

# MAXIMA OF DIRICHLET AND TRIANGULAR ARRAYS OF GAMMA VARIABLES

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ABSTRACT. Consider a rowwise independent triangular array of gamma random variables with varying parameters. Under several different conditions on the shape parameter, we show that the sequence of row-maximums converges weakly after linear or power transformation. Depending on the parameter combinations, we obtain both Gumbel and non-Gumbel limits.

The weak limits for maximum of the coordinates of certain Dirichlet vectors are also obtained using the gamma representation.

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## 1. INTRODUCTION

Suppose  $\{Y_n\}$  is a sequence of i.i.d. random variables and  $M_n = \max\{Y_1, \dots, Y_n\}$ . Necessary and sufficient conditions for the weak convergence of  $M_n$  under linear normalisation are well known. See for example, Fisher and Tippett (1928), Gnedenko (1943), de Haan (1970). In particular, let  $Y_n$  be i.i.d. standard normal variables and let  $G$  denote the Gumbel distribution

$$G(x) = \exp(-e^{-x}).$$

Then (cf. Leadbetter et al., 1983, Theorem 1.5.3),

$$\lim_{n \rightarrow \infty} P[M_n \leq c_n x + d_n] = G(x),$$

where

$$c_n = \frac{1}{\sqrt{2 \log n}} \quad \text{and} \quad d_n = \sqrt{2 \log n} - \frac{\log \log n + \log(4\pi)}{2\sqrt{2 \log n}}. \quad (1)$$

Now let  $(Y_{1n}, \dots, Y_{nn})$  be a triangular sequence of random variables and let  $M_n = \max\{Y_{1n}, \dots, Y_{nn}\}$ . The question of convergence of  $M_n$  has been addressed under a variety of conditions. For example, let  $Y_{in}$  be i.i.d. with  $Y_{1n} = (\sum_{1 \leq j \leq \alpha_n} U_j - \alpha_n \mu) / (\sigma \alpha_n^{1/2})$ , where  $U_j$  are i.i.d. with mean  $\mu$  and standard deviation  $\sigma$ . Assuming that  $U_j$  has a finite moment generating function in an open interval containing the origin and  $\log n = o(\alpha_n^{(R+1)/(R+3)})$  for some integer  $R \geq 0$ , Anderson et al. (1997) showed that

$$\lim_{n \rightarrow \infty} P[M_n \leq c_n x + d_n] = G(x)$$

for  $c_n$  as in (1) and some suitable sequences  $d_n$ .

Nadarajah and Mitov (2002) considered the maximums of a triangular array of binomial, negative binomial and discrete uniform variables. The case of binomial triangular array is discussed with increasing number of trials  $m_n$  and fixed probability of success,  $p$ .

Bose et al. (2006) considered the row-maximum of a triangular array with dependent rows. More precisely, for  $n$ -dimensional multinomial random variable with equally likely cells, the maximum of the coordinates converges to Gumbel law if number of trials increases fast enough.

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Now let  $\{Y_{in}\}$  be a triangular sequence such that, for each  $n$ ,  $Y_{in}$  are i.i.d. random variables having Gamma  $(\alpha_n, 1)$  distribution. Also let  $\mathbf{X}_n = (X_{1n}, \dots, X_{nn})$  be an  $n$ -dimensional Dirichlet distribution with parameters  $\alpha_n, \dots, \alpha_n, \beta_n$  supported on the  $n$ -dimensional simplex  $\{\mathbf{x} : 0 \leq \sum_{i=1}^n x_i \leq 1\}$  and with density

$$\frac{\Gamma(n\alpha_n + \beta_n)}{\Gamma(\alpha_n)^n \Gamma(\beta_n)} \left( \prod_{i=1}^n x_i \right)^{\alpha_n} \left( 1 - \sum_{i=1}^n x_i \right)^{\beta_n}.$$

We investigate the problem of existence of the weak limits of the maximums

$$M_n = \vee_{i=1}^n Y_{in}, \quad \widetilde{M}_n = \vee_{i=1}^n X_{in}$$

under linear or power transformation.

In Section 2, we study the behavior of  $M_n$ . When  $n\alpha_n$  has a positive limit, a linear transformation works. When  $n\alpha_n \rightarrow \infty$ , the limit will be Gumbel. If further  $\alpha_n/\log n \rightarrow 0$ , we can take the scaling to be 1. In Theorem 2.6, we show that if  $n\alpha_n$  has a positive, finite limit,  $M_n$  itself has a non-Gumbel limit. In Theorem 2.7, we show that under the assumption  $n\alpha_n \rightarrow 0$ , a power transformation leads to uniform limit.

