FINITE DIAGONAL RANDOM MATRICES

ARUP BOSE [*]
Statistics and Mathematics Unit
Indian Statistical Institute
INDIA
bosearu@gmail.com

SANCHAYAN SEN [†]
Courant Institute of Mathematical Sciences
USA
sen@cims.nyu.edu

March 14, 2011

Abstract

The goal of this article is to extend some results of Popescu (2009, Prob. Theory. Rel. Fields) in several directions. We establish the limiting spectral distribution for r-diagonal matrices under reduced moment conditions compared to those required by Popescu. We also deal with the joint convergence of several sequences of such matrices. In particular, we show that there is a large class of such matrices where the joint limit is not free while the marginals are semicircular. We also consider matrices of the form $X_nX_n^T$ where X_n is a sequence of nonsymmetric r-diagonal random matrices and establish their LSD.

AMS 2010 Subject Classification Primary 60B20, Secondary 60B10, 46L53, 46L54.

1 Introduction

Suppose λ_1, λ_2,...., λ_n are the eigenvalues of a real symmetric matrix $M_{n\times n}$ then its *Empirical Spectral Distribution (ESD)* is defined as,

$$F^{M_n}(x,y) = n^{-1}\sum_{i=1}^{n}\mathbb{I}(\lambda_i \leq\ x). \tag{1.1}$$

The *Limiting Spectral Distribution (or measure) (LSD)* of the sequence is defined as the weak limit of the sequence $\{F^{M_n}\}$, if it exists, either almost surely or in probability.

LSD of various random matrices have been studied in the literature. See for example Bose, Hazra and Saha (2010) for a list of common symmetric patterned matrices and their LSD. More examples of LSD may be found in Bai (1997).

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[*] Research supported by J.C.Bose National Fellowship, Dept. of Science and Technology, Govt. of India. Part of the work was done while visiting Dept. of Economics, University of Cincinnati.

[†] Supported by NYU graduate fellowship under Henry M. MacCracken Program.

Tridiagonal matrix models are heavily used in numerical recipes. Popescu (2009) discussed an interesting class of symmetric tridiagonal matrix models under some nonstandard conditions on the entries. He proved the convergence of the ESD and computed the limiting distributions in some particular cases. He also obtained the limit of the trace of monomials formed by several such independent matrices. Extensions were also provided for the r-diagonal model where the k-th superdiagonals and subdiagonals are zero for $k > r$.

The goal of this article is to extend some of Popescu's results in several directions. First, we relax some of the moment conditions that are required by Popescu. We also deal with the joint convergence of sequences of such matrices. In particular, we show that the joint limit need not be free while the marginals may still be semicircular. Matrices of the form $X_n X_n^T$ are generalizations of the sample variance covariance matrix. We also consider such matrices where X_n is a sequence of nonsymmetric r-diagonal random matrices and establish their LSD.

2 Main results

2.1 Popescu's result and an extension

We first restate Popescu's main result in a convenient manner and then state an extension. Let $\{b_{ij}\}$ be a sequence of random variables. Further conditions on this sequence shall be imposed as needed. Consider for any fixed r, the sequence of r-diagonal symmetric random matrices $\{A_n\}$ given by

$$(A_n)_{ij} = \begin{cases} b_{j-i,i}, & \text{if } i \le j \le r+i \\ 0, & \text{if } r+i < j \le n \\ (A_n)_{ji}, & \text{if } j < i \end{cases} \tag{2.1}$$

and let for some $\alpha > 0$,

$$X_n = \frac{A_n}{n^\alpha}. \tag{2.2}$$

Let

$$\begin{aligned} \Gamma_k &= \{\gamma = (j_0, \ldots, j_k) | j_0 = j_k, \ \ \max \ \gamma_j = 0, |j_{i+1} - j_i| \le r\} \\ l_{i,s}(\gamma) &= \#\{u : 0 \le u \le k-1 \ \text{ such that either } \ j_u = i, j_{u+1} = s \text{ or } j_u = s, j_{u+1} = i\}. \end{aligned}$$

Theorem 1. *(Popescu(2006)). Suppose* $\{b_{i,j}\}$, $j \ge 0, r \ge i \ge 0$ *are independent and for some* $\alpha > 0$,

$$\lim_{n\to\infty} \mathbf{E} \left(\frac{b_{j,n}}{n^\alpha} \right)^k = m_{jk}, \ \forall \ k = 1, 2, \ldots.$$

(a) Then as $n \to \infty$, *for all* $k = 1, 2, \ldots$,

$$\mathbf{E}\frac{Tr(X_n^k)}{n} \to L_k \quad \textit{and} \quad \frac{Tr(X_n^k)}{n} \to L_k, \quad \textit{almost surely,}$$

where

$$L_k = \frac{1}{\alpha k + 1} \sum_{\gamma \in \Gamma_k} \prod_{0 \le i \le r, \ j \le 0} m_{i, l_{j,j+i}(\gamma)}. \tag{2.3}$$

(b) Further if for every $0 \leq j \leq r$, $\{m_{jk}\}_{k \geq 0}$ are the moments of a compactly supported measure μ_j, then there exists a unique measure μ, which is compactly supported and has moments $\{L_k\}_{k \geq 0}$. Hence $F^{X_n} \longrightarrow \mu$ almost surely.

(c) If further $m_{jk} = a_j^k$, $\forall\ k \geq 0, 0 \leq j \leq r$, then μ is the distribution of

$U_1^{\alpha}[a_0 + 2\sum_{k=1}^{r} a_k \cos(2\pi k U_2)]$ where U_1 and U_2 are i.i.d. uniformly distributed on $[0,1]$.

Note that in the above result, convergence of the ESD is guaranteed if $\{L_k\}$ determines a distribution uniquely. The compact support assumption on $\{\mu_j\}$ achieves this. To state our extension of this result, let $\{a_{in}\}$, $0 \leq i \leq r, n \geq 1$ be a sequence of real numbers and let $\{x_{in}\}$, $0 \leq i \leq r, n \geq 1$ be a sequence of independent random variables.

