LIMITING SPECTRAL DISTRIBUTION OF k-CIRCULANT MATRIX WITH DEPENDENT ENTRIES

ARUP BOSE* AND KOUSHIK SAHA

This article is subsumed in the Technical Report R6/2009, Stat-Math Unit.

Abstract. Formula for the eigenvalues of circulant matrices with general shift by k places is available in the literature. Under suitable restrictions on $\{k = k(n)\}$, the limiting spectral distribution (LSD) of suitably scaled eigenvalues is known when the input sequence of the matrix are i.i.d. with finite variance. We derive the LSD of the *k-circulant matrix* when the input sequence is a stationary, two sided moving average process of infinite order.

Keywords Large dimensional random matrix, eigenvalues, k-circulant matrix, empirical spectral distribution, limiting spectral distribution, moving average process, convergence in distribution, convergence in probability, normal approximation.

AMS 2000 Subject Classification 60F99, 62E20, 60G57.

1. Introduction

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of an $n \times n$ matrix A_n, then the *empirical spectral distribution function (ESDF)* of A_n is defined as

$$F_{A_n}(x, y) = n^{-1} \sum_{i=1}^{n} I\{Re\lambda_i \leq x,\ Im\lambda_i \leq y\}.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices with the corresponding ESDF $\{F_{A_n}\}_{n=1}^{\infty}$. The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence $\{F_{A_n}\}_{n=1}^{\infty}$, if it exists. If $\{A_n\}$ are random, the limit is in some probabilistic sense, such as "almost surely" or "in probability". Suppose elements of $\{A_n\}$ are defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, that is $\{A_n\}$ are random. Let F be a distribution function. We say the ESD of A_n converges in probability to F if for every $\epsilon > 0$ and at all continuity points (x, y) of F,

$$\mathbb{P}\big(|F_{A_n}(x, y) - F(x, y)| > \epsilon\big) \to 0 \text{ as } n \to \infty.$$

For detailed information on LSD of large dimensional random matrices see [Bai(1999)] and [Bose and Sen (2008)].

*Research supported by J.C.Bose Fellowship, Dept. of Science and Technology, Govt. of India.
*Research supported by CSIR Fellowship, Dept. of Science and Technology, Govt. of India.

In this article we focus on the LSD of k-circulant matrices. Suppose $\{x_i,\ i = 0, 1, 2, ...\}$ is a sequence of numbers (called the *input sequence*). For positive integers k and n, define the $n \times n$ square matrix

$$A_{k,n} = \begin{bmatrix} x_0 & x_1 & x_2 & \dots & x_{n-2} & x_{n-1} \\ x_{n-k} & x_{n-k+1} & x_1 & \dots & x_{n-k-2} & x_{n-k-1} \\ x_{n-2k} & x_{n-2k+1} & x_0 & \dots & x_{n-2k-2} & x_{n-2k-1} \\ & & & \vdots & & \end{bmatrix}_{n \times n}.$$

We emphasize that all subscripts appearing in the entries above are calculated modulo n. The first row of $A_{k,n}$ is $(x_0, x_1, x_2, ..., x_{n-1})$ and for $1 \le j < n-1$, its $(j+1)$-th row is obtained by giving its j-th row a right circular shift by k positions (equivalently, k mod n positions).

Establishing the LSD for general k-circulant matrices appears to be a difficult problem. For certain combinations of n and k, a large proportion of eigenvalues are zero and the limit distribution is point mass at zero. For instance, [Bose, Mitra and Sen (2008)] show that if $\{x_i\}$ are i.i.d $N(0,1)$, $k = n^{o(1)}$ (≥ 2) and $gcd(k,n) = 1$ then the LSD of $F_{n^{-1/2}A_{k,n}}$ is degenerate at zero, in probability. Nondegenerate LSD in a few special cases are also derived in [Bose, Mitra and Sen (2008)]. In particular, suppose $\{x_i\}$ are i.i.d. with mean zero and variance one and $E|x_1|^{2+\delta} < \infty$ for some $\delta > 0$. Let $\{E_i\}$ be i.i.d. $Exp(1)$, U_1 be uniformly distributed over $(2g)$-th roots of unity, U_2 be uniformly distributed over the unit circle where $\{U_i\}$, $\{E_i\}$ are mutually independent. Then

(i) for $k^g = -1 + sn,\ g \ge 1,\ s = o(n^{1/3})$, $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to $U_1(\prod_{i=1}^{g} E_i)^{1/2g}$ as $n \to \infty$,

(ii) for $k^g = 1 + sn,\ g \ge 1$ and

$$s = \begin{cases} o(n) & \text{if } g \text{ is even} \\ o(n^{\frac{g+1}{g-1}}) & \text{if } g \text{ is odd,} \end{cases}$$

$F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to $U_2(\prod_{i=1}^{g} E_i)^{1/2g}$.

Now let $\{x_n; n \ge 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. random variables with mean zero and variance one and consider the matrix $A_{k,n}$ with these $\{x_k\}$. If $\mathbb{E}(|\epsilon_i|^{2+\delta}) < \infty$ for some $\delta > 0$ then the following results are known (see [Bose and Saha (2008)(a)] and [Bose and Saha (2008)(b)]).

(i) If $k = n-1$, the LSD of the k-circulant (which is also known as the reverse circulant), when scaled by $n^{-1/2}$ is

$$H(x) = \begin{cases} 1 - \int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x > 0 \\ \int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x \le 0. \end{cases}$$

(ii) If $k = 1$ or equivalently $k = n+1$, the LSD of the k-circulant (which is also known simply as the circulant), when scaled by $n^{-1/2}$ (assuming $\inf_{\omega \in [0,2\pi]} f(\omega) > 0$) is

$$F(x,y) = \iint \mathbb{I}_{\{(v_1,v_2)\le(x,y)\}} \Big[\int_0^1 \frac{1}{2\pi^2 f(2\pi s)} e^{-\frac{v_1^2+v_2^2}{2\pi f(2\pi s)}} ds \Big] dv_1 dv_2.$$

In this paper we extend the above result to show that the LSD of $n^{-1/2}A_{k,n}$ continues to exist in this dependent situation when $n = k^g + 1$ or $n = k^g - 1$ for $g \ge 2$. See Theorems 2.1 and 2.5. The proofs exploit known eigenvalues formulae and their properties as well as appropriate normal approximations.

2. Main results and proofs

The structure of eigenvalues of a k-circulant matrix is already known, see for example Zhou (1996). More detailed analysis of eigenvalues and related properties of $A_{k,n}$, useful in the present context, have been developed in Section 2 of [Bose, Mitra and Sen (2008)]. We use some of their development. Let

$$\nu = \nu_n := \cos(2\pi/n) + i\sin(2\pi/n),\ \ i^2 = -1 \ \text{ and } \ \lambda_k = \sum_{l=0}^{n-1} x_l \nu^{kl},\ \ 0 \le j < n. \tag{2.1}$$

For any two positive integers k and n, let $p_1 < p_2 < \ldots < p_c$ be all their common prime factors so that,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \quad \text{and} \quad k = k' \prod_{q=1}^{c} p_q^{\alpha_q}.$$

Here $\alpha_q,\ \beta_q \ge 1$ and n', k', p_q are pairwise relatively prime. For any positive integer s, let $\mathbb{Z}_s = \{0, 1, 2, \ldots, s-1\}$. Define the following sets

$$S(x) = \{xk^b \bmod n' : b \ge 0\},\ 0 \le x < n'.$$

Let $g_x = |S(x)|$. Define

$$\upsilon_{k,n'} := |\{x \in \mathbb{Z}_{n'} : g_x < g_1\}|.$$

We observe the following about the sets $S(x)$.

(1) $S(x) = \{xk^b \bmod n' : 0 \le b < |S(x)|\}$.

(2) For $x \neq u$, either $S(x) = S(u)$ or, $S(x) \cap S(u) = \phi$. As a consequence, the distinct sets from the collection $\{S(x) : 0 \leq x < n'\}$ forms a partition of $\mathbb{Z}_{n'}$.

We shall call $\{S(x)\}$ the *eigenvalue partition* of $\{0, 1, 2, \ldots, n-1\}$ and we will denote the partitioning sets and their sizes by

$$\{\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{l-1}\}, \text{ and } \quad n_i = |\mathcal{P}_i|, \ 0 \leq i < l.$$

Define

$$y_j := \prod_{t \in \mathcal{P}_j} \lambda_{ty}, \quad j = 0, 1, \ldots, l-1 \quad \text{where} \quad y = n/n'.$$

Then the characteristic polynomial of $A_{k,n}$ is given by

$$\chi\left(A_{k,n}\right) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{n_j} - y_j\right), \tag{2.2}$$

and this provides a formula solution for the eigenvalues.