In Section 3,  $\widetilde{M}_n$  is taken up. When  $n\alpha_n + \beta_n \rightarrow \infty$  and  $n\alpha_n$  converges to a positive limit, the limit will still be Gumbel. When  $n\alpha_n$  and  $n\alpha_n + \beta_n$  both have finite, positive limits,  $\widetilde{M}_n$  itself will converge, but to a non-standard limit, cf. Theorem 3.1. Finally, in Theorem 3.2, we show that a power transformation of  $\widetilde{M}_n$  will converge to a mixture of uniform distribution on  $(0, 1)$  and a point mass at 1.

## 2. MAXIMUM OF TRIANGULAR ARRAY OF GAMMA RANDOM VARIABLES

The centering and scaling depends on the nature of the sequence  $\alpha_n$ . The first case is similar to Proposition 2 of Anderson et al. (1997).

**Theorem 2.1.** *Assume that  $\alpha_n/\log n \rightarrow \infty$ . Then*

$$\sqrt{\frac{2\log n}{\alpha_n}}(M_n - \alpha_n - b_n\sqrt{\alpha_n}) \Rightarrow G,$$

where  $b_n$  is the unique solution, in the region  $b_n \sim \sqrt{2\log n}$ , of

$$\log z + \frac{1}{2} \log(2\pi) + z\sqrt{\alpha_n} - \alpha_n \log \left( 1 + \frac{z}{\sqrt{\alpha_n}} \right) = \log n. \quad (2)$$

Observe that in this case,  $Y_{1n}$  can be considered to be the “sum” of  $\alpha_n$  many i.i.d. random variables, each of which is distributed as unit Exponential random variable. This set up is similar to that of Proposition 2 of Anderson et al. (1997) mentioned earlier but we have the added advantage that the random variables are gamma distributed. It may also be noted that the condition  $\alpha_n/\log n \rightarrow \infty$ , is the limiting form ( $R = \infty$ ) of their condition. Almost verbatim repetition of their argument yields the proof of Theorem 2.1. We omit the details but point out that their Lemma 2 continues to hold if we replace the degree  $R$  polynomial with the corresponding power series ( $R = \infty$ ). Using the moment generating function of the gamma distribution, the  $j$ -th coefficient of the power series simplifies to  $(-1)^{j+1}/(j+2)$ , for  $j \geq 1$ . This yields the defining equation for  $b_n$  given in (2) above.

Let the centering and scaling required in general be  $d_n$  and  $c_n$  respectively and let

$$x_n = c_n x + d_n. \quad (3)$$

Suppose  $x_n$  is such that  $P[Y_{1n} > x_n] \rightarrow 0$ . Then

$$-\log P[M_n \leq x_n] = -\log P[Y_{1n} < x_n]^n \sim nP[Y_{1n} > x_n]. \quad (4)$$

Motivated by the above, and noting that  $Y_{1n}$  has the Gamma( $\alpha_n, 1$ ) distribution, define

$$A_n = \frac{n}{\Gamma(\alpha_n)} \int_{x_n}^{\infty} e^{-u} u^{\alpha_n-1} du \sim nP[Y_{1n} > x_n].$$

Integrating by parts, we immediately have

$$A_n = B_n + (\alpha_n - 1)C_n, \quad (5)$$

where

$$B_n = \frac{n}{\Gamma(\alpha_n)} e^{-x_n} x_n^{\alpha_n-1} \quad (6)$$

and

$$C_n = \frac{n}{\Gamma(\alpha_n)} \int_{x_n}^{\infty} e^{-u} u^{\alpha_n-2} du \leq \frac{A_n}{x_n}, \quad (7)$$

which provides us with an upper bound for  $A_n$ :

$$A_n \leq \frac{1}{1 - \frac{|\alpha_n-1|}{x_n}} B_n. \quad (8)$$

For fixed  $k$ , if  $\alpha_n > k$ , we can also obtain a lower bound integrating by parts  $k$  times repeatedly:

$$\begin{aligned} A_n &= B_n \left[ 1 + \sum_{j=1}^{k-1} \prod_{i=1}^j \frac{\alpha_n - i}{x_n} \right] + \prod_{i=1}^k (\alpha_n - i) \frac{n}{\Gamma(\alpha_n)} \int_{x_n}^{\infty} e^{-u} u^{\alpha_n-k-1} du \\ &\geq B_n \left[ 1 + \sum_{j=1}^{k-1} \left( \frac{\alpha_n - k}{x_n} \right)^j \right] = \frac{B_n}{1 - \frac{\alpha_n - k}{x_n}} \left[ 1 - \left( \frac{\alpha_n - k}{x_n} \right)^k \right] \geq \frac{B_n}{1 - \frac{\alpha_n - k}{x_n}} \left[ 1 - \left( \frac{\alpha_n}{x_n} \right)^k \right]. \end{aligned} \quad (9)$$

When  $\alpha_n / \log n$  remains bounded away from both 0 and  $\infty$ , we have  $\alpha_n \rightarrow \infty$ . The proof of the following Theorem requires careful use of both the bounds (8) and (9).

**Theorem 2.2.** *Assume that  $\alpha_n / \log n$  remains bounded away from 0 and  $\infty$ . Then*

$$\left( 1 - \frac{\alpha_n}{\zeta_n} \right) (M_n - \zeta_n) + \log \left( 1 - \frac{\alpha_n}{\zeta_n} \right) \Rightarrow G,$$

where  $\zeta_n / \alpha_n$  is the unique solution bigger than 1 of

$$z = 1 + \frac{\log n}{\alpha_n} - \frac{\log \sqrt{2\pi} + \frac{1}{2} \log \alpha_n}{\alpha_n} + \left( 1 - \frac{1}{\alpha_n} \right) \log z. \quad (10)$$

*Proof.* We start with the solution of (10). Observe that  $(\log \sqrt{2\pi} + \frac{1}{2} \log \alpha_n) / \alpha_n \rightarrow 0$ , since  $\alpha_n \rightarrow \infty$ . Further, since  $\log n / \alpha_n$  is bounded away from 0, for all large  $n$ ,  $1 + \log n / \alpha_n - (\log \sqrt{2\pi} + \frac{1}{2} \log \alpha_n) / \alpha_n$  is bounded away from 1. Thus, (10) has unique solution  $\zeta_n / \alpha_n$  bigger than 1 for all large  $n$  and hence  $\zeta_n \rightarrow \infty$ .

We have,  $\zeta_n / \alpha_n > 1$ , for all large  $n$ . If for some subsequence  $\{n_k\}$ ,  $\zeta_{n_k} / \alpha_{n_k} \rightarrow 1$ , then using (10), we also have  $\log n_k / \alpha_{n_k} \rightarrow 0$ , which is a contradiction. If for some subsequence  $\{n_k\}$ ,  $\zeta_{n_k} / \alpha_{n_k} \rightarrow \infty$ , then using (10), we have

$$1 = \frac{1}{\zeta_{n_k} / \alpha_{n_k}} + \frac{\log n_k / \alpha_{n_k}}{\zeta_{n_k} / \alpha_{n_k}} - \frac{\log \sqrt{2\pi} + \frac{1}{2} \log \alpha_{n_k}}{(\zeta_{n_k} / \alpha_{n_k}) \cdot \alpha_{n_k}} + \left( 1 - \frac{1}{\alpha_{n_k}} \right) \frac{\log(\zeta_{n_k} / \alpha_{n_k})}{\zeta_{n_k} / \alpha_{n_k}}$$

and the right side converges to 0, which is again a contradiction. Thus,  $\zeta_n / \alpha_n$  is bounded away from both 1 and  $\infty$ .

Corresponding to the choice of centering and scaling, we have

$$x_n = \frac{x - \log\left(1 - \frac{\alpha_n}{\zeta_n}\right)}{1 - \frac{\alpha_n}{\zeta_n}} + \zeta_n. \quad (11)$$

First observe that, since  $\zeta_n/\alpha_n$  is bounded away from both 1 and  $\infty$ , we have  $x_n \sim \zeta_n$ . Hence,

$$\log x_n = \log \zeta_n + \frac{x - \log\left(1 - \frac{\alpha_n}{\zeta_n}\right)}{\zeta_n\left(1 - \frac{\alpha_n}{\zeta_n}\right)} + O\left(\frac{1}{\zeta_n^2}\right). \quad (12)$$