Theorem 2. *Let A_n and X_n be as in (2.1) and (2.2) with $b_{in} = a_{in} x_{in}$.*

(a) Suppose for some $\alpha > 0$, and for $0 \leq i \leq r$,

$$\frac{a_{in}}{n^{\alpha}} \overset{n\to\infty}{\longrightarrow} a_i. \tag{2.4}$$

Suppose that either (a1) for each $0 \leq i \leq r$, $\{x_{in}\}_{n \geq 1}$ is i.i.d with finite second moment or (a2) there exists a sequence of positive numbers $\{c_n\}$ increasing to infinity such that:

(i) $\displaystyle\sum_{j=1}^{\infty} \frac{\mathbb{E}(x_{ij}^4)}{j^2} < \infty$ *for* $0 \leq i \leq r$,

(ii) $\lim_n \mathbb{E}(x_{in}\mathbf{1}(|x_{in}| < c_l))^k$ *exists for each $k, l \geq 1$ and $0 \leq i \leq r$,*

(iii) $\displaystyle\lim_{l\to\infty} \limsup_{n\to\infty} \frac{\sum_{j=1}^{n} \mathbb{E}(x_{ij}^2 \mathbf{1}(|x_{ij}| \geq c_l))}{n} = 0$ *for* $0 \leq i \leq r$.

Then almost surely, X_n has a nonrandom limiting spectral distribution, say γ.

(b) Suppose $\mathbb{E}(|x_{in}^k|) < \infty$, $\forall\ i, n, k$ and $\lim_n \mathbb{E}(n^{-\alpha} b_{in})^k = a_i^k \lim_n \mathbb{E}(x_{in})^k$ exists and equals m_{ik}. Then L_k in (2.3) are the moments of γ.
Further if for some $0 \leq i_0 \leq r$, $\{m_{i_0 k}\}_{k \geq 1}$ are moments of some distribution having noncompact support and either m_{ik} are nonnegative $\forall i, k$ or $m_{ik} = 0$ whenever $i \neq 0, i_0$ (i.e. all the diagonals except the main diagonal and the i_0-th diagonal are "switched off"), then γ also has noncompact support.

Remark 1.

(a) Note that condition (ii) in part (b) holds whenever there exists random variables x_i for $0 \leq i \leq r$ such that $x_{in} \overset{w}{\to} x_i$ as $n \to \infty$.

(b) In Theorem 2 we have managed to remove the compactness assumption of Theorem 1 in exchange for some other conditions on the entries. As a consequence the LSD can have noncompact support. For instance, if for at least one j, the moment sequences $\{m_{jk}\}$ corresponds to a measure μ_j with noncompact support then the LSD also has noncompact support.

2.2 r-diagonal covariance type matrix

Suppose that A_n is a sequence of nonsymmetric square matrices and consider the symmetric matrix $A_nA_n^T$. If A_n is the i.i.d. matrix (all entries are i.i.d.), then $n^{-1}A_nA_n^T$ is the well studied sample variance covariance matrix; it is known that under appropriate conditions $F^{n^{-1}A_nA_n^T}$ converges to the Marčenko-Pastur law. Moreover the LSD of the symmetric version of $n^{-1/2}A_n$ is the semicircular law and distribution of the square of a semicircle random variable is the Marčenko-Pastur law. As observed in Bose, Gangopadhyay and Sen (2010), this phenomenon of squaring may or may not hold for general patterned matrices.

Now let A_n be the nonsymmetric square r-diagonal matrix defined as

$$(A_n)_{ij} = \begin{cases} b_{j-i,i}, & \text{if } |j-i| \le r \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Let

$$S_n = \frac{A_nA_n^T}{n^{2\alpha}}. \tag{2.6}$$

Then under suitable conditions, F^{S_n} converges almost surely (see Theorem 3). The moments of the LSD are equal to the moments of a trigonometric polynomial of a uniform random variable. Further, under suitable conditions, the squaring relation mentioned above holds. See Corollary 3.1.

We need the following notation to state our next theorem. Define

$$l_{i\to s,R}(\gamma) = \#\{u : 1 \le u \le k, u \text{ is odd } j_{u-1} = i, j_u = s\}, \text{ and similarly}$$

$$l_{i\to s,B}(\gamma) = \#\{u : 1 \le u \le k, u \text{ is even } j_{u-1} = i, j_u = s\}.$$

So we think of this as if every path γ is being colored using colors R and B, the segment (j_{u-1}, j_u) is colored R if u is odd and is colored B if u is even.

Theorem 3. *(a) Let r be a positive integer, $\alpha > 0$ and let $\{b_{i,j}\}$ be independent such that for some $\alpha > 0$,*

$$\lim_{n\to\infty} \mathbf{E}\left(\frac{b_{j,n}}{n^\alpha}\right)^k = m'_{jk}, \ \forall\ k \ge 0,\ |j| \le r.$$

Then as $n \to \infty$,

$$\mathbf{E}\frac{Tr(S_n^k)}{n} \to L'_k \quad \textit{and} \quad \frac{Tr(S_n^k)}{n} \to L'_k, \quad \textit{almost surely,}$$

where

$$L'_k = \frac{1}{2\alpha k + 1} \sum_{\gamma\in\Gamma_{2k}} \prod_{|h|\le r,\ i\le 0} m'_{h,l_{i\to i+h,R}(\gamma)+l_{i+h\to i,B}(\gamma)}. \tag{2.7}$$

(b) If $\{m'_{jk}\}_{k\ge0}$ are the moments of compactly supported measures μ'_j, then there exists a compactly supported measure μ' whose moments are $\{L'_k\}_{k\ge0}$. Hence $F^{S_n} \to \mu'$ almost surely.

(c) If $m'_{jk} = a_j^k, \ \forall\ k \geq 0, |j| \leq r,$ *then* μ' *is the distribution of* $U_1^{2\alpha}[c_0 + 2\sum_{k=1}^{r} c_k \cos(2\pi U_2)]$ *where* U_1 *and* U_2 *are i.i.d. uniformly distributed on* $[0,1]$ *and*

$$c_h = 2\sum_{i=-r}^{r-h} a_i a_{i+h} \quad \text{if} \quad 1 \leq h \leq 2r \quad \text{and} \quad c_0 = \sum_{|i|\leq r} a_i^2.$$

From this the following corollary follows.