Now we turn to the process $\{x_n\}$. Define $\gamma_h = Cov\,(x_{t+h}, x_t)$. Then it is easy to see that $\sum_{j \in \mathbb{Z}} |\gamma_j| < \infty$ and the *spectral density function* of $\{x_n\}$ is given by

$$f(\omega) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi} \big[\gamma_0 + 2 \sum_{k \geq 1} \gamma_k \cos(k\omega)\big] \text{ for } \ \omega \in [0, 2\pi].$$

Let

$$G_d(x) = \mathbb{P}\big(\prod_{i=1}^{d} E_i \leq x\big),$$

where $\{E_i\}$ are i.i.d. $Exp(1)$. For any integer $d \geq 1$, define $H_d(x, \omega_1, \ldots, \omega_d)$ on $\mathbb{R}_{\geq 0} \times [0, 2\pi]^d$ as

$$H_d(x, \omega_1, \ldots, \omega_d) = \begin{cases} G_d\Big(\frac{x^4}{(2\pi)^d \prod_{i=1}^d f(\omega_i)}\Big) & \text{if} \quad \prod_{i=1}^d f(\omega_i) \neq 0, \ x \neq 0 \\ 1 & \text{if} \quad \prod_{i=1}^d f(\omega_i) = 0, \ x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Since G_d is continuous on $\overline{\mathbb{R}}$ and $f(\omega)$ is uniformly continuous on $[0, 2\pi]$, it is easy to see that for fixed x, $H_d(x, \omega_1, \ldots, \omega_d)$ is uniformly continuous on $[0, 2\pi]^d$. It then easily follows that F_d defined below is a valid distribution function.

$$F_d(x) \quad = \quad \int_0^1 \cdots \int_0^1 H_d(x, t_1, \ldots, t_d) \prod_{i=1}^{d} dt_i. \tag{2.3}$$

Theorem 2.1. Suppose $\{\epsilon_i\}$ is an i.i.d. sequence with mean zero, variance 1 and $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Suppose $n = k^g + 1$ for some $g \geq 2$. Then as $n \to \infty$, $F_{n^{-1/2} A_{k,n}}$ converges weakly in probability to the LSD $U_1 (\prod_{i=1}^2 E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. with distribution function F_g and U_1 is uniformly distributed over the $(2g)$th roots of unity, independent of the $\{E_i\}$.

Remark 2.1. Using the expression (2.2) for the characteristic polynomial, it is then not difficult to manufacture $\{k = k(n)\}$ such that the LSD of $n^{-1/2}A_{k,n}$ has some positive mass at the origin. For example, suppose the sequences k and n satisfy $k^g = -1 + sn$ where $g \geq 1$ is fixed and $s = o(n^{1/3})$. Fix primes $p_1, p_2, \ldots, p_t$ and positive integers $\beta_1, \beta_2, \ldots, \beta_t$. Define

$$\widetilde{n} = p_1^{\beta_1} p_2^{\beta_2} \ldots p_t^{\beta_t} n.$$

Suppose $k = p_1 p_2 \ldots p_t m \to \infty$. Then the ESD of $\widetilde{n}^{-1/2} A_{k,\widetilde{n}}$ converges weakly in probability to the LSD which has $1 - \left(\Pi_{s=1}^t p_s^{\beta_s}\right)^{-1}$ mass at zero, and rest of the probability mass is distributed as $U_1(\prod_{i=1}^g E_i)^{1/2g}$ where U_1 and $\{E_i\}$ are as above.

Remark 2.2. If $\{x_i\}$ are i.i.d, then $f(\omega) = 1/2\pi$ for all $\omega \in [0, 2\pi]$ and the LSD is $U_1(\prod_{i=1}^g E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. $Exp(1)$, U_1 is as above and independent of $\{E_i\}$. This agrees with [Bose, Mitra and Sen (2008)].

The proof of the theorem draws substantially from the method of [Bose and Mitra (2002)] and mainly depends on following two lemmas. The proof of Lemma 2.2 is given in [Fan and Yao (2003)] (Theorem 2.14(ii), page 63). The proof of Lemma 2.3 follows easily from [Bhattacharya and Ranga Rao (1976)] (Corollary 18.1, page 181).

Lemma 2.2. Let $x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j}$ for $t \geq 0$, where $\{\epsilon_t\}$ are i.i.d. with mean zero and variance one and $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then for $k = 1, 2, \cdots, n$,

$$\omega_k = \frac{2\pi k}{n} \quad \text{and} \quad \frac{1}{n}|\lambda_k|^2 = L_n(\omega_k) + R_n(\omega_k)$$

where

$$L_n(\omega_k) = 2\pi f(\omega_k)\big[\big(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} \epsilon_l \cos(\frac{2\pi kl}{n})\big)^2 + \big(\frac{1}{\sqrt{n}}\sum_{l=0}^{n-1} \epsilon_l \sin(\frac{2\pi kl}{n})\big)^2\big]$$

and $\max_{1 \leq k \leq [\frac{n-1}{2}]} E|R_n(\omega_k)| \to 0$ as $n \to \infty$.

Lemma 2.3. Let $X_1, \ldots, X_k$ be independent random vectors with values in $\mathbb{R}^d$, having zero means and an average positive-definite covariance matrix $V_k = k^{-1}\sum_{j=1}^k Cov(X_j)$. Let G_k denote the distribution of $k^{-1/2}T_k(X_1 + \ldots + X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \geq 1$. If for some $\delta > 0$, $E\|X_j\|^{(2+\delta)} < \infty$, then there exists $C_1,\ C_2 > 0$ (depending only on d), such that for any Borel set A,

$$\begin{aligned} |G_k(A) - \Phi_d(A)| &\leq C_1 k^{-\delta/2}\big[k^{-1}\sum_{j=1}^k E \parallel T_k X_j \parallel^{(2+\delta)}\big] + 2\sup_{y \in \mathbb{R}^d} \Phi_d((\partial A)^\eta - y), \\ &\leq C_1 k^{-\delta/2}(\lambda_{\min}(V_k))^{-(2+\delta)}\rho_{2+\delta} + 2\sup_{y \in \mathbb{R}^d} \Phi_d((\partial A)^\eta - y), \end{aligned}$$

where $\rho_{2+\delta} = k^{-1}\sum_{j=1}^k E \parallel X_j \parallel^{(2+\delta)}$ and $\eta = C_2 \rho_{2+\delta} n^{-\delta/2}$.

Proof of Theorem 2.1: For simplicity we first prove the result when $g = 2$. Note that $gcd(k, n) = 1$ and hence in this case $n' = n$ in (2.2). Thus the index of each eigenvalue belongs to *exactly one* of the sets $\mathcal{P}_l$ in the eigenvalue partition of $\{0, 1, 2, ..., n-1\}$. Recall that $v_{k,n}$ is the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{P}_l$ and $|\mathcal{P}_l| < g_1$. In view of Lemma 7 of [Bose, Mitra and Sen (2008)], we have $v_{k,n}/n \leq 2/n \to 0$ and hence these eigenvalues do not contribute to the LSD.

Hence it remains to consider only the eigenvalues corresponding to the sets $\mathcal{P}_l$ which have size *exactly equal* to g_1. It also follows from the argument in the proof of Lemma $3(i)$ of [Bose, Mitra and Sen (2008)] that $g_1 = 4$.

Recall the quantities $n_j = |\mathcal{P}_j|$, $y_j = \prod_{t \in \mathcal{P}_l} \lambda_t$, where $\lambda_j = \sum_{l=0}^{n-1} x_l \nu^{jl}$, $0 \leq j < n$. Also, for every integer $t \geq 0$, $tk^2 = -t \mod n$, so that, λ_t and λ_{n-t} belong to same partition block $S(t) = S(n-t)$. Thus each y_t is real. Let us define

$$I_n = \{l : |\mathcal{P}_l| = 4\}.$$

It is clear that $n/|I_n| \to 4$. Without any loss, let $I_n = \{1, 2, ..., |I_n|\}$.

Let $1, \omega, \omega^2, \omega^3$ be all the fourth roots of unity. Note that for every j, the eigenvalues of $A_{k,n}$ corresponding to the set $\mathcal{P}_j$ are: $y_j^{1/4}, y_j^{1/4}\omega, y_j^{1/4}\omega^2, y_j^{1/4}\omega^3$. Hence it suffices to consider only the modulus of eigenvalues $y_j^{1/4}$ as j varies: if these have an LSD F, say, then the LSD of the whole sequence will be (r, θ) in polar coordinates where r is distributed uniformly across all the fourth roots of unity and r are θ are independent. With this in mind, and remembering the scaling $\sqrt{n}$, we consider, for $x > 0$,

$$F_n(x) = |I_n|^{-1} \sum_{i=1}^{|I_n|} \mathbb{I}\left(\left[\frac{y_j}{n^2}\right]^{\frac{1}{4}} \leq x\right).$$

Since the set of λ values corresponding to any $\mathcal{P}_j$ is closed under conjugation, there exists a set $\mathcal{A}_i \subset \mathcal{P}_i$ of size 2 such that

$$\mathcal{P}_i = \{x : x \in \mathcal{A}_i \text{ or } n - x \in \mathcal{A}_i\}.$$

Combining each λ_j with its conjugate, we may write y_j in the form,

$$y_j = \prod_{t \in \mathcal{A}_j} (b_{t,n}^2 + c_{t,n}^2)$$

where $\{b_{t,n}\}$ and $\{c_{t,n}\}$ are of the form

$$b_{t,n} = \sum_{l=0}^{n-1} x_l \cos\left(\frac{2\pi tl}{n}\right), \quad c_{t,n} = \sum_{l=0}^{n-1} x_l \sin\left(\frac{2\pi tl}{n}\right), \quad t \in \mathcal{A}_j, \quad 1 \leq j \leq |I_n|.$$

We show that for every x,

$$\mathbb{E}[F_n(x)] \to F_2(x) \text{ and } Var[F_n(x)] \to 0.$$

This will prove that the ESD converges to the required LSD in probability when $g = 2$.

