(i-1), \pi_e(i)) = (\pi_e(j_i-1), \pi_e(j_i)) \\ -1 & \text{if } (\pi_e(i-1), \pi_e(i)) = (\pi_e(j_i), \pi_e(j_i-1)). \end{cases}$$

Define m_i to be any of 1 or -1 if both the conditions are satisfied simultaneously (this happens when $\pi_e(i-1) = \pi_e(i)$). As $\pi \in \Pi_{L_T, nk}^*(w)$ and $\phi_A(\pi) = \pi_e, \phi_B(\pi) = \pi_b$, $m_i = 1 \Rightarrow \pi_b(i-1) - \pi_b(i) = \pi_b(j_i-1) - \pi_b(j_i)$, and $m_i = -1 \Rightarrow \pi_b(i-1) - \pi_b(i) = -(\pi_b(j_i-1) - \pi_b(j_i))$. Let $l^* = (m_{i_1}, \dots, m_{i_t})$. Clearly then, $\pi_e \in \Pi_{L_W, n, l^*}^*(w)$. And it follows from the above argument that, $\pi_b \in \Pi_{L_T, k, l^*}^*(w)$. By hypothesis, $l^* \in \{l^1, l^2, \dots, l^m\}$ and hence, $\pi_e \in \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w)$.

Now the map

$$\phi_B^{-1}(\pi_b) \rightarrow \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w) : \pi \rightarrow \pi_e$$

is clearly injective. To show that it is surjective, it is easy to see that if $\pi_b \in \Pi_{L_T, k, l}^*(w)$, then for any $\pi_e \in \Pi_{L_W, n, l}^*(w)$, the circuit determined by (π_b, π_e) is indeed in $\Pi_{L_T, nk}^*(w)$. This shows $\# \phi_B^{-1}(\pi_b) = \# \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w)$.

(ii) Proof of this part is similar to that of the previous one. Let $\pi \in \phi_A^{-1}(\pi_e)$. Then we claim that $\pi_b = \phi_B(\pi) \in \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w)$. To prove this, let i_1, i_2, \dots, i_t be all the non-zero generating vertices for w . Since $\pi_b \in \Pi_{L_T, k}^*(w)$, then, $(\pi_b(i-1) - \pi_b(i)) = \pm(\pi_b(j_i - 1) - \pi_b(j_i))$. For $i \in \{i_1, i_2, \dots, i_t\}$, define

$$m_i = \begin{cases} 1 & \text{if } (\pi_b(i-1) - \pi_b(i)) = (\pi_b(j_i - 1) - \pi_b(j_i)) \\ -1 & \text{if } (\pi_b(i-1) - \pi_b(i)) = -(\pi_b(j_i) - \pi_b(j_i - 1)). \end{cases}$$

As $\pi \in \Pi_{L, nk}^*(w)$ and $\phi_B(\pi) = \pi_b, \phi_A(\pi) = \pi_e, m_i = 1 \Rightarrow (\pi_b(i-1), \pi_b(i)) = (\pi_b(j_i - 1), \pi_b(j_i))$, and $m_i = -1 \Rightarrow (\pi_b(i-1), \pi_b(i)) = (\pi_b(j_i), \pi_b(j_i - 1))$. Let $l^* = (m_{i_1}, \dots, m_{i_t})$. It follows from the above argument that, $\pi_e \in \Pi_{L_W, n, l^*}^*(w)$. Now, $l^* \in \{l^1, l^2, \dots, l^m\}$ by hypothesis, and hence, $\pi_b \in \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w)$ as in the previous lemma. Now the map

$$\phi_A^{-1}(\pi_e) \mapsto \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w) : \pi \mapsto \pi_b$$

is as before a bijection, and hence the lemma is proved. \square

Now we are ready to prove the theorem.

In view of Proposition 3.1 (1), without loss of generality, we assume the input sequences to be uniformly bounded. Since the blocks $A_{n,i}$ satisfy Assumption B, it is clear that $B_k \star A_{n,i}$ satisfies Property B. Hence Proposition 3.1 (3) implies that it is enough to show that, for every $t > 0$ and for every $w \in \mathcal{W}_{2t}(2)$, $p(w)$ exists.