Corollary 3.1. *If* $m'_{jk} = a_{|j|}^k, \ \forall\ k \geq 0, |j| \leq r,$ *then* $L'_k = L_{2k}, \ \forall\ k \geq 0,$ *where* $\{L_k\}$ *are as in Theorem 1 obtained from* $m_{jk} = a_j^k, 0 \leq j \leq r, k \leq 0$ *and hence if the random variable* Z *is distributed as* μ*, then* Z^2 *is distributed as* μ'*.*

Remark 2.

(a) It is clear from the expressions for the moments that in general we cannot say that μ' *is the "folded" distribution corresponding to* μ*.*

(b) In Theorem 1 (resp. Theorem 3) we can set $m_{jk} = 0, \ \forall\ k \geq 0$ *(resp.* $m'_{jk} = 0, \ \forall\ k \geq 0$*) to "switch off" the* j*-th diagonal and we can get different patterns in the matrix models. The corresponding LSDs will be given by the moments* $\{L_k\}$ *(resp.* $\{L'_k\}$*) by plugging in zeros in (2.3) (resp. (2.7)) in place of* m_{jk} *(resp.* m'_{jk}*) in the paths* $\gamma \in \Gamma_k$ *(resp.* Γ_{2k}*).*

We can relax the assumption of existence of all moments as we did in Theorem 2. This is done in the following

Corollary 3.2. *Let* A_n *and* S_n *be as in (2.5) and (2.6) with* $b_{in} = a_{in}x_{in}$*.*

(a) Suppose for some $\alpha > 0$*, (2.4) holds for* $|i| \leq r$*, and either (a1) of Theorem 2 holds for* $|i| \leq r$ *or conditions (i), (ii) and (iii) of (a2) of Theorem 2 hold for* $|i| \leq r$*.*
Then almost surely, X_n *has a nonrandom limiting spectral distribution, say* γ'*.*

(b) Suppose $\mathbb{E}(|x_{in}^k|) < \infty, \ \forall\ i, n, k$ *and* $\lim_n \mathbb{E}(n^{-\alpha} b_{in})^k = a_i^k \lim_n \mathbb{E}(x_{in})^k$ *exists and equals* m_{ik}*. Then* L'_k *in (2.7) are the moments of* γ'*.*
Further if for some $0 \leq i_0 \leq r$*,* $\{m_{i_0 k}\}_{k\geq 1}$ *are moments of some distribution having noncompact support and* m_{ik} *are nonnegative* $\forall i, k$ *then* γ' *also has noncompact support.*

2.3 Joint convergence

Suppose now that we have J independent sequences of r-diagonal random matrices. The appropriate notion of joint convergence now is the convergence of the trace of all monomials formed from this sequence. We make this precise in the following.

A *noncommutative probability space* is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is an algebra over $\mathbb{C}$ having a unit and $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\phi(1) = 1$. Elements of the algebra $\mathcal{A}$ are called noncommutative random variables.

Let $\mathcal{A}$ be the space of $n \times n$ complex random matrices with entries being random variables defined on a fixed probability space and having all moments finite. We can define two functionals on $\mathcal{A}$ as

$$\phi_1(a) = \frac{1}{n} Tr(A) \quad \text{and} \quad \phi_2(a) = \frac{1}{n} \mathbb{E}(Tr(A)).$$

Then $(\mathcal{A}, \phi_i)$, $i = 1, 2$ are noncommutative probability spaces.

Let $\{a_i\}_{i \in I}$ be a set of elements in $(\mathcal{A}, \phi)$. Denote by $\mathbb{C}[\{X_i\}_{i\in I}]$ the polynomial algebra in the noncommutative indeterminates $\{X_i\}_{i\in I}$. The *joint law or distribution* of $\{a_i\}_{i\in I}$ is given by the linear functional

$$\begin{aligned} \mu: \ & \mathbb{C}[\{X_i\}_{i\in I}] \to \mathbb{C} \\ & p \mapsto \phi(p\{a_i\}). \end{aligned}$$

Let $(\mathcal{A}_n, \phi_n)_{n\geq 1}$ and $(\mathcal{A}, \phi)$ be noncommutative probability spaces and let $\{a_{i,n}\}_{i\in I} \in \mathcal{A}_n$ for each n, $\{a_i\}_{i\in I} \in \mathcal{A}$. Then $\{a_{i,n}\}_{i\in I}$ *converges in law (or in distribution)* to $\{a_i\}_{i\in I}$ if $\forall p \in \mathbb{C}[\{X_i\}_{i\in I}]$,

$$\lim_n \mu_{\{a_{i,n}\}_{i\in I}}(p) = \mu_{\{a_i\}_{i\in I}}(p).$$

To state our result on joint convergence, we first generalize the notation used in the proof of Theorem 3. As before, our generic notation for a path $(j_0, \ldots, j_l)$ is γ. For a sequence of "colors" $c = \{1, \ldots, l\}$, let

$$\begin{aligned} \Gamma_l^c &= \{\gamma \in \Gamma_l | \ \forall\ u \ \text{ the segment } (j_{u-1}, j_u) \text{ is colored by } u\} \text{ and for } \gamma \in \Gamma_l^c \text{ let} \\ l_{j\to j+s;R,t}(\gamma) &= \#\{u | (j_{u-1}, j_u) = (j, j+s), \ u \text{ is odd and } (j_{u-1}, j_u) \text{ is a } t \text{ colored segment}\}, \\ l_{j\to j+s;B,t}(\gamma) &= \#\{u | (j_{u-1}, j_u) = (j, j+s), \ u \text{ is even and } (j_{u-1}, j_u) \text{ is a } t \text{ colored segment}\} \text{ and} \\ l_{j,j+s;t}(\gamma) &= \#\{u | \text{ either } (j_u, j_{u+1}) = (j, j+s) \text{ or} \\ & \qquad (j_u, j_{u+1}) = (j+s, j) \text{ and } (j_u, j_{u+1}) \text{ is a } t \text{ colored segment}\}. \end{aligned}$$