Note that for $x > 0$,

$$\mathbb{E}[F_n(x)] = |I_n|^{-1} \sum_{j=1}^{|I_n|} \mathbb{P}\big(\frac{y_j}{n^2} \le x^4\big).$$

Define

$$\bar{b}_{t,n} = \sum_{l=0}^{n-1} \epsilon_l \cos\left(\frac{2\pi tl}{n}\right), \quad \bar{c}_{t,n} = \sum_{l=0}^{n-1} \epsilon_l \sin\left(\frac{2\pi tl}{n}\right), \quad t \in \mathcal{A}_j, \quad 1 \le j \le |I_n|$$

$$\overline{y}_j = \prod_{t \in \mathcal{A}_j} (\bar{b}_{t,n}^2 + \bar{c}_{t,n}^2), \quad 1 \le j \le |I_n|,$$

$$\omega_t = \frac{2\pi t}{n}, \quad f_j = \prod_{t \in \mathcal{A}_j} f(\omega_t), \quad 1 \le j \le |I_n|.$$

Let $\mathcal{A}_j = \{j_1, j_2\}$. From Lemma 2.2, it is intutively clear that for large n, $\frac{y_j}{n^2} \sim 4\pi^2 f_j \frac{\overline{y}_j}{n^2}$. So first we show that for large n,

$$\frac{1}{|I_n|} \sum_{k=1}^{|I_n|} \mathbb{P}(\frac{y_j}{n^2} \le x^4) \sim \frac{1}{|I_n|} \sum_{k=1}^{|I_n|} \mathbb{P}(4\pi^2 f_j \frac{\overline{y}_j}{n^2} \le x^4).$$

Let $4\pi^2 f_j \frac{y_j}{n^2} = L_{n,j} + R_{n,j}$ for $1 \le j \le |I_n|$, where

$$L_{n,j} = 4\pi^2 f_j \frac{\overline{y}_j}{n^2},$$

$$R_{n,j} = L_n(\omega_{j_1}) R_n(\omega_{j_2}) + L_n(\omega_{j_2}) R_n(\omega_{j_1}) + R_n(\omega_{j_1}) R_n(\omega_{j_2}),$$

$$L_n(\omega_{j_1}) = 2\pi f(\omega_{j_1})[n^{-1}(\bar{b}_{j_1,n}^2 + \bar{c}_{j_1,n}^2)], \quad L_n(\omega_{j_2}) = 2\pi f(\omega_{j_2})[n^{-1}(\bar{b}_{j_2,n}^2 + \bar{c}_{j_2,n}^2)].$$

Without loss we may choose ϵ small enough such that $x - \epsilon > 0$. Then

$$\begin{aligned}
\Big| \sum_{k=1}^{|I_n|} \Big[\mathbb{P}(L_{n,j} + R_{n,j} \le x^4) - \mathbb{P}(L_{n,j} \le x^4) \Big] \Big| &\le \sum_{k=1}^{|I_n|} \mathbb{P}(|R_{n,j}| \ge \epsilon) \\
&+ \sum_{k=1}^{|I_n|} \big| \mathbb{P}(L_{n,j} \le x^4) - \mathbb{P}(L_{n,j} \le x^4 \pm \epsilon) \big| \\
&\le |I_n|(T_1 + T_2 + T_3 + T_4), \text{ say,}
\end{aligned}$$

where

$$T_1 = \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \mathbb{P}(|R_{n,j}| \ge \epsilon), \quad T_2 = \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \big| \mathbb{P}(L_{n,j} \le x^4) - \Phi_4(A_{n,j}) \big|,$$

$$T_3 = \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \big| \mathbb{P}(L_{n,j} \le x^4 \pm \epsilon) - \Phi_4(A_{n,j}^{\epsilon}) \big|, \quad T_4 = \frac{1}{|I_n|} \sum_{k=1}^{|I_n|} \big| \Phi_4(A_{n,j}) - \Phi_4(A_{n,j}^{\epsilon}) \big|,$$

$$A_{n,j} = \Big\{ (a_1, b_1, a_2, b_2) : \prod_{i=1}^{2} [2^{-1}(a_i^2 + b_i^2)] \le \frac{x^4}{4\pi^2 f_j} \Big\},$$

$$A_{n,j}^{\epsilon} = \Big\{(a_1, b_1, a_2, b_2) : \prod_{i=1}^{2}[2^{-1}(a_i^2 + b_i^2)] \leq \frac{x^4 \pm \epsilon}{4\pi^2 f_j}\Big\}.$$

Note that $x > 0$ and without loss we may choose ϵ small enough such that $x^4 - \epsilon > 0$. It may be note that if $f_j = 0$ for any j, then $A_{n,j}^{\epsilon}$ and $A_{n,j}$ are the whole spaces. We first show that $T_1 \to 0$ as $n \to \infty$. Now

$$\begin{aligned}\mathbb{P}(|R_{n,j}| \geq \epsilon) \quad &\leq \quad \mathbb{P}(|L_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3) + \mathbb{P}(|L_n(\omega_{j_2})R_n(\omega_{j_1})| \geq \epsilon/3) \\ &\quad + \mathbb{P}(|R_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3).\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}(|R_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3) \quad &\leq \quad \mathbb{P}(|R_n(\omega_{j_2})| \geq \epsilon/3) + \mathbb{P}(|R_n(\omega_{j_1})| \geq 1) \\ &\leq \quad \frac{3}{\epsilon}E(|R_n(\omega_{j_1})|) + E(|R_n(\omega_{j_2})|)\end{aligned}$$

Now by Lemma 2.2 as $n \to \infty$

$$\frac{1}{|I_n|}\sum_{j=0}^{|I_n|}\mathbb{P}(|R_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3) \leq [\frac{3}{\epsilon} + 1] \sup_{1 \leq j \leq [\frac{n-1}{2}]} E|R_n(\omega_j)| \to 0.$$

$$\begin{aligned}\mathbb{P}(|L_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3) \quad &\leq \quad \mathbb{P}(|L_n(\omega_{j_1})| > \frac{\epsilon M}{3}) + \mathbb{P}(|R_n(\omega_{j_2})| \geq \frac{1}{M}) \\ &\leq \quad \frac{3}{M\epsilon}E(L_n(\omega_{j_1})) + M\ E|R_n(\omega_{j_2})|\end{aligned}$$

Note that $\sup_{1 \leq j \leq n} E[L_n(\omega_{j_1})] \leq 4\pi \max_{\omega \in [0,2\pi]} f(\omega) < \infty$. Hence

$$\frac{1}{|I_n|}\sum_{j=0}^{|I_n|}\mathbb{P}(|L_n(\omega_{j_1})R_n(\omega_{j_2})| \geq \epsilon/3) \leq \frac{3}{M\epsilon}4\pi \max_{\omega \in [0,2\pi]} f(\omega) + M \sup_{1 \leq j \leq [\frac{n-1}{2}]} E|R_n(\omega_j)|$$

Now as $n \to \infty$, the right side of the above expression can be made smaller than any given positive number by choosing M large enough. Similarly

$$\frac{1}{|I_n|}\sum_{j=0}^{|I_n|}\mathbb{P}(|L_n(\omega_{j_2})R_n(\omega_{j_1})| \geq \epsilon/3)$$

can be made arbitrarily small. Therefore $T_1 \to 0$ as $n \to \infty$.

We now show $T_4 \leq C_x\epsilon$ for any n, where C_x is a constant depending only on x. It is understood that if $f_j = 0$ for any j, then in the string of equations and inequalities below, that term is absent.

$$
\begin{aligned}
T_4 &= \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}|\Phi_4(A_{n,j})-\Phi_4(A^{\epsilon}_{n,j})| \\
&\leq \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\Big|\int_0^{\infty}\frac{1}{2}e^{-t/2}\Big[\int_{\frac{x^4}{\pi^2 f_j t}\leq s\leq \frac{x^4+\epsilon}{\pi^2 f_j t}}\frac{1}{2}e^{-s/2}ds\Big]dt\Big| \\
&\leq \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\Big|\int_0^{\infty}\frac{1}{2}e^{-t/2}[e^{-\frac{x^4}{2\pi^2 f_j t}}-e^{-\frac{x^4+\epsilon}{2\pi^2 f_j t}}]dt\Big| \\
&\leq \int_0^{\infty}\frac{1}{2}e^{-t/2}\frac{1}{|I_n|}\sum_{k=1}^{|I_n|}e^{-\frac{x^4}{2\pi^2 f_j t}}\frac{\epsilon}{2\pi^2 f_j t}dt \\
&\leq \frac{\epsilon}{x^4}, \quad (\because\ |e^{-\frac{x^4}{2\pi^2 f_j t}}\frac{1}{2\pi^2 f_j t}|\leq \frac{1}{x^4}).
\end{aligned}
$$