(i) Let k be fixed and let $n \rightarrow \infty$. Let $w \in \mathcal{W}_{2t}(2)$. We need to show that $\lim_{n \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w)$ exists. Note that,

$$\#\Pi_{L, nk}^*(w) = \sum_{\pi_b \in \Pi_{L_T, k}^*(w)} \#\phi_B^{-1}(\pi_b). \quad (3.2)$$

Let $\pi_b \in \Pi_{L_T, k}^*(w)$ and let l^1, \dots, l^m as in Lemma 3.2 (1). Hence,

$$\frac{\#\phi_B^{-1}(\pi_b)}{(nk)^{t+1}} = \frac{1}{k^{t+1}} \frac{\#\bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_W, n, l}^*(w)}{n^{t+1}}. \quad (3.3)$$

Hence, from Lemma 3.1 (3) and (3.3), it is clear that

$$\lim_{n \rightarrow \infty} \frac{\#\phi_B^{-1}(\pi_b)}{(nk)^{t+1}} = \begin{cases} \frac{1}{k^{t+1}} & \text{if } w \text{ is Catalan and } \pi_b \in \Pi_{L_T, k, l_0}^*(w) \\ 0, & \text{otherwise.} \end{cases} \quad (3.4)$$

Now from (3.2), it is clear that,

$$p(w) = \lim_{n \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w) = \begin{cases} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} & \text{if } w \text{ Catalan} \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $p(w)$ exists for all pair-matched w and hence LSD exists. All the odd moments are 0 and the $(2t) - th$ moment of the limiting distribution is given by

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} p(w) = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}}.$$

(ii) Now let n be fixed and let $k \rightarrow \infty$. Let $w \in \mathcal{W}_{2t}(2)$. We need to show that $\lim_{k \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w)$ exists. Note that,

$$\#\Pi_{L, nk}^*(w) = \sum_{\pi_e \in \Pi_{L_W, n}^*(w)} \#\phi_A^{-1}(\pi_e). \quad (3.5)$$

Let $\pi_e \in \Pi_{L_W, n}^*(w)$ and let l^1, \dots, l^m as in Lemma 3.2. Hence,

$$\frac{\#\phi_A^{-1}(\pi_e)}{(nk)^{t+1}} = \frac{1}{n^{t+1}} \frac{\#\bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w)}{k^{t+1}}. \quad (3.6)$$

Hence, from Lemma 3.1 and (3.6), it is clear that

$$p(w) = \lim_{k \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w) = \frac{\#\Pi_{L_W, n, l_0}^*(w)}{n^{t+1}} p_T(w).$$

Hence LSD of $\frac{1}{\sqrt{nk}} B_k \star A_{n, i}$ exists if n is fixed and $k \rightarrow \infty$, is symmetric and is determined by its even moments

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_W, n, l_0}^*(w)}{n^{t+1}} p_T(w).$$

(iii) From the proofs of (i) and (ii) it is clear that, for any fixed n and k ,

$$(\#\Pi_{L_W, n, l_0}^*(w))(\#\Pi_{L_T, k, l_0}^*(w)) \leq \#\Pi_{L, nk}^*(w) \leq \#(\Pi_{L_W, n}^*(w))(\#\Pi_{L_T, k}^*(w)).$$

Now, we know that $p_T(w) = 1$ if w is a Catalan word. Hence, using Lemma 3.1,

$$\lim_{n, k \rightarrow \infty} \frac{\#\Pi_{L_W, n, l_0}^*(w)}{n^{t+1}} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} = \lim_{n, k \rightarrow \infty} \frac{\#\Pi_{L_W, n}^*(w)}{n^{t+1}} \frac{\#\Pi_{L_T, k}^*(w)}{k^{t+1}} = 1 \text{ or } 0$$

according as w is Catalan or non-Catalan.

Sandwiching we get $p(w) = \lim_{n, k \rightarrow \infty} \frac{\#\Pi_{L, nk}^*(w)}{(nk)^{t+1}} = 1$ or 0 according as w is Catalan or non-Catalan. As a consequence, the LSD exists and is semicircular and the proof is complete.

3.4. Proof of Theorem 3.2. Let $w \in \mathcal{W}_{2t}(2)$. Let $\pi \in \Pi_{L, nk}^*(w)$. Let π_b be the block address of π . Then $\pi_b \in \Pi_{L_T, k}^*(w)$ where $\Pi_{L_T, k}^*(w)$ is as in the previous case. As before let us denote by ϕ_B , the block address function $\Pi_{L, nk}^*(w) \mapsto \Pi_{L_T, k}^*(w) : \pi \mapsto \pi_b$.

Now let $\pi \in \Pi_{L, nk}^*(w)$, Let π_e be the entry address of π . It follows that

$$L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j)) \Rightarrow L_{2, m_1}(\pi_e(i-1), \pi_e(i)) = L_{2, m_2}(\pi_e(j-1), \pi_e(j))$$

where $m_1 = L_1(\pi_b(i-1), \pi_b(i))$ and $m_2 = L_1(\pi_b(j-1), \pi_b(j))$. Now $L_{2, m_1}(\pi_e(i-1), \pi_e(i)) = L_{2, m_2}(\pi_e(j-1), \pi_e(j))$ implies $m_1 = \pm m_2$ and hence $(\pi_e(i-1) - \pi_e(i)) = \pm(\pi_e(j-1) - \pi_e(j))$ and hence $\pi_e \in \Pi_{L_T, n}^*(w)$. As before let us denote by ϕ_A , the entry address function $\Pi_{L, nk}^*(w) \mapsto \Pi_{L_T, n}^*(w) : \pi \mapsto \pi_e$.