Let $\{b_{t,s}^{(i)}\}$ be random variables and define the following matrix sequences,

$$(A^{(n,i)})_{st} = \begin{cases} b_{t-s,s}^{(i)}, & \text{if } s \leq t \leq r+s \\ 0, & \text{if } r+s < t \leq n \\ (A^{(n,i)})_{ts}, & \text{if } t < s, \end{cases} \qquad (\overline{A}^{(n,i)})_{st} = \begin{cases} b_{t-s,s}^{(i)}, & \text{if } |s-t| \leq r \\ 0, & \text{otherwise} \end{cases}$$

$$X^{(n,i)} = \frac{A^{(n,i)}}{n^{\alpha_i}}. \qquad S^{(n,i)} = \frac{\overline{A}^{(n,i)} \overline{A}^{(n,i)t}}{n^{2\alpha_i}}.$$

Theorem 4. *Let r, m be positive integers, $\alpha_i > 0$ for $1 \leq i \leq m$ and let $\{b_{i,j}^{(s)}\}$, be independent such that for some $\{\alpha_i\}$,*

$$\lim_{n\to\infty} \mathbf{E} \left(\frac{b_{j,n}^{(i)}}{n^{\alpha_i}} \right)^k = m_{jk}^{(i)}, \ \forall\ k \geq 0, -r \leq j \leq r, 1 \leq i \leq m.$$

(a) The joint law of $\{X^{(n,i)}\}_{1\leq i\leq m}$ converges almost surely and in expectation to the joint law of $\{a_i\}_{1\leq i\leq m}$ whose joint distribution is given by

$$\phi(a_{i_1} \ldots a_{i_l}) = \frac{1}{\alpha_{i_1} + \ldots + \alpha_{i_l} + 1} \sum_{\gamma \in \Gamma_l^c} \prod_{\substack{1\leq t\leq m \\ 0\leq s\leq r, \ j\leq 0}} m_{s, l_{j,j+s;t}(\gamma)}^{(t)}. \tag{2.8}$$

where c is the sequence of colors $(i_1, \ldots, i_l)$.

(b) The joint law of $\{S^{(n,i)}\}_{1\leq i\leq m}$ *converges almost surely and in expectation to the joint law of* $\{a'_i\}_{1\leq i\leq m}$ *whose joint distribution is given by*

$$\phi(a'_{i_1}\ldots a'_{i_l}) = \frac{1}{2(\alpha_{i_1}+\ldots+\alpha_{i_l})+1}\sum_{\gamma\in\Gamma^c_{2l}}\prod_{\substack{1\leq t\leq m\\ |s|\leq r,\ j\leq 0}} m^{(t)}_{s,l_{j\to j+s;R,t}(\gamma)+l_{j+s\to j;B,t}(\gamma)}. \tag{2.9}$$

where c *is the sequence of colors* $(i_1, i_1, i_2, i_2, \ldots, i_l, i_l)$.

Suppose $(\mathcal{A}, \phi)$ is a noncommutative probability space. A family $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras of $\mathcal{A}$ are called *freely independent* (or simple free) if $\forall n$ positive, indices $k_1, \ldots, k_n \in I$ with $k_j \neq k_{j+1}$ and $a_j \in \mathcal{A}_{k_j}$, $1 \leq j \leq n$ with $\phi(a_j) = 0$, we have $\phi(a_1 \ldots a_n) = 0$. For positive integers m, $(m_k)_{1\leq k\leq m}$, the sets $\{a_{1,p}, \ldots, a_{m_p,p}\}_{1\leq p\leq m}$ are freely independent if the algebras they generate are freely independent.

Remark 3. *In the setup of Theorem 1, when* $r = 1$, $\alpha = \frac{1}{2}$ *and* $m_{ik} = \delta_{i1}$ *for* $i = 0, 1$*; we get* $L_{2k} = \frac{1}{k+1}\begin{pmatrix} 2k \\ k \end{pmatrix}$, *i.e. the LSD is the semicircular law. Under the same parameter values in part (a) of Theorem 4 we get the joint law to be*

$$\phi(a_{i_1}\ldots a_{i_l}) = \begin{cases} 0 & , \text{ if } l \text{ is odd} \\ \frac{1}{\frac{l}{2}+1}\begin{pmatrix} l \\ \frac{l}{2} \end{pmatrix} & , \text{ if } l \text{ is even} \end{cases}$$

hence $\phi(a_{i_1}\ldots a_{i_l}) = \phi(a_1^l)$. *This is an example where the marginals converge to the semicircular law but the joint law is not a free product of semicircular laws.*

3 Proof of theorems

3.1 Auxiliary results

In the course of our proofs, we will need to estimate the distance between different spectral measures. This shall be done via the bounded Lipschitz metric d_{BL}, which is a complete metric defined on the space of probability measures on any Polish space (X, d), topologising the weak convergence of probability measures (see Dudley (2002)):

$$d_{BL}(\mu,\ \nu) = \sup\{\int f d\mu - \int f d\nu : ||f||_\infty + ||f||_L \leq 1\}$$

where $||f||_\infty = \sup_x |f(x)|$, $||f||_L = \sup_{x\neq y} |f(x) - f(y)|/d(x, y)$.

We shall also need the following Lemma. Its proof may be found in Bai and Silverstein (2006) or Bai (1999) and uses Lidskii's theorem (see Bhatia, 1997, page 69).

Lemma 1. *(a) Suppose* A, B *are* $n \times n$ *real symmetric matrices. Then*

$$d^2_{BL}(F^A, F^B) \leq \frac{1}{n}Tr(A - B)^2. \tag{3.1}$$

(b) Suppose A *and* B *are* $p \times n$ *real matrices. Let* $X = AA^T$ *and* $Y = BB^T$. *Then*

$$d^2_{BL}(F^X, F^Y) \leq \frac{2}{p^2}Tr(X + Y)Tr[(A - B)(A - B)^T]. \tag{3.2}$$

3.2 Proof of Theorem 2

Proof of (a): Define $x_{in}^{(l)} = x_{in}\mathbf{1}(|x_{in}| < c_l)$ and let

$$(A_n^{(l)})_{ij} = \begin{cases} a_{j-i,i}x_{j-i,i}^{(l)}, & \text{if } i \le j \le r+i \\ 0, & \text{if } r+i < j \le n \\ (A_n^{(l)})_{ji}, & \text{if } j < i. \end{cases}$$

Using (2.4) and condition (ii) in case (a2), we see that in both the cases (a1) and (a2),

$$\lim_n \mathbb{E}\left(\frac{a_{in}x_{in}^{(l)}}{n^\alpha}\right)^k = m_{ik}^{(l)} \text{ exists and } m_{ik}^{(l)} \le (max_i\ \{a_i\}c_l)^k.$$

Hence for each l and $0 \le i \le r$, $\{m_{ik}^{(l)}\}_{k\ge 0}$ is the moment sequence of a compactly supported measure. From Theorem 1, $F^{\frac{A_n^{(l)}}{n^\alpha}}$ converges almost surely to a nonrandom compactly supported probability measure $\gamma^{(l)}$.