To show $T_2, T_3 \to 0$ we will use normal approximation. First note the following well known trigonometric identities where $1 \leq l < l' \leq n$,

$$
\sum_{t=0}^{n-1}\cos\big(\frac{2\pi lt}{n}\big)\cos\big(\frac{2\pi l't}{n}\big) = \begin{cases} 0 & \text{if} \quad l+l'\neq n, \\ n/2 & \text{if} \quad l+l'=n, \end{cases}
$$

$$
\sum_{t=0}^{n-1}\sin\big(\frac{2\pi lt}{n}\big)\sin\big(\frac{2\pi l't}{n}\big) = \begin{cases} 0 & \text{if} \quad l+l'\neq n, \\ -n/2 & \text{if} \quad l+l'=n, \end{cases}
$$

$$
\sum_{t=0}^{n-1}\cos\big(\frac{2\pi lt}{n}\big)\sin\big(\frac{2\pi l't}{n}\big) = 0,
$$

$$
\sum_{t=0}^{n-1}\cos^2\big(\frac{2\pi lt}{n}\big) = n-\sin^2\big(\frac{2\pi lt}{n}\big) = \begin{cases} n/2 & \text{if} \quad 2l\neq n, \\ n & \text{if} \quad 2l=n. \end{cases}
$$

Towards using an appropriate Berry-Esséen bound, define

$$
X_{l,j} = 2^{1/2}\left(\epsilon_l\cos\left(\frac{2\pi tl}{n}\right),\ \ \epsilon_l\sin\left(\frac{2\pi tl}{n}\right),\ \ t\in\mathcal{A}_j\right) \quad 0\leq l<n, 1\leq j\leq |I_n|.
$$

Using the trigonometric identities we have

$$
E(X_{l,j}) = 0 \ \text{ and } \ n^{-1}\sum_{l=0}^{n-1}Cov(X_{l,j}) = I_4 \ \text{ for } \ 1\leq j\leq |I_n|.
$$

Note that

$$
\{4\pi^2 f_j\frac{\overline{y}_j}{n^2}\leq x^4\} \ = \ \{n^{-1/2}\sum_{l=1}^{n-1}X_{l,j}\in A_{n,j}\}.
$$

Now using Lemma 2.3 and Lemma 4 of [Bose, Mitra and Sen (2008)], we get for small $\epsilon > 0$ and some constant C,

$$
\Big|\mathbb{P}\big(4\pi^2 f_j\frac{\overline{y}_j}{n^2}\leq x^4\big)-\Phi_4(A_{n,j})\Big| \leq Cn^{-\delta/2}\rho_{2+\delta}+C(\rho_{2+\delta}n^{-\delta/2})^{1-\epsilon}
$$

where

$$\rho_{2+\delta} = n^{-1}\sum_{l=0}^{n-1} E\|X_{l,j}\|^{2+\delta} \quad \text{and} \quad \sup_{n}\sup_{1\le j\le n} \rho_{2+\delta} < \infty.$$

Therefore

$$|I_n|^{-1}\sum_{l=1}^{|I_n|}\Big|\mathbb{P}\big(\frac{y_j}{n^2}\le x^4\big) - \Phi_4(A_{n,j})\Big| \to 0.$$

Hence $T_2 \to 0$ and similarly $T_3 \to 0$. Therefore

$$E[F_n(x)] = \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}\Big(\frac{y_j}{n^2}\le x^4\Big) \sim \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}\Big(4\pi^2 f_j\frac{\overline{y}_j}{n^2}\le x^4\Big),$$

and also by similar argument as above

$$\Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\big[\mathbb{P}(4\pi^2 f_j\frac{\overline{y}_j}{n^2}\le x^4) - \Phi_4(A_{n,j})\big]\Big| \to 0 \text{ as } n\to\infty.$$

Now it remains to identify the limit

$$\lim_{n\to\infty}\frac{1}{|I_n|}\sum_{l=1}^{|I_n|}\Phi_4(A_{n,j}).$$

To do so we recall the structure of the sets $S(x), \mathcal{P}_j, \mathcal{A}_j$ and their properties. Since $|I_n|/n \to 1/4$, $v_{k,n}\le 2$ and either $S(x) = S(u)$ or $S(x)\cap S(u) = \phi$, we have

$$\lim_{n\to\infty}\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Phi_4(A_{n,j}) = \lim_{n\to\infty}\frac{1}{n}\sum_{j=1,|\mathcal{A}_j|=2}^{n}\Phi_4(A_{n,j}) \tag{2.4}$$

Also for $n = k^2+1$ we can write $\{0,1,2,\ldots,n-1\}$ as $\{ak+b;\ 0\le a\le k-1,\ 1\le b\le k\}$ and using the construction of $S(x)$ we have $\mathcal{A}_j = \{ak+b, bk-a\}$ for $j = ak+b;\ 0\le a\le k-1,\ 1\le b\le k$ (except for atmost two values of j).

Recall that for fixed x, $H_2(x,\omega,\omega')$ is uniformly continuous on $[0,2\pi]\times[0,2\pi]$, . Therefore given any positive number δ we can choose N large enough such that for all $n = k^2+1 > N$,

$$\sup_{0\le a\le k-1,\ 1\le b\le k}\Big|H_2\Big(x,\frac{2\pi(ak+b)}{n},\frac{2\pi(bk-a)}{n}\Big) - H_2\Big(x,\frac{2\pi a}{\sqrt{n}},\frac{2\pi b}{\sqrt{n}}\Big)\Big| < \delta. \tag{2.5}$$

Finally using $(1.3), (1.4)$ we have

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\Phi_4(A_{n,j}) &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\Phi_4(A_{n,j}) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}G_2\big(\frac{x^4}{4\pi^2 f_j}\big) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{b=1}^{[\sqrt{n}]}\sum_{a=0}^{[\sqrt{n}]}H_2\Big(x,\frac{2\pi(ak+b)}{n},\frac{2\pi(bk-a)}{n}\Big) \\
&= \lim_{n\to\infty}\frac{1}{n}\sum_{b=1}^{[\sqrt{n}]}\sum_{a=0}^{[\sqrt{n}]}H_2\Big(x,\frac{2\pi a}{\sqrt{n}},\frac{2\pi b}{\sqrt{n}}\Big) \\
&= \int_0^1\int_0^1 H_2(x,2\pi s,2\pi t)ds\ dt = F_2(x).
\end{aligned}$$

To show that $Var[F_n(x)]\to 0$, since the variables involved are all bounded, it is enough to show that

$$n^{-2}\sum_{j\neq j'}Cov\left(\mathbb{I}\big(\frac{y_j}{n^2}\le x^4\big),\ \mathbb{I}\big(\frac{y_{j'}}{n^2}\le x^4\big)\right)\to 0.$$

Along the lines of the proof used to show $E[F_n(x)]\to F_2(x)$, one may now extend the vectors with 4 coordinates defined above to ones with 8 coordinates and proceed exactly as above to verify this. We omit the routine details. This proves the Theorem when $g=2$.

The above argument essentially can be extended to cover the general $(g>2)$ case. Here we highlight only a few of the technicalities and omit the other details. We now have the following lemma.

Lemma 2.4. Given any $\epsilon,\eta>0$ there exist a $N\in\mathbb{N}$ such that

$$\mathbb{P}\big(\big|\prod_{i=1}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>\epsilon\big)\le\frac{2\pi}{M}(s-1+\frac{1}{\epsilon})\max_{0\le\omega\le 2\pi}f(\omega)+(M^s+g-s-1)\max_{1\le k\le n}E|R_n(\omega_k)|<\eta$$

for all $n\ge N$, where $L_n(\omega_j),\ R_n(\omega_j)$ are as defined in Lemma 2.2.

Proof. Note

$$\begin{aligned}
\mathbb{P}\big(\big|\prod_{i=1}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>\epsilon\big) &\le \mathbb{P}\big(\big|L_n(\omega_{j_1})\big|\ge M\epsilon\big) \\
&\quad +\mathbb{P}\big(\big|\prod_{i=2}^{s}L_n(\omega_{j_i})\prod_{i=s+1}^{g}R_n(\omega_{j_i})\big|>1/M\big),
\end{aligned}$$

and iterating this argument,

$$\begin{aligned} & P\big(\big|\prod_{i=2}^{s} L_n(\omega_i) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M\big) \\ \leq\ & P\big(\big|L_n(\omega_{j_2})\big| \geq M\big) + P\big(\big|\prod_{i=3}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^2\big). \end{aligned}$$

Continuing in this way we have

$$\begin{aligned} \mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) \ &\leq\ \mathbb{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) + \sum_{i=2}^{s} \mathbb{P}\big(\big|L_n(\omega_{j_i})\big| \geq M\big) \\ &\quad + \mathbb{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) \end{aligned}$$