Analogous to Lemma 3.2 we have the following lemma whose proof is omitted.

Lemma 3.3. *Let $w \in \mathcal{W}_{2t}(2)$.*

- (1) *Let $\pi_b \in \Pi_{L_T, k}^*(w)$ and l^1, l^2, \dots, l^m be all the distinct values of l such that, $\pi_b \in \Pi_{L_T, k, l}^*(w)$. Then,*

$$\#\phi_B^{-1}(\pi_b) = \# \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, n, l}^*(w).$$

- (2) *Let $\pi_e \in \Pi_{L_T, n}^*(w)$ and l^1, l^2, \dots, l^m be all the distinct values of l such that, $\pi_e \in \Pi_{L_T, n, l}^*(w)$. Then,*

$$\#\phi_A^{-1}(\pi_e) = \# \bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, k, l}^*(w).$$

The proof of this theorem is along the same lines as the proof of the previous theorem.

(i) Let k be fixed and let $n \rightarrow \infty$. Let $w \in \mathcal{W}_{2t}(2)$. We need to show that $\lim_{n \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w)$ exists. Note that,

$$\#\Pi_{L, nk}^*(w) = \sum_{\pi_b \in \Pi_{L_T, k}^*(w)} \#\phi_B^{-1}(\pi_b). \quad (3.7)$$

Let $\pi_b \in \Pi_{L_T, k}^*(w)$ and let l^1, \dots, l^m as in Lemma 3.3 (1). Hence,

$$\frac{\#\phi_B^{-1}(\pi_b)}{(nk)^{t+1}} = \frac{1}{k^{t+1}} \frac{\#\bigcup_{l \in \{l^1, \dots, l^m\}} \Pi_{L_T, n, l}^*(w)}{n^{t+1}}. \quad (3.8)$$

Now using Lemma 3.1,

$$\lim_{n \rightarrow \infty} \frac{\#\phi_B^{-1}(\pi_b)}{(nk)^{t+1}} = \frac{1}{k^{t+1}} p_T(w) \text{ if } \pi_b \in \Pi_{L_T, k, l_0}^*(w) \quad (3.9)$$

$$\lim_{n \rightarrow \infty} \frac{\#\phi_B^{-1}(\pi_b)}{(nk)^{t+1}} = 0, \text{ otherwise.} \quad (3.10)$$

Now from (3.7), it is clear that,

$$p(w) = \lim_{n \rightarrow \infty} \frac{1}{(nk)^{t+1}} \#\Pi_{L, nk}^*(w) = \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} p_T(w). \quad (3.11)$$

In particular, $p(w)$ exists for all pair-matched w and hence LSD exists. All the odd moments are 0 and the $2t - th$ moment of the limiting distribution is given by

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} p(w) = \sum_{w \in \mathcal{W}_{2t}(2)} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} p_T(w).$$

(ii) Proof of (ii) is exactly similar to that of (i), except that we use Lemma 3.3(2) instead of Lemma 3.3 (1).

(iii) It is clear that for any fixed n and k ,

$$(\#\Pi_{L_T, n, l_0}^*(w))(\#\Pi_{L_T, k, l_0}^*(w)) \leq \#\Pi_{L, nk}^*(w) \leq (\#\Pi_{L_T, n}^*(w))(\#\Pi_{L_T, k}^*(w)).$$

Also, using Lemma 3.1,

$$\lim_{n, k \rightarrow \infty} \frac{\#\Pi_{L_T, n, l_0}^*(w)}{n^{t+1}} \frac{\#\Pi_{L_T, k, l_0}^*(w)}{k^{t+1}} = \lim_{n, k \rightarrow \infty} \frac{\#\Pi_{L_T, n}^*(w)}{n^{t+1}} \frac{\#\Pi_{L_T, k}^*(w)}{k^{t+1}} = (p_T(w))^2$$

for every pair-matched w of length $(2t)$.

Sandwiching, we get $p(w) = \lim_{n,k \rightarrow \infty} \frac{\#\Pi_{L,nk}^*(w)}{(nk)^{t+1}} = (p_T(w))^2$ for every $w \in \mathcal{W}_{2t}(2)$. Hence the LSD in this case exists and is identified by the even moments

$$\beta_{2t} = \sum_{w \in \mathcal{W}_{2t}(2)} (p_T(w))^2.$$

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