From (2.4), for some $M > 0$, $\frac{a_{ij}^2}{j^{2\alpha}} \le M,\ \forall\ i,j$. Hence an application of (3.1) gives

$$\begin{aligned} d_{Bl}^2(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) &\le \tfrac{1}{n^{2\alpha+1}} Tr(A_n - A_n^{(l)})^2 \\ &\le \tfrac{2}{n^{2\alpha+1}} \sum_{i=1}^{r}\sum_{j=1}^{n} \frac{(a_{ij}x_{ij}\mathbf{1}(x_{ij} \ge c_l))^2}{n^{2\alpha+1}} \\ &\le 2M \sum_{i=1}^{r}\sum_{j=1}^{n} \frac{(x_{ij}\mathbf{1}(x_{ij} \ge c_l))^2}{n}. \end{aligned}$$

Under condition (a1)

$$\sum_{i=1}^{r}\sum_{j=1}^{n} \frac{(x_{ij}\mathbf{1}(x_{ij} \ge c_l))^2}{n} \xrightarrow{a.s} \sum_{i=1}^{r} \mathbb{E}((x_{i1}\mathbf{1}(x_{i1} \ge c_l))^2)$$

and hence $\lim_l \lim_n d_{Bl}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) = 0,\ a.s.$.

Under condition (a2),

$$\sum_{j=1}^{\infty} \frac{\mathbf{Var}(x_{ij}^2\mathbf{1}(x_{ij} > c_l))}{j^2} \le \sum_{j=1}^{\infty} \frac{\mathbb{E}(x_{ij}^4)}{j^2} < \infty.$$

By Kolmogorov SLLN,

$$\sum_{j=1}^{n} \frac{1}{n}\left(x_{ij}^2\mathbf{1}(x_{ij} > c_l) - \mathbb{E}(x_{ij}^2\mathbf{1}(x_{ij} > c_l))\right) \xrightarrow{a.s.} 0.$$

From (iii) it follows that

$$\lim_l \limsup_n \sum_{j=1}^{n} \frac{x_{ij}^2\mathbf{1}(x_{ij} > c_l)}{n} \xrightarrow{a.s.} 0 \text{ and hence } \lim_l \limsup_n d_{Bl}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) = 0,\ a.s..$$

Since d_{Bl} metrizes weak convergence of probability measures on $\mathbb{R}$, $\lim\limits_{n} d_{BL}(F^{\frac{A_n^{(l)}}{n^\alpha}}, \gamma^{(l)}) = 0$, $a.s.$ $\forall\ l$, i.e. $\{F^{\frac{A_n^{(l)}}{n^\alpha}}\}_n$ is almost surely Cauchy. Hence

$$\begin{aligned} d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_m}{m^\alpha}}) &\leq d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) + d_{BL}(F^{\frac{A_n^{(l)}}{n^\alpha}}, F^{\frac{A_m^{(l)}}{m^\alpha}}) + d_{BL}(F^{\frac{A_m^{(l)}}{m^\alpha}}, F^{\frac{A_m}{m^\alpha}}) \\ &\Rightarrow \limsup_{m,n} d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_m}{m^\alpha}}) \leq 2 \limsup_{n} d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}), \text{ for each } l \\ &\Rightarrow \limsup_{m,n} d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_m}{m^\alpha}}) \leq 2 \lim_{l} \limsup_{n} d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) = 0, \end{aligned}$$

i.e. $\{F^{\frac{A_n}{n^\alpha}}\}$ is almost surely Cauchy. Since d_{BL} is complete on the space of probability measures on $\mathbb{R}$, there exists a probability measure γ such that $F^{\frac{A_n}{n^\alpha}} \xrightarrow{w} \gamma$.

Also

$$\lim_{l}\ d_{BL}(\gamma, \gamma^{(l)}) = \lim_{l} \lim_{n}\ d_{BL}(F^{\frac{A_n}{n^\alpha}}, F^{\frac{A_n^{(l)}}{n^\alpha}}) = 0,$$

being the weak limit of the nonrandom sequence $\{\gamma^{(l)}\}$, γ itself is nonrandom. This completes the proof of (a).

Proof of (b): We have established that

$$F^{X_n} \xrightarrow{w} \gamma\ w.p.\ 1.$$

Also from Theorem 1, $\int x^k\ d\gamma_n \overset{n\to\infty}{\longrightarrow} L_k$ $w.p.$ 1, γ_n being the ESD of X_n. In particular

$$\sup_{n} \int x^{2k}\ d\gamma_n < \infty\ w.p.\ 1,\ \forall\ k.$$

Hence if X_n follows γ_n, then almost surely $\{\gamma_n\}$ are uniformly integrable. Hence we have

$$\int x^k\ d\gamma = \lim_{n} \int x^k\ d\gamma_n = L_k.$$

For the other part, consider the path $\gamma_0 \in \Gamma_{2k}$ given by $\gamma_0 = (-i_0, 0, -i_0, 0, \ldots, -i_0, 0)$. Then

$$\begin{aligned} L_{2k} &= \tfrac{1}{2\alpha k+1} \textstyle\sum_{\gamma\in\Gamma_{2k}} \prod_{0\leq i\leq r,\ j\leq 0} m_{i,l_{j,j+i}(\gamma)} \\ &\geq \tfrac{1}{2\alpha k+1} \prod_{0\leq i\leq r,\ j\leq 0} m_{i,l_{j,j+i}(\gamma_0)} = \frac{1}{2\alpha k+1} m_{i_0,l_{-i_0,0}(\gamma_0)} = \frac{1}{2\alpha k+1} m_{i_0,2k}. \end{aligned}$$