Also note that

$$\begin{aligned} \mathbb{P}\big(\big|\prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) \ &\leq\ \mathbb{P}\big(\big|\prod_{i=s+2}^{g} R_n(\omega_{j_i})\big| > 1/M^s\big) + \mathbb{P}\big(\big|R_n(\omega_{j_{s+1}})\big| > 1\big) \\ &\leq\ \mathbb{P}\big(\big|R_n(\omega_{j_g})\big| > 1/M^s\big) + \sum_{i=1}^{g-1} \mathbb{P}\big(\big|R_n(\omega_{j_i})\big| > 1\big) \\ &\leq\ \big(M^s + g - s - 1\big) \max_{1 \leq k \leq n} E|R_n(\omega_k)|. \end{aligned}$$

Combining all the above we get

$$\begin{aligned} \mathbb{P}\big(\big|\prod_{i=1}^{s} L_n(\omega_{j_i}) \prod_{i=s+1}^{g} R_n(\omega_{j_i})\big| > \epsilon\big) \ &\leq\ \mathbb{P}\big(\big|L_n(\omega_{j_1})\big| \geq M\epsilon\big) + \sum_{i=2}^{s} \mathbb{P}\big(\big|L_n(\omega_{j_i})\big| \geq M\big) \\ &\quad + \big(M^s + g - s - 1\big) \max_{1 \leq k \leq n} E|R_n(\omega_k)| \\ &\leq\ \frac{1}{M}(s - 1 + 1/\epsilon)4\pi \max_{\omega \in [0,2\pi]} f(\omega) \\ &\quad + \big(M^s + g - s - 1\big) \max_{1 \leq k \leq n} E|R_n(\omega_k)|. \end{aligned}$$

First term in the right side can be made smaller than $\eta/2$ by choosing M large enough and since $\max_{1 \leq k \leq n} E|R_n(\omega_k)| \to 0$ as $n \to \infty$, we can choose $N \in \mathbb{N}$ such that the second term is less than $\eta/2$ for all $n \geq N$, proving the lemma. □

For general $g \geq 2$, as before, $n' = n$, the index of each eigenvalue belongs to one of the sets $\mathcal{P}_l$ in the eigenvalue partition of $\{0, 1, 2, ..., n-1\}$ and $v_{k,n}/n \to 0$.

Hence it remains to consider only the eigenvalues corresponding to the sets $\mathcal{P}_l$ which have size exactly equal to g_1 and it follows from the argument in the proof of Lemma 3(i) of

[Bose, Mitra and Sen (2008)] that $g_1 = 2g$. We can now proceed as in $g = 2$ case. Now,

$$\frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}\big(\frac{y_j}{n^g} \le x^4\big) \sim \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}\big((2\pi)^g f_j \frac{\overline{y}_j}{n^g} \le x^4\big),$$

where

$$(2\pi)^g f_j \frac{\overline{y}_j}{n^g} = L_{n,j} + R_{n,j} \quad \text{for } 1 \le j \le |I_n|, \; L_{n,j} = \prod_{t \in \mathcal{A}_j} L_n(\omega_t).$$

Note that

$$\begin{aligned}
\frac{1}{|I_n|}\sum_{k=1}^{|I_n|}|\Phi_2 g(A_{n,j}) - \Phi_2 g(A^{\epsilon}_{n,j})| &\le \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\Big|\int_0^\infty h(t)[e^{-\frac{x^4}{2\pi^g f_j t}} - e^{-\frac{x^4+\epsilon}{2\pi^g f_j t}}]dt\Big| \\
&\le \int_0^\infty h(t)\frac{1}{|I_n|}\sum_{k=1}^{|I_n|} e^{-\frac{x^4}{2\pi^g f_j t}}\frac{\epsilon}{2\pi^2 f_j t}dt \\
&\le \frac{\epsilon}{x^4},
\end{aligned}$$

since $|e^{-\frac{x^4}{2\pi^g f_j t}}\frac{1}{2\pi^g f_j t}| \le \frac{1}{x^4}$ and $h(t)$ is the density function of product of $(g-1)$ i.i.d. exp(1). Now by previous observation and Lemma 2.4 and using normal approximation appropriately, we can show as $n \to \infty$

$$\Big|\frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}(L_{n,j} + R_{n,j} \le x^4) - \frac{1}{|I_n|}\sum_{k=1}^{|I_n|}\mathbb{P}(L_{n,j} \le x^4)\Big| \to 0.$$

Now note that for $n = k^g + 1$ we can write $\{0, 1, 2, \ldots, n-1\}$ as $\{b_1 k^{g-1} + b_2 k^{g-2} + \cdots + b_{g-1}k + b_g;\; 0 \le b_i \le k-1, \text{ for } 1 \le i \le k-1;\; 1 \le b_g \le k\}$. So we can write the sets $\mathcal{A}_j$ explicitly using this decomposition of $\{0, 1, 2, \ldots, n-1\}$ as done in $g = 2$ that is, $n = k^2 + 1$ case. For example if $g = 3$, $\mathcal{A}_j = \{b_1k^2 + b_2k + b_3,\; b_2k^2 + b_3k - b_1,\; b_3k^2 - b_1k - b_2\}$ for $j = b_1k^2 + b_2k + b_3$ (except for finitely many j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we conclude that the LSD is now $F_g(x)$. □

We now state and prove another result which also describes the LSD for k-circulant matrices but where $n = k^g - 1$, $g \ge 2$. Define

$$C_0 = \{\omega \in [0, 2\pi] : f(\omega) = 0\},$$

$$B(\omega_1, \omega_2) = \begin{pmatrix} a_1(e^{i\omega_1}) & -a_2(e^{i\omega_1}) & 0 & 0 \\ a_2(e^{i\omega_1}) & a_1(e^{i\omega_1}) & 0 & 0 \\ 0 & 0 & a_1(e^{i\omega_2}) & -a_2(e^{i\omega_2}) \\ 0 & 0 & a_2(e^{i\omega_2}) & a_1(e^{i\omega_2}) \end{pmatrix} = \begin{pmatrix} B_1 & \bigcirc \\ \bigcirc & B_2 \end{pmatrix},$$

where $a_1(e^{i\omega}) = \mathcal{R}[a(e^{i\omega})]$, $a_2(e^{i\omega}) = \mathcal{I}[a(e^{i\omega})]$, $a(e^{i\omega})$ is same as defined in Lemma 2.6 and for $z \in \mathbb{C}$, $\mathcal{R}, \mathcal{I}$ denote the real and imaginary part of z respectively. It is easy to see that

$$|a(e^{i\omega})|^2 = a_1(e^{i\omega})^2 + a_2(e^{i\omega})^2 = 2\pi f(\omega).$$

Define for $\{z_i, w_i \in \mathbb{R}\}$ and $(\omega_1, \omega_2) \in [0, 2\pi]^2$,

$$H_2(\omega_i, z_i, w_i, i = 1, 2) = \begin{cases} \mathbb{P}\big(B(\omega_1, \omega_2)N \leq \sqrt{2}(z_1, w_1, z_2, w_2)'\big) & \text{if} \quad f(\omega_1)f(\omega_2) \neq 0 \\ \mathbb{P}\big(B_1(N_1, N_2)' \leq \sqrt{2}(z_1, w_1)'\big)\mathbb{I}(z_2 \geq 0, w_2 \geq 0) & \text{if} \quad f(\omega_1) \neq 0, f(\omega_2) = 0 \\ \mathbb{P}\big(B_2(N_3, N_4)' \leq \sqrt{2}(z_2, w_2)'\big)\mathbb{I}(z_1 \geq 0, w_1 \geq 0) & \text{if} \quad f(\omega_2) \neq 0, f(\omega_1) = 0 \\ \mathbb{I}(z_i \geq 0, w_i \geq 0, i = 1, 2) & \text{if} \quad f(\omega_1), f(\omega_2) = 0. \end{cases}$$

Since $a(e^{i\omega})$ is continuous on $[0, 2\pi]$, it is easy to verify that for fixed $\{z_i, w_i, i = 1, 2\}$, H_2 is bounded continuous function in (ω_1, ω_2). Hence we may define

$$\mathcal{F}_2(z_1, w_1, z_2, w_2) = \int_0^1 \int_0^1 H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt.$$

$\mathcal{F}_2$ is a proper distribution function.

For any Borel set B, let $\lambda(B)$ denote the corresponding Lebesgue measure. It is easy to see that

(i) if $\lambda(C_0) = 0$ then $\mathcal{F}_2$ is continuous everywhere and,

(ii) if $\lambda(C_0) \neq 0$ then $\mathcal{F}_2$ is discontinuous *only* on $D_2 = \{(z_1, w_1, z_2, w_2) : \prod_{i=1}^2 z_i w_i = 0\}$.

Extending the above concepts, define for any $g \geq 2$,

$$B(\omega_1, \omega_2, .., \omega_g) = \begin{pmatrix} a_1(e^{i\omega_1}) & -a_2(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ a_2(e^{i\omega_1}) & a_1(e^{i\omega_1}) & 0 & 0 & \cdots & 0 \\ 0 & 0 & a_1(e^{i\omega_2}) & -a_2(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & a_2(e^{i\omega_2}) & a_1(e^{i\omega_2}) & \cdots & 0 \\ 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \cdots & a_1(e^{i\omega_g}) & -a_2(e^{i\omega_g}) \\ 0 & 0 & 0 & \cdots & a_2(e^{i\omega_g}) & a_1(e^{i\omega_g}) \end{pmatrix}.$$

For $\{z_i, w_i \in \mathbb{R}, i = 1, 2, \ldots, g\}$, define

$$H_g(\omega_i, z_i, w_i, i = 1, \ldots, g) = \mathbb{P}\big(B(\omega_1, \omega_2, .., \omega_g)(N_1, ..., N_{2g})' \leq (z_i, w_i, i = 1, 2, .., g)'\big).$$

where N_i i.i.d. $N(0, 1)$.