Now, using Stirling's approximation, (6), (11), (12) and the fact  $\alpha_n - 1 = O(\zeta_n) = o(\zeta_n^2)$ , we have, after collecting terms,

$$\begin{aligned} -\log B_n &= x \left[ \frac{1}{1 - \frac{\alpha_n}{\zeta_n}} - \frac{\frac{\alpha_n - 1}{\zeta_n}}{1 - \frac{\alpha_n}{\zeta_n}} \right] + \zeta_n - \frac{\log\left(1 - \frac{\alpha_n}{\zeta_n}\right)}{1 - \frac{\alpha_n}{\zeta_n}} - \log n \\ &\quad - (\alpha_n - 1) \log \zeta_n + \frac{\alpha_n - 1}{\zeta_n} \frac{\log\left(1 - \frac{\alpha_n}{\zeta_n}\right)}{1 - \frac{\alpha_n}{\zeta_n}} + \log \sqrt{2\pi} \\ &\quad - \alpha_n + \left(\alpha_n - \frac{1}{2}\right) \log \alpha_n + o(1) \\ &= \alpha_n \left[ \frac{\zeta_n}{\alpha_n} - 1 - \frac{\log n}{\alpha_n} + \frac{\log \sqrt{2\pi} + \frac{1}{2} \log \alpha_n}{\alpha_n} - \left(1 - \frac{1}{\alpha_n}\right) \log \frac{\zeta_n}{\alpha_n} \right] \\ &\quad + x \left[ 1 + \frac{1}{\zeta_n\left(1 - \frac{\alpha_n}{\zeta_n}\right)} \right] - \frac{\log\left(1 - \frac{\alpha_n}{\zeta_n}\right)}{1 - \frac{\alpha_n}{\zeta_n}} \left[ 1 - \frac{\alpha_n}{\zeta_n} + \frac{1}{\zeta_n} \right] + o(1) \\ &= x(1 + o(1)) - \log\left(1 - \frac{\alpha_n}{\zeta_n}\right) + o(1), \end{aligned} \quad (13)$$

where the first term of (13) vanishes since  $\zeta_n/\alpha_n$  satisfies (10). Thus, we have

$$\left(1 - \frac{\alpha_n}{\zeta_n}\right)^{-1} B_n \sim e^{-x}. \quad (14)$$

Hence, using the upper bound (8) and the facts  $\alpha_n \rightarrow \infty$  and  $x_n \sim \zeta_n$ , we have

$$\limsup A_n \leq \lim \frac{B_n}{1 - \frac{\alpha_n - 1}{x_n}} = \lim \frac{B_n}{1 - \frac{\alpha_n}{\zeta_n}} = e^{-x}. \quad (15)$$

Also, since  $\alpha_n/\zeta_n$  is bounded away from 1 and  $x_n \sim \zeta_n$ , given any  $\varepsilon > 0$ , we can fix a positive integer  $K$ , such that  $(\alpha_n/x_n)^K < \varepsilon$ , for all large  $n$ . Hence, using the lower bound (9), since for all large  $n$ ,  $\alpha_n > K$  and  $(\alpha_n/x_n)^K < \varepsilon$  hold, we have

$$\liminf A_n \geq \liminf \frac{B_n}{1 - \frac{\alpha_n - K}{x_n}} (1 - \varepsilon) = (1 - \varepsilon)e^{-x}.$$

Since  $\varepsilon > 0$  is arbitrary, we get

$$\liminf A_n \geq e^{-x}.$$

Combining with (15), we get  $A_n \rightarrow e^{-x}$ , which completes the proof.  $\square$

When  $n\alpha_n \rightarrow \infty$ , but  $\alpha_n = o(\log n)$ , it turns out that  $c_n = 1$  and the limiting distribution is  $G$ . However, the choices of  $d_n$  vary according to the specific limiting behavior of  $\alpha_n$ . In general, we have the following lemma, which is used repeatedly in the subsequent developments.