This implies

$$\limsup_{k} (L_{2k})^{\frac{1}{2k}} \geq \limsup_{k} (\frac{1}{2\alpha k+1} m_{i_0,2k})^{\frac{1}{2k}} = \infty,$$

and hence γ has unbounded support. This completes the proof. $\square$

3.3 Proof of Theorem 3

(a) First we prove convergence in expectation. Let

$$S = \{\gamma : \gamma = (j_0, \ldots, j_{2k}),\ j_0 = j_{2k},\ 1 \le j_u \le n,\ |j_{u+1} - j_u| \le r\ \forall\ u\}.$$

Then

$$\begin{aligned}\tfrac{Tr(S_n^k)}{n} &= \tfrac{1}{n^{2\alpha k+1}} Tr(A_n A_n^T)^k \\ &= \tfrac{1}{n^{2\alpha k+1}} \sum_{\gamma \in S} a_{j_0 j_1} a_{j_2 j_1} \ldots a_{j_{2k-2} j_{2k-1}} a_{j_{2k} j_{2k-1}} \\ &= \tfrac{1}{n^{2\alpha k+1}} \sum_{p=1}^{n} \sum_{\gamma\in S,\ \max\ \gamma_j = p} a_{j_0 j_1} a_{j_2 j_1} \ldots a_{j_{2k-2} j_{2k-1}} a_{j_{2k} j_{2k-1}}.\end{aligned}$$

Denoting the inner sum by S_p, we write

$$\frac{Tr(S_n^k)}{n} = \frac{1}{n^{2\alpha k+1}} \sum_{p=1}^{n} S_p \tag{3.3}$$

Now $max\ \gamma_j = p$, $j_0 = j_{2k}$ and $|j_{u+1} - j_u| \le r$ together imply $\min \gamma_j \ge p - rk$. Hence for $p \ge rk+1$, we have $\{\gamma : j_0 = j_{2k}, 1 \le j_u \le n, |j_{u+1} - j_u| \le r, \max \gamma_j = p\} = \{\gamma + p : \gamma \in \Gamma_{2k}\}$.

For $p \ge rk + 1$,

$$\begin{aligned}\mathbf{E}(\tfrac{S_p}{p^{2\alpha k}}) &= \frac{1}{p^{2\alpha k}} \sum_{\gamma\in\Gamma_{2k}} \mathbf{E}(a_{j_0+p,j_1+p} a_{j_2+p,j_1+p} \ldots a_{j_{2k-2}+p,j_{2k-1}+p} a_{j_{2k}+p,j_{2k-1}+p}) \\ &= \tfrac{1}{p^{2\alpha k}} \sum_{\gamma\in\Gamma_{2k}} \mathbf{E} \prod_{|h|\le 2r, i\le 0} b_{h,i+p}^{l_{i\to i+h,R}(\gamma)+l_{i+h\to i,B}(\gamma)}\end{aligned}$$

Now, we have $\mathbb{E}\left(\frac{b_{h,i+p}}{p}\right)^{l_{i\to i+h,R}(\gamma)+l_{i+h\to i,B}(\gamma)} \to m'_{h,l_{i\to i+h,R}(\gamma)+l_{i+h\to i,B}(\gamma)}$.

Since $\sum_{|h|\le 2r, i\le 0} (l_{i\to i+h,R}(\gamma) + l_{i+h\to i,B}(\gamma)) = 2k$, we have

$$\mathbf{E}(\frac{S_p}{p^{2\alpha k}}) \overset{p\to\infty}{\longrightarrow} \sum_{\gamma\in\Gamma_{2k}} \prod_{|h|\le r,\ i\le 0} m'_{h,l_{i\to i+h,R}(\gamma)+l_{i+h\to i,B}(\gamma)}.$$

Hence it follows that $\mathbf{E}(\frac{Tr(S_n^k)}{n}) \to L'_k$.

To establish almost sure convergence, we shall show that $\frac{1}{n^{2\alpha k+1}} \sum_{p=1}^{n} (S_p - \mathbf{E}(S_p)) \overset{a.s}{\to} 0$.

Consider $p \ge rk + 1$, then we have

$$\begin{aligned}\tfrac{1}{p^{4\alpha k}} \mathbf{Var(S_p)} &\le \tfrac{1}{p^{4\alpha k}} \mathbf{E}(S_p^2) \\ &= \tfrac{1}{p^{4\alpha k}} \mathbf{E}(\sum_{\gamma\in\Gamma_{2k}} a_{j_0+p,j_1+p} a_{j_2+p,j_1+p} \ldots a_{j_{2k-2}+p,j_{2k-1}+p} a_{j_{2k}+p,j_{2k-1}+p})^2 \\ &\le C \tfrac{1}{p^{4\alpha k}} \sum_{\gamma\in\Gamma_{2k}} \mathbf{E}(a^2_{j_0+p,j_1+p} a^2_{j_2+p,j_1+p} \ldots a^2_{j_{2k-2}+p,j_{2k-1}+p} a^2_{j_{2k}+p,j_{2k-1}+p})\end{aligned}$$

where $C = |\Gamma_{2k}|$ is free of p. Hence

$$\begin{aligned} \tfrac{1}{p^{4\alpha k}}\mathbf{Var}(S_p) &\leq C\tfrac{1}{p^{4\alpha k}} \sum_{\gamma \in \Gamma_{2k}} \mathbf{E} \prod_{|h|\leq 2r, i \leq 0} b_{h,i+p}^{2l_{i\to i+h,R}(\gamma)+2l_{i+h\to i,B}(\gamma)} \\ &\to \sum_{\gamma\in\Gamma_{2k}} \prod_{|h|\leq r,\ i\leq 0} m'_{h,2l_{i\to i+h,R}(\gamma)+2l_{i+h\to i,B}(\gamma)}. \end{aligned}$$

Then we have $\frac{1}{p^{4\alpha k+2}}\mathbf{Var}(S_p) = O(\frac{1}{p^2})$ and hence

$$\sum_{1}^{\infty} \frac{1}{p^{4\alpha k+2}}\mathbf{Var}(S_p) < \infty. \tag{3.4}$$