As before, we can easily see that H_g is bounded continuous in $(\omega_1, \ldots, \omega_g)$ for fixed $\{z_i, w_i, i = 1, \ldots, g\}$. Therefore we can define $\mathcal{F}_g$ as

$$\mathcal{F}_g(z_i, w_i, i = 1, \ldots, g) = \int_0^1 \cdots \int_0^1 H_g(2\pi t_i, z_i, w_i, i = 1, \ldots, g) \prod dt_i$$

and $\mathcal{F}_g$ is a proper distribution function. Again,

(i) if $\lambda(C_0) = 0$ then $\mathcal{F}_g$ is continuous everywhere and,

(ii) if $\lambda(C_0) \neq 0$ then $\mathcal{F}_g$ is discontinuous on $D_g = \{(z_1, w_1, i = 1, \ldots, g) : \prod_{i=1}^g z_i w_i = 0\}$.

Theorem 2.5. Suppose $\{\epsilon_i\}$ is an i.i.d. sequence with mean zero and variance 1 and $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$. Suppose $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Suppose $n = k^g - 1$ for some $g \geq 2$. Then as $n \to \infty$, $F_{n^{-1/2}A_{k,n}}$ converges weakly in probability to the LSD $(\prod_{i=1}^{g} G_i)^{1/g}$ such that $(Re\ G_i, Im\ G_i;\ i = 1, 2, \dots g)$ has the distribution $\mathcal{F}_g$ given by

$$\mathcal{F}_g(z_i, w_i, i = 1, .., g) = \int_0^1 \cdots \int_0^1 H_g(2\pi t_i, z_i, w_i, i = 1, \dots, g) \prod dt_i.$$

Further, if $\lambda(C_0) = 0$ then

$$\begin{aligned} &\mathcal{F}_g(z_i, w_i, i = 1, .., g) \\ = \quad &\int \cdots \int \mathbb{I}_{\{\mathbf{t} \leq (z_k, w_k, k=1,..,g)\}} \Big[\int_0^1 \cdots \int_0^1 \frac{\mathbb{I}_{\{\prod f(2\pi u_i) \neq 0\}}}{(2\pi)^g \prod_{i=1}^{g} [\pi f(2\pi u_i)]} \prod_{i=1}^{g} e^{-\frac{1}{2} \frac{t_{2i-1}^2 + t_{2i}^2}{\pi f(2\pi u_i)}} \prod du_i \Big] \prod dt_i. \end{aligned}$$

Remark 2.3. If $\inf_{\omega \in [0,2\pi]} f(\omega) > 0$, we can write $\mathcal{F}_g$ in the following form

$$\begin{aligned} &\mathcal{F}_g(z_k, w_k, k = 1, 2, \dots, g) \\ = \quad &\int \cdots \int \mathbb{I}_{\{\mathbf{t} \leq (z_k, w_k, k=1,\dots,g)\}} \Big[\int_0^1 \cdots \int_0^1 \frac{1}{(2\pi)^g \prod_{i=1}^{g} [\pi f(2\pi u_i)]} \prod_{i=1}^{g} e^{-\frac{1}{2} \frac{t_{2i-1}^2 + t_{2i}^2}{\pi f(2\pi u_i)}} \prod du_i \Big] \prod dt_i. \end{aligned}$$

Remark 2.4. If $\{x_i\}$ are i.i.d, then $f(\omega) = 1/2\pi$ for all $\omega \in [0, 2\pi]$ and the LSD simplifies to $U_2(\prod_{i=1}^{g} E_i)^{1/2g}$ where $\{E_i\}$ are i.i.d. $Exp(1)$, U_2 is uniformly distributed over the unit circle independent of $\{E_i\}$. This agrees with [Bose, Mitra and Sen (2008)].

Proof. First we assume $\lambda(C_0) = 0$. Note that $\gcd(k, n) = 1$. Since $k^g = 1 + n = 1 \mod n$, we have $g_1 | g$. If $g_1 < g$, then $g_1 \leq g/\alpha$ where $\alpha = 2$ if g is even and $\alpha = 3$ if g is odd. In either case, it is easy to check that

$$k^{g_1} \leq k^{g/\alpha} \leq (1 + n)^{1/\alpha} = o(n).$$

Hence, $g = g_1$. By Lemma 3(ii) of [Bose, Mitra and Sen (2008)] the total number of eigenvalues γ_j of $A_{k,n}$ such that $j \in \mathcal{A}_l$ and $|\mathcal{A}_l| < g$ is asymptotically negligible.

Unlike the previous theorem, here the partition sets $\mathcal{A}_l$ are not necessarily self-conjugate. However, the number of indices l such that $\mathcal{A}_l$ is self-conjugate is asymptotically negligible compared to n. To show this, we need to bound the cardinality of the following set for $1 \leq l < g$:

$$D_l = \{t \in \{1, 2, \dots, n\} : tk^l = -t \mod n\} = \{t \in \{1, 2, \dots, n\} : n | t(k^l + 1)\}.$$

Note that $t_0 = n/\gcd(n, k^l + 1)$ is the minimum element of D_l and every other element is a multiple of t_0. Thus

$$|D_l| \leq \frac{n}{t_0} \leq \gcd(n, k^l + 1).$$

Let us now estimate $\gcd(n, k^l + 1)$. For $l > [g/2]$,

$$\gcd(n, k^l + 1) \le \gcd(k^g - 1, k^l + 1) = \gcd\big(k^{g-l}(k^l + 1) - (k^{g-l} - 1), k^l + 1\big) \le k^{g-l},$$

which implies $\gcd(n, k^l + 1) \le k^{[g/2]}$ for all $1 \le l < g$. Therefore,

$$\frac{\gcd(n, k^l + 1)}{n} = \frac{k^{[g/2]}}{(k^g - 1)} \le \frac{2}{k^{[(g+1)/2]}} \le \frac{2}{((n)^{1/g})^{[(g+1)/2]}} = o(1),$$

So, we can ignore the partition sets $\mathcal{P}_j$ which are self-conjugate. For other $\mathcal{P}_j$,

$$x_j = \prod_{t \in \mathcal{P}_j} (b_{t,n} + ic_{t,n})$$

will be complex.

Now for simplicity we will provide the detailed argument assuming that $g = 2$. Then, $n = k^2 - 1$ and we can write $\{0, 1, 2, \ldots, n\}$ as $\{ak + b;\ 0 \le a \le k - 1,\ 0 \le b \le k - 1\}$ and using the construction of $S(x)$ we have $\mathcal{P}_j = \{ak + b, bk + a\}$ and $|\mathcal{P}_j| = 2$ for $j = ak + b;\ 0 \le a \le k - 1,\ 0 \le b \le k - 1$ (except for finitely many j and hence such indices do not contribute to the LSD).

Let us define

$$I_n = \{l : |\mathcal{P}_i| = 2\}.$$

It is clear that $n/|I_n| \to 2$. Without any loss, let $I_n = \{1, 2, \ldots, |I_n|\}$. Suppose $\mathcal{P}_j = \{j_1, j_2\}$. We first find the limiting distribution of the empirical distribution of $\frac{1}{\sqrt{n}}(b_{j_1,n}, c_{j_1,n}, b_{j_1,n}, c_{j_1,n})$ for those j for which $|\mathcal{P}_j| = 2$ and show the convergence in L_2. Let $F_n(x, y, z, w)$ be the ESD of $\frac{1}{\sqrt{n}}(b_{j_1,n}, c_{j_1,n}, b_{j_1,n}, c_{j_1,n})$, that is

$$F_n(x_1, y_1, x_2, y_2) = \frac{2}{n} \sum_{j=1}^{[\frac{n}{2}]} \mathbb{I}\big(\frac{1}{\sqrt{n}} b_{j_k,n} \le x_k, \frac{1}{\sqrt{n}} c_{j_k,n} \le y_k,\ k = 1, 2\big).$$

We show that for $z_1, w_1, z_2, w_2 \in \mathbb{R}$,

$$\mathbb{E}[F_n(z_1, w_1, z_2, w_2)] \to \mathcal{F}_2(z_1, w_1, z_2, w_2) \text{ and } Var[F_n(z_1, w_1, z_2, w_2)] \to 0. \tag{2.6}$$

We state the following Lemma which is similar to Lemma 2.2. We omit the proof.

Lemma 2.6. Let $x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j}$ for $t \ge 0$, where $\{\epsilon_t\}$ are i.i.d random variables with mean 0, variance 1 and $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then for $k = 1, 2, \cdots, n$,

$$\frac{1}{\sqrt{n}}(b_{j,n} + ic_{j,n}) = a(e^{i\omega_j})[\frac{1}{\sqrt{n}}(\bar{b}_{j,n} + i\bar{c}_{j,n})] + Y_n(\omega_j),$$

where $a(e^{i\omega_j}) = \sum_{l=-\infty}^{\infty} a_l e^{i\omega_j l}$ and $\max_{0 \le j < n} E|Y_n(\omega_j)|^2 \to 0$ as $n \to \infty$.