**Lemma 2.1.** *Suppose  $c_n$  and  $d_n$  are such that for all  $x \in \mathbb{R}$ ,*

$$B_n \rightarrow -\log F(x), \quad (16)$$

and

$$(\alpha_n - 1) = o(x_n). \quad (17)$$

Then  $A_n \rightarrow -\log F(x)$  and hence

$$\frac{M_n - d_n}{c_n} \Rightarrow F.$$

*Proof.* From the upper bound (8), we have

$$\limsup A_n \leq \lim_{n \rightarrow \infty} \frac{1}{1 - \frac{|\alpha_n - 1|}{x_n}} \lim_{n \rightarrow \infty} B_n = -\log F(x).$$

Thus  $A_n$  is bounded. Also, since  $(\alpha_n - 1)/x_n \rightarrow 0$ , we have, using (7),

$$|\alpha_n - 1|C_n \leq \frac{|\alpha_n - 1|}{x_n} A_n \rightarrow 0.$$

Then, (5) gives us  $\lim A_n = \lim B_n = -\log F(x)$ .  $\square$

As an illustration, suppose  $\alpha_n = \alpha$  for all  $n$ . It is well-known that, in this case, the limiting distribution is Gumbel with the centering,  $d_n = \log n + (\alpha - 1) \log \log n - \log \Gamma(\alpha)$ ,  $c_n = 1$  and  $M_n - d_n \Rightarrow G$ . See for example, Resnick (1987, pp. 72-73). This follows from the above Lemma since  $x_n = x + d_n \sim \log n$  and

$$-\log B_n = -\log n + \log \Gamma(\alpha) + x_n - (\alpha - 1) \log x_n = x + (\alpha - 1) \log \frac{\log n}{x_n} \rightarrow x.$$

We begin with the case where  $\alpha_n \rightarrow \infty$ , but  $\alpha_n = o(\log n)$ .

**Theorem 2.3.** *Assume that  $\alpha_n \rightarrow \infty$ , such that  $\alpha_n = o(\log n)$ . Then*

$$M_n - \log n - (\alpha_n - 1) \log \log n - \xi_n + \log \Gamma(\alpha_n) \Rightarrow G,$$

where  $\xi_n / \log n$  is the unique positive solution of

$$z = \frac{\alpha_n - 1}{\log n} \log \left[ 1 - \frac{\alpha_n}{\log n} \log \frac{\alpha_n}{\log n} + \frac{\alpha_n - \log \log n}{\log n} + \frac{1}{2} \frac{\log \alpha_n}{\log n} + z \right]. \quad (18)$$

*Proof.* First we consider the solution of (18). Define

$$\begin{aligned} \varepsilon_n &= \frac{\alpha_n - \log \log n}{\log n} + \frac{1}{2} \frac{\log \alpha_n}{\log n} - \frac{\alpha_n}{\log n} \log \frac{\alpha_n}{\log n} \\ &= \frac{\alpha_n}{\log n} - \frac{\alpha_n - 1}{\log n} \log \frac{\alpha_n}{\log n} - \frac{1}{2} \frac{\log \alpha_n}{\log n} \\ &\sim -\frac{\alpha_n}{\log n} \log \frac{\alpha_n}{\log n}, \end{aligned}$$

since  $\alpha_n \rightarrow \infty$ , but  $\alpha_n = o(\log n)$ . Thus we have  $\varepsilon_n > 0$  eventually, but  $\varepsilon_n \rightarrow 0$ . Also  $m_n := \log n / (\alpha_n - 1) \rightarrow \infty$ . With these notations, (18) becomes

$$e^{m_n z} = 1 + \varepsilon_n + z.$$

Since  $\varepsilon_n > 0$  eventually, there will be a unique positive solution  $z_n$ . For positive  $z_n$ , we have  $1 + \varepsilon_n + z_n = e^{m_n z_n} > 1 + m_n z_n$ , so that  $z_n < \varepsilon_n / (m_n - 1) \rightarrow 0$ . Hence,

$$\xi_n = o(\log n). \quad (19)$$

Using Stirling's formula, we write

$$x_n = x + d_n = x + \log n + (\alpha_n - 1) \log \log n + \xi_n - \log \Gamma(\alpha_n) \quad (20)$$

$$\begin{aligned}
&= x + \log n + (\alpha_n - 1) \log \log n + \xi_n - \log \sqrt{2\pi} + \alpha_n \\
&\quad - (\alpha_n - 1/2) \log \alpha_n + o(1) \\
&= \log n \left[ 1 - \frac{\alpha_n}{\log n} \log \frac{\alpha_n}{\log n} + \frac{\xi_n}{\log n} + \frac{\alpha_n - \log \log n}{\log n} \right. \\
&\quad \left. + \frac{1}{2} \frac{\log \alpha_n}{\log n} + \frac{x - \log \sqrt{2\pi}}{\log n} + o\left(\frac{1}{\log n}\right) \right] \\
&= R_n \log n \left[ 1 + \frac{x - \log \sqrt{2\pi}}{R_n \log n} + o\left(\frac{1}{R_n \log n}\right) \right], \tag{21}
\end{aligned}$$