Now the paths $\gamma = (j_0, \ldots, j_{2k})$ contributing to S_p satisfy $min\ \gamma_j \geq p - rk$. Hence $\forall\ t \geq 0$, S_t and S_{t+rk+1} are independent. Hence by (3.4) we have, for each fixed $t \leq rk$, the sequence $\{Y_n\}$ given by

$$Y_n = \sum_{p=0}^{n} \frac{S_{t+p(rk+1)} - \mathbf{E}(S_{t+p(rk+1)})}{(t+p(rk+1))^{2\alpha k+1}}$$

is an L^2 bounded martingale. Hence for every $t \leq rk$, $\sum_{p=0}^{\infty} \frac{S_{t+p(rk+1)}-\mathbf{E}(S_{t+p(rk+1)})}{(t+p(rk+1))^{2\alpha k+1}}$ converges almost surely and hence $\sum_{p=1}^{\infty} \frac{S_p - \mathbf{E}(S_p)}{p^{2\alpha k+1}}$ converges almost surely. Finally we use Kronecker's lemma to conclude that $\frac{1}{n^{2\alpha k+1}} \sum_{p=1}^{n}(S_p - \mathbf{E}(S_p)) \overset{a.s}{\to} 0$ which completes the proof of part (a).

(b) Following Popescu's arguments for the tridiagonal model we can easily construct a bounded operator on a Hilbert space such that L'_k's will be the moments of a probability measure concentrated on the spectrum of the operator.

Assume μ'_j is supported on I_j, $|j| \leq r$ and let $I = \prod_{|j|\leq r} I_j$ and that $\mu' = \bigotimes_{|j|\leq r} \mu'_j$.

Let $\Omega = \prod_{i\in\mathbb{Z}} I_i$, $P = \bigotimes_{\mathbb{Z}} \mu$. Let $H = \bigoplus_{\mathbb{Z}} L^2(\Omega, P)$.

Let us denote a typical $\omega \in \Omega$ as $\omega = (\omega_i)_{i\in\mathbb{Z}}$ where $\omega_i = (\omega_{i,-r}, \ldots, \omega_{i,r}) \in I$. We define the operator

$$\begin{aligned} T:\ & H \to H \\ & x \mapsto y \end{aligned}$$

where $y_j(\omega) = \sum_{|h|\leq r} \omega_{j,h} x_{j+h}(\omega)$ if $x = (x_j)_{j\in\mathbb{Z}}$. Since each I_j is bounded, T is a bounded operator. T can be represented by the following matrix

$$A = \left[\begin{array}{ccccccccccc} & & & \cdots & & & & \cdots & & & \\ \cdots & X_{-r,-1} & X_{-r+1,-1} & \cdots & X_{0,-1} & X_{1,-1} & X_{2,-1} & \cdots & 0 & 0 & \cdots \\ \cdots & 0 & X_{-r,0} & \cdots & X_{-1,0} & X_{0,0} & X_{1,0} & \cdots & X_{r,0} & 0 & \cdots \\ \cdots & 0 & 0 & \cdots & X_{-2,1} & X_{-1,1} & X_{0,1} & \cdots & X_{r-1,1} & X_{r,1} & \cdots \\ & & & \cdots & & & & \cdots & & & \end{array}\right] \tag{3.5}$$

where $X_{h,n}(\omega) = \omega_{n,h}$ for $|h| \leq r, n \in \mathbb{Z}$. We note that $\int X_{h,n}^k dP = m'_{hk}$. Define U to be the self-adjoint, bounded linear operator $U = TT^*$. Let $\mathbf{e} \in H$ be the element given by

$\mathbf{e_j} \equiv 0,\ \forall\ j \neq 0$ and $\mathbf{e_0} \equiv 1$. Then

$$\begin{aligned} < U^k\mathbf{e}, \mathbf{e} > &= \textstyle\int (AA^T)^k_{0,0} dP \\ &= \textstyle\int \displaystyle\sum_{\substack{j_i \in \mathbb{Z} \\ |j_{i+1}-j_i| \le r}} a_{0,j_1} a_{j_2,j_1} \ldots a_{2k-2,2k-1} a_{0,j_{2k-1}} dP \\ &= \textstyle\int \displaystyle\sum_{\gamma \in \Gamma_{2k}} \prod_{\substack{|h| \le r \\ i \le 0}} X_{h,i}^{l_{i \to i+h,R}(\gamma) + l_{i+h \to i,B}(\gamma)} dP \\ &= \sum_{\gamma \in \Gamma_{2k}} \prod_{\substack{|h| \le r \\ i \le 0}} m'_{h, l_{i \to i+h,R}(\gamma) + l_{i+h \to i,B}(\gamma)}. \end{aligned}$$

The spectral theorem furnishes a probability measure Λ supported on the spectrum of U such that $< U^k\mathbf{e}, \mathbf{e} >= \int x^k d\Lambda$ (Λ is a probability measure as $\|\mathbf{e}\| = 1$). Hence if Z is distributed according to Λ and Y is a uniform random variable independent of Z, then the k-th moment of $Y^{2\alpha}Z$ is L'_k, $\forall\ k \ge 0$.

Since $Y^{2\alpha}Z$ is compactly supported, we conclude that the sequence L'_k is the moment sequence of a unique compactly supported probability measure and hence the ESD's of the sequence S_n converges almost surely to μ'.

(c) If $m'_{jk} = a_j^k$ then $I_j = \{a_j\}$, Ω consists of a singleton ω with $\omega_{n,h} = a_h$, $\forall\ n$, $H = l^2$, U is given by the matrix AA^T whose i,j-th entry is $(AA^T)_{ij} = \sum_{-r}^{r-h} a_i a_{i+h} = c_h$ (say) where $h = |j-i|$.

Identifying $L^2(S^1)$ with l^2 via the isometry which sends $f \in L^2$ to the sequence of its Fourier coefficients $(\widehat{f}(n))$, we see that U on l^2 corresponds to the operator M on $L^2(S^1)$ given by $Mf = p'f$ where $p'(x) = c_0 + \sum_{k=1}^r c_k(e^{ikx} + e^{-ikx})$. Since $\mathbf{e}$ corresponds to the function which is identically 1, we get

$$< U^k\mathbf{e}, \mathbf{e} >=< M^k\mathbf{1}, \mathbf{1} >= \int_{S^1} (p'(x))^k d\lambda$$

and the rest is immediate from part (b).