Define for $k = 1, 2, \cdots, n$,

$$\eta_j = \frac{1}{\sqrt{n}}(\bar{b}_{j_1,n}, \bar{c}_{j_1,n}, \bar{b}_{j_2,n}, \bar{c}_{j_2,n})',$$

and let $Y_n(\omega_j) = Y_{1n}(\omega_j) + iY_{2n}(\omega_j)$, $\; a(e^{i\omega_j}) = a_1(e^{i\omega_j}) + ia_2(e^{i\omega_j})$, where $a(e^{i\omega_j}), Y_n(\omega_j)$ are same as defined in Lemma 2.6 and $a_1(e^{i\omega_j}), a_2(e^{i\omega_j}), Y_{1n}(\omega_j), Y_{2n}(\omega_j)$ are all real valued. Define $Y_{n,j} = \big(Y_{1n}(\omega_{j_1}), Y_{2n}(\omega_{j_1}), Y_{1n}(\omega_{j_2}), Y_{2n}(\omega_{j_2})\big)$, $N = \big(N_1, N_2, N_3, N_4\big)'$ and

$$B_j = \begin{pmatrix} a_1(e^{i\omega_{j_1}}) & -a_2(e^{i\omega_{j_1}}) & 0 & 0 \\ a_2(e^{i\omega_{j_1}}) & a_1(e^{i\omega_{j_1}}) & 0 & 0 \\ 0 & 0 & a_1(e^{i\omega_{j_2}}) & -a_2(e^{i\omega_{j_2}}) \\ 0 & 0 & a_2(e^{i\omega_{j_2}}) & a_1(e^{i\omega_{j_2}}) \end{pmatrix},$$

where $\{N_i\}$ are i.i.d. $N(0, 1)$. Denote

$$|Y_{n,j}| = \big(|Y_{1n}(\omega_{j_1})|, |Y_{2n}(\omega_{j_1})|, |Y_{1n}(\omega_{j_2})|, |Y_{2n}(\omega_{j_2})|\big)$$

and

$$\big(|Y_{n,j}| \le \epsilon\big) = \big(|Y_{1n}(\omega_{j_1})| \le \epsilon, |Y_{2n}(\omega_{j_1})| \le \epsilon, |Y_{1n}(\omega_{j_2})| \le \epsilon, |Y_{2n}(\omega_{j_2})| \le \epsilon\big).$$

Then $\frac{1}{\sqrt{n}}(b_{j_1,n}, c_{j_1,n}, b_{j_1,n}, c_{j_1,n}) = B_j\eta_j + Y'_{n,j}$. From Lemma 2.6, it is intuitively clear that for large n, $[b_{j,n} + ic_{j,n}] \sim a(e^{i\omega_j})[\bar{b}_{j,n} + i\bar{c}_{j,n}]$. So first we show that for large n

$$\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}\big(\frac{1}{\sqrt{n}}b_{j_k,n} \le z_k, \frac{1}{\sqrt{n}}c_{j_k,n} \le w_k,\ k = 1, 2\big) \sim \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}P(B_j\eta_j \le (z_1, w_1, z_2, w_2)').$$

Note

$$\begin{aligned} & \Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}\big(\frac{1}{\sqrt{n}}b_{j_k,n} \le z_k, \frac{1}{\sqrt{n}}c_{j_k,n} \le w_k,\ k = 1, 2\big) - \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j \le (z_1, w_1, z_2, w_2)')\Big| \\ = & \Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j + Y'_{n,j} \le (z_1, w_1, z_2, w_2)') - \mathbb{P}(B_j\eta_j \le (z_1, w_1, z_2, w_2)')\Big| \\ \le & \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}((|Y_{1n}(\omega_{j_1})|, |Y_{2n}(\omega_{j_1})|, |Y_{1n}(\omega_{j_2})|, |Y_{2n}(\omega_{j_2})|) \ge (\epsilon, \epsilon, \epsilon, \epsilon)) \\ + & \Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j + Y'_{n,j} \le (z_1, w_1, z_2, w_2)', |Y_{n,j}| \le \epsilon) - P(B_j\eta_j \le (z_1, w_1, z_2, w_2)')\Big| \\ = & T_1 + T_2, \text{ say.} \end{aligned}$$

Now using Lemma 2.6, as $n \to \infty$

$$T_1 \le \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(|Y_n(\omega_{j_1})|^2 > 2\epsilon^2) \le \frac{1}{2\epsilon^2}\sup_j E|Y_n(\omega_{j_1})|^2 \to 0.$$

$$T_2 \leq \max\Big\{\Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j + Y'_{n,j} \leq (z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon)') - \mathbb{P}(B_j\eta_j \leq (z_1,w_1,z_2,w_2)')\Big|,$$
$$\Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j \leq (z_1-\epsilon, w_1-\epsilon, z_2-\epsilon, w_2-\epsilon)') - \mathbb{P}(B_j\eta_j \leq (z_1,w_1,z_2,w_2)')\Big|\Big\}$$

and

$$\Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j + Y'_{n,j} \leq (z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon)') - \mathbb{P}(B_j\eta_j \leq (z_1,w_1,z_2,w_2)')\Big|$$
$$\leq T_3 + T_4 + T_5.$$

where

$$T_3 = \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\big|\mathbb{P}(B_j\eta_j \leq (z_1,w_1,z_2,w_2)') - \mathbb{P}(B_jN \leq (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)')\big|,$$

$$T_4 = \frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\big|\mathbb{P}(B_j\eta_j \leq (z_k+\epsilon, w_k+\epsilon,\ k=1,2)') - \mathbb{P}(B_jN \leq (\sqrt{2}z_k+\sqrt{2}\epsilon, \sqrt{2}w_k+\sqrt{2}\epsilon,\ k=1,2)')\big|,$$

$$T_5 = \Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_jN \leq (\sqrt{2}z_k+\sqrt{2}\epsilon, \sqrt{2}w_k+\sqrt{2}\epsilon,\ k=1,2)') - \mathbb{P}(B_jN \leq (\sqrt{2}z_k, \sqrt{2}w_k,\ k=1,2)')\Big|.$$

To show $T_3, T_4 \to 0$, define

$$X_{l,k} = 2^{1/2}\left(\epsilon_l \cos\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_1 l}{n}\right), \epsilon_l \cos\left(\frac{2\pi j_2 l}{n}\right), \epsilon_l \sin\left(\frac{2\pi j_2 l}{n}\right)\right)'.$$

Note that

$$E(X_{l,j}) = 0 \ \ \forall \ \ l,\ j,\ n. \tag{2.7}$$

$$n^{-1}\sum_{l=0}^{n-1} Cov(X_{l,j}) = I \ \ \forall \ \ j,\ n. \tag{2.8}$$

Note that for

$$\{B_j\eta_j \leq (z_1,w_1,z_2,w_2)'\} = \{B_j(n^{-1/2}\sum_{l=0}^{n-1}X_{l,j}) \leq (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\}.$$

Since $\{(r_1,r_2,r_3,r_4) : B_j(r_1,r_2,r_3,r_4)' \leq (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)'\}$ is a convex set in $\mathbb{R}^2$ and $\{X_{l,k},\ l = 0,1,\ldots(n-1)\}$ satisfies (2.7) and (2.8), we can show using Lemma 1.4

[Bose and Saha (2008)(a)] that $T_3,\ T_4 \to 0$ as $n \to \infty$ and hence

$$
\begin{aligned}
& \lim_{n\to\infty} \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} P(B_j N \leq (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} P(B_j N \leq (\sqrt{2}z_1, \sqrt{2}w_1, \sqrt{2}z_2, \sqrt{2}w_2)') \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H(\omega_{j_1}, \omega_{j_2}, z_1, w_1, z_2, w_2) \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{[\sqrt{n}]} \sum_{b=1}^{[\sqrt{n}]} H\big(\frac{2\pi(ak+b)}{n}, \frac{2\pi(bk+a)}{n}, z_1, w_1, z_2, w_2\big) \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{a=0}^{[\sqrt{n}]} \sum_{b=1}^{[\sqrt{n}]} H\big(\frac{2\pi a}{\sqrt{n}}, \frac{2\pi b}{\sqrt{n}}, z_1, w_1, z_2, w_2\big) \\
= & \int_0^1 \int_0^1 H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \lim_{n\to\infty} T_5 \\
= & \Big| \int_0^1 \int_0^1 H(2\pi s, 2\pi t, z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon) ds\ dt - \int_0^1 \int_0^1 H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt \Big| \\
\leq & \int_0^1 \int_0^1 \big| H(2\pi s, 2\pi t, z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon) - H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) \big| ds\ dt.
\end{aligned}
$$

Note that

$$\big| H(2\pi s, 2\pi t, z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon) - H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) \big| \leq 2$$

and for fixed $\{z_k, w_k; k = 1, 2\}$ as $\epsilon \to 0$,

$$\big| H(2\pi s, 2\pi t, z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon) - H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) \big| \to 0. \tag{2.9}$$

Hence by DCT $\lim_{\epsilon\to 0} \lim_{n\to\infty} T_5 = 0$ and

$$\lim_{\epsilon\to 0} \lim_{n\to\infty} \Big| \frac{1}{|I_n|} \sum_{j=1}^{|I_n|} \mathbb{P}(B_j\eta_j + Y'_{n,j} \leq (z_1+\epsilon, w_1+\epsilon, z_2+\epsilon, w_2+\epsilon)') - \mathbb{P}(B_j\eta_j \leq (z_1, w_1, z_2, w_2)') \Big| = 0.$$

Now note that for fixed $\{z_k, w_k; k = 1, 2\}$ as $\epsilon \to 0$,

$$\big| H(2\pi s, 2\pi t, z_1-\epsilon, w_1-\epsilon, z_2-\epsilon, w_2-\epsilon) - H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) \big| \to 0, \tag{2.10}$$

outside the measure zero set $\{(2\pi s, 2\pi t) : f(2\pi s) = 0 \text{ or } f(2\pi t) = 0\}$. Using this fact, proceeding as above we can show that

$$\lim_{\epsilon\to 0}\lim_{n\to\infty}\Big|\frac{1}{|I_n|}\sum_{j=1}^{|I_n|}\mathbb{P}(B_j\eta_j+Y'_{n,j} \leq (z_1-\epsilon, w_1-\epsilon, z_2-\epsilon, w_2-\epsilon)')-\mathbb{P}(B_j\eta_j \leq (z_1, w_1, z_2, w_2)')\Big| = 0$$

and hence $\lim_{\epsilon\to 0}\lim_{n\to\infty} T_2 \to 0$. Therefore as $n \to \infty$,

$$\begin{aligned}\mathbb{E}[F_n(z_1, w_1, z_2, w_2)] &\sim \frac{1}{|I_n|}\sum_{j=1}^{|I_n|} P(B_j\eta_j \leq (z_1, w_1, z_2, w_2)') \\ &\to \int_0^1\int_0^1 H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt\end{aligned}$$

and since $\lambda(C_0) = 0$, we have

$$\begin{aligned}&\int_0^1\int_0^1 H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt \\ =\ &\int_0^1\int_0^1 \mathbb{I}_{\{f(2\pi s)f(2\pi t)\neq 0\}} H(2\pi s, 2\pi t, z_1, w_1, z_2, w_2) ds\ dt \\ =\ &\int_0^1\int_0^1 \mathbb{I}_{\{f(2\pi s)f(2\pi t)\neq 0\}} \iiiint \mathbb{I}_{\{B_j\mathbf{u}'\leq(\sqrt{2}z_1,\sqrt{2}w_1,\sqrt{2}z_2,\sqrt{2}w_2)'\}}\frac{1}{(2\pi)^2}e^{-\frac{\sum_{i=1}^4 u_i^2}{2}}\prod du_i\ dsdt \\ =\ &\int_0^1\int_0^1 \mathbb{I}_{\{f(2\pi s)f(2\pi t)\neq 0\}} \iiiint \mathbb{I}_{\{\mathbf{v}\leq(z_1,w_1,z_2,w_2)\}}\frac{e^{-\frac{1}{2}\left[\frac{\sum_{i=1}^2 v_i^2}{\pi f(2\pi s)}+\frac{\sum_{i=3}^4 v_i^2}{\pi f(2\pi t)}\right]}}{(2\pi)^2\pi f(2\pi u)\pi f(2\pi v)}\prod dv_i\ dsdt \\ =\ &\iiiint \mathbb{I}_{\{\mathbf{v}\leq(z_1,w_1,z_2,w_2)\}}\Big[\int_0^1\int_0^1 \mathbb{I}_{\{f(2\pi s)f(2\pi t)\neq 0\}}\frac{e^{-\frac{1}{2}\left[\frac{\sum_{i=1}^2 v_i^2}{\pi f(2\pi s)}+\frac{\sum_{i=3}^4 v_i^2}{\pi f(2\pi t)}\right]}}{(2\pi)^2\pi f(2\pi u)\pi f(2\pi v)}dsdt\Big]\prod dv_i \\ =\ &\mathcal{F}_2(z_i, w_i, i = 1, 2).\end{aligned}$$

Similarly we can show $Var(F_n) \to 0$ as $n \to \infty$.
Hence the empirical distrubution of y_j for those j for which $|\mathcal{P}_j| = 2$ converges to the distribution of $\prod_{i=1}^2 G_i$ such that $(Re\ G_i, Im\ G_i;\ i = 1, 2)$ has distribution $\mathcal{F}_2$ in distribution. Hence the LSD of $n^{-1/2}A_{k,n}$ is $\left(\prod_{i=1}^2 G_i\right)^{1/2}$, proving the result when $g = 2$ and $\lambda(C_0) = 0$.

When $\lambda(C_0) \neq 0$, we have to show (2.6) only at the continuity points of $\mathcal{F}_2$. Recall that $\mathcal{F}_2$ is continuous on complement of D_2. All the above steps in the proof will go through for all $\{z_i, w_i; i = 1, 2\}$ except (2.9),(2.10) but on complement of D_2 (2.9),(2.10) also holds. Hence if $\lambda(C_0) \neq 0$, we have our required LSD. This proves the Theorem when $g = 2$.

For general $g > 2$, note that we can write $\{0, 1, 2, \ldots, n\}$ as $\{b_1k^{g-1} + b_2k^{g-2} + \cdots + b_{g-1}k + b_g;\ 0 \leq b_i \leq k-1,\ \text{for } 1 \leq i \leq k\}$. So we can write the sets $\mathcal{A}_j$ explicitly using this decomposition of $\{0, 1, 2, \ldots, n\}$ as done in $n = k^2 - 1$ case. For example if $g = 3$, $\mathcal{A}_j = \{b_1k^2+b_2k+b_3,\ b_2k^2+b_3k+b_1,\ b_3k^2+b_1k+b_2\}$ for $j = b_1k^2+b_2k+b_3$ (except for finitely many

j, bounded by $v_{k,n}$ and they do not contribute to this limit). Using this fact and proceeding as before we will have the LSD as $\left(\prod_{i=1}^{g} G_i\right)^{1/g}$ such that $(Re\ G_i, Im\ G_i;\ i = 1, 2, \ldots g)$ has distribution $\mathcal{F}_g$. □

References

[Bai(1999)] Z. D. Bai. Methodologies in spectral analysis of large-dimensional random matrices, a review. *Statist. Sinica*, 9(3):611–677, 1999. ISSN 1017-0405. With comments by G. J. Rodgers and Jack W. Silverstein; and a rejoinder by the author.

[Bhattacharya and Ranga Rao (1976)] R. N. Bhattacharya and R. Ranga Rao. *Normal approximation and asymptotic expansions*. John Wiley & Sons, New York-London-Sydney, 1976. Wiley Series in Probability and Mathematical Statistics.

[Bose and Mitra (2002)] Arup Bose and Joydip Mitra. Limiting spectral distribution of a special circulant. *Statist. Probab. Lett.*, 60(1):111–120, 2002. ISSN 0167-7152.

[Bose and Sen (2008)] Arup Bose and Arnab Sen. Another look at the moment method for large dimensional random matrices. *Electron. J. Probab.*, 13:no. 21, 588–628, 2008. ISSN 1083-6489.

[Bose, Mitra and Sen (2008)] Arup Bose, Joydip Mitra and Arnab Sen. Large dimensional random k-circulants. *Technical Report No.R10/2008, Stat-Math Unit, Indian Statistical Institute, Kolkata.*

[Bose and Saha (2008)(a)] Arup Bose and Koushik Saha. Limiting spectral distribution of reverse circulant matrix with dependent entries. *Technical Report No.R9/2008, Stat-Math Unit, Indian Statistical Institute, Kolkata*.

[Bose and Saha (2008)(b)] Arup Bose and Koushik Saha. Limiting spectral distribution of circulant matrix with dependent entries. *Technical Report No.R11/2008, Stat-Math Unit, Indian Statistical Institute, Kolkata.*

[Brockwell and Davis (2002)] Peter J. Brockwell and Richard A. Davis. *Introduction to time series and forecasting*. Springer Texts in Statistics. Springer-Verlag, New York, second edition, 2002. ISBN 0-387-95351-5.

[Fan and Yao (2003)] Jianqing Fan and Qiwei Yao. *Nonlinear time series*. Springer Series in Statistics. Springer-Verlag, New York, 2003. ISBN 0-387-95170-9. Nonparametric and parametric methods.

[Zhou (1996)] Zhou, Jin Tu (1996). A formula solution for the eigenvalues of g circulant matrices. *Math. Appl. (Wuhan)*, 9, No. 1, 53-57.

(Arup Bose) Stat-Math Unit, Indian Statistical Institute, 203 B. T. Rd., Calcutta 700108, India, E-mail: abose@isical.ac.in, bosearu@gmail.com

(Koushik Saha) Stat-Math Unit, Indian Statistical Institute, 203 B. T. Rd., Calcutta 700108, India, E-mail: koushik_r@isical.ac.in