Proof of Corollary 3.1: For a path $\gamma = (j_0, \ldots, j_k)$, let $t_h(\gamma) = \#\{u : |j_{u+1} - j_u| = h\}$, for $h \ge 0$, i.e. the number of transitions with jump h.

Then under the assumptions of Corollary 3.1,

$$\begin{aligned} L_k &= \frac{1}{\alpha k + 1} \sum_{\gamma \in \Gamma_k} \prod_{0 \le h \le r,\ j \le 0} a_h^{l_{j,j+h}(\gamma)} = \frac{1}{\alpha k + 1} \sum_{\gamma \in \Gamma_k} \prod_{0 \le h \le r} a_h^{t_h(\gamma)} \\ L'_k &= \frac{1}{2\alpha k + 1} \sum_{\gamma \in \Gamma_{2k}} \prod_{|h| \le r,\ i \le 0} a_{|h|}^{l_{i \to i+h,R}(\gamma) + l_{i+h \to i,B}(\gamma)} = \frac{1}{2\alpha k + 1} \sum_{\gamma \in \Gamma_{2k}} \prod_{0 \le h \le r} a_h^{t_h(\gamma)}. \end{aligned}$$

Hence $L'_k = L_{2k}$, since all the measures are compactly supported, the result follows. □

Proof of Corollary 3.2: This can be done in the exact same way as in the proof of Theorem 2. We define $S_n^{(l)} = n^{-2\alpha} A_n^{(l)} {A_n^{(l)}}^t$ ($A_n^{(l)}$ being the matrices with truncated entries). Using

(3.2) we get

$$\begin{aligned} d_{BL}^2(F^{S_n}, F^{S_n^{(l)}}) &\leq \tfrac{2}{n^2} Tr(S_n + S_n^{(l)}) \tfrac{Tr((A_n - A_n^{(l)})(A_n - A_n^{(l)})^t)}{n^{2\alpha}} \\ &\leq 4 \tfrac{Tr(S_n)}{n} \tfrac{Tr((A_n - A_n^{(l)})(A_n - A_n^{(l)})^t)}{n^{2\alpha+1}}. \end{aligned}$$

Under the assumptions made $n^{-1}Tr(S_n)$ remains bounded almost surely and we manipulate the other factor as in proof of Theorem 2 to get the result. □

3.4 Proof of Theorem 4

Let $c_0 = \{1, \ldots, m\}$ be a set of colors corresponding to $X^{(n,1)}, \ldots, X^{(n,m)}$. Let $c = (i_1, \ldots, i_l)$ be the color sequence corresponding to $X^{(n,i_1)}, \ldots, X^{(n,i_l)}$.

$$\begin{aligned} \mathbf{E}\tfrac{1}{n}(Tr(X^{(n,i_1)}) \ldots X^{(n,i_l)}) &= \tfrac{1}{n^{\alpha_{i_1} + \ldots + \alpha_{i_l} + 1}} \sum_{\substack{\gamma : j_0 = j_l \\ 1 \leq j_u \leq n \\ |j_{u+1} - j_u| \leq r}} \mathbf{E}(a_{j_0 j_1}^{(n,i_1)} \ldots a_{j_{l-1} j_l}^{(n,i_l)}) \\ &= \tfrac{1}{n^{\alpha_{i_1} + \ldots + \alpha_{i_l} + 1}} \sum_{p=1}^{n} \sum_{\substack{\gamma : j_0 = j_l \\ 1 \leq j_u, \ \max \gamma = p \\ |j_{u+1} - j_u| \leq r}} \mathbf{E}(a_{j_0 j_1}^{(n,i_1)} \ldots a_{j_{l-1} j_l}^{(n,i_l)}) \\ &= \tfrac{1}{n^{\alpha_{i_1} + \ldots + \alpha_{i_l} + 1}} \sum_{p=1}^{n} S_p \text{ (say)}. \end{aligned}$$

As in the proof of Theorem 3, for $p \geq \frac{rl}{2} + 1$,

$$\{\gamma : j_0 = j_l, 1 \leq j_u, \ \max \gamma = p, |j_{u+1} - j_u| \leq r\} = \{\gamma + p : \gamma \in \Gamma_l^c\}.$$

So we consider $p \geq \frac{rl}{2} + 1$. Then

$$\begin{aligned} \tfrac{S_p}{p^{\alpha_{i_1} + \ldots + \alpha_{i_l}}} &= \tfrac{1}{p^{\alpha_{i_1} + \ldots + \alpha_{i_l}}} \sum_{\gamma \in \Gamma_l^c} \mathbf{E}(a_{j_0+p, j_1+p}^{(n,i_1)} \ldots a_{j_{l-1}+p, j_l+p}^{(n,i_l)}) \\ &= \tfrac{1}{p^{\alpha_{i_1} + \ldots + \alpha_{i_l}}} \sum_{\gamma \in \Gamma_l^c} \prod_{\substack{t \in \{i_1, \ldots, i_l\} \\ 0 \leq s \leq r, \ j \leq 0}} {b_{s,j+p}^{(t)}}^{l_{j,j+s;t}(\gamma)} \end{aligned}$$

Since

$$\sum_{0 \leq s \leq r, \ j \leq 0} l_{j,j+s;t}(\gamma) = \#\{k | i_k = t\},$$

we get

$$\frac{S_p}{p^{\alpha_{i_1} + \ldots + \alpha_{i_l}}} \to \sum_{\gamma \in \Gamma_l^c} \prod_{\substack{1 \leq t \leq m \\ 0 \leq s \leq r, \ j \leq 0}} m_{s, l_{j,j+s;t}(\gamma)}^{(t)}.$$

Hence an application of Cesaro sums establishes converges in expectation.

The almost sure convergence can be proved using arguments similar to those used in the proof of (a) of Theorem 3.

The proof of part (b) follows along the same line of argument as in part (a) and is omitted. □

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