where

$$R_n = 1 - \frac{\alpha_n}{\log n} \log \frac{\alpha_n}{\log n} + \frac{\alpha_n - \log \log n}{\log n} + \frac{1}{2} \frac{\log \alpha_n}{\log n} + \frac{\xi_n}{\log n} \rightarrow 1, \tag{22}$$

using (19). Hence

$$x + d_n = x_n \sim d_n \sim \log n \tag{23}$$

and  $\alpha_n - 1 \sim \alpha_n = o(x_n)$ , which gives us (17). Since  $\xi_n/\log n$  is a solution of (18), we have, using (22), and  $\alpha_n/\log n \rightarrow 0$ ,

$$\xi_n = (\alpha_n - 1) \log R_n. \tag{24}$$

Also, using (6), (20) and (21), we have

$$\begin{aligned}
-\log B_n &= x + \xi_n - (\alpha_n - 1) \log \frac{x_n}{\log n} \\
&= x + \xi_n - (\alpha_n - 1) \log R_n \\
&\quad - (\alpha_n - 1) \log \left[ 1 + \frac{x - \log \sqrt{2\pi}}{R_n \log n} + o\left(\frac{1}{R_n \log n}\right) \right] \\
&= x - (\alpha_n - 1) \log \left[ 1 + \frac{x - \log \sqrt{2\pi}}{R_n \log n} + o\left(\frac{1}{R_n \log n}\right) \right] \\
&\sim x - \frac{\alpha_n - 1}{\log n} (x - \log \sqrt{2\pi}) \rightarrow x,
\end{aligned}$$

which gives us (16) and completes the proof using Lemma 2.1.  $\square$

Next we consider the cases where  $\alpha_n$  is bounded above, but  $n\alpha_n \rightarrow \infty$ . Here the centering  $d_n$  depends on the limiting behavior of  $\log \alpha_n/\log n$ . We separate out two cases, depending on whether the ratio  $\log \alpha_n/\log n$  converges to 0, or is bounded away from 0. We consider the latter case first, which includes the case  $\alpha_n = \alpha$ , discussed earlier.

**Theorem 2.4.** *Suppose  $\alpha_n$  is bounded above, but  $\log \alpha_n = o(\log n)$ . Then*

$$M_n - \log n - (\alpha_n - 1) \log \log n + \log \Gamma(\alpha_n) \Rightarrow G.$$

*Proof.* In this case,

$$x_n = x + d_n = x + \log n + (\alpha_n - 1) \log \log n - \log \Gamma(\alpha_n). \tag{25}$$

If  $\alpha_n \rightarrow 0$ , then  $\log \Gamma(\alpha_n) = -\log \alpha_n + o(1) = o(\log n)$ . Otherwise,  $\alpha_n$  is bounded away from both 0 and  $\infty$ . Hence  $\log \Gamma(\alpha_n)$  is bounded and is  $o(\log n)$ . In either case, we have

$$x_n \sim \log n. \tag{26}$$

Also note that

$$d_n \sim x_n \sim \log n \sim \log(n\alpha_n). \tag{27}$$

As  $\alpha_n$  is bounded, we have  $|\alpha_n - 1| = o(x_n)$ , which gives us (17). Using (6) and (25) and the fact  $\alpha_n$  is bounded and (26), we have

$$-\log B_n = x - (\alpha_n - 1) \log \frac{x_n}{\log n} \rightarrow x.$$

This shows (16) and completes the proof of the theorem using Lemma 2.1.  $\square$

Next we consider the case where  $\alpha_n$  is bounded above and  $\log \alpha_n / \log n$  is bounded away from 0.

**Theorem 2.5.** *Assume that  $\alpha_n$  is bounded above,  $n\alpha_n \rightarrow \infty$  and  $\log \alpha_n / \log n$  is bounded away from 0. Then*

$$M_n - \log(n\alpha_n) - (\alpha_n - 1) \log \log(n\alpha_n) \Rightarrow G.$$

*Proof.* From the given conditions, we have  $\log \alpha_n \rightarrow -\infty$  and hence  $\alpha_n \rightarrow 0$ . Here

$$x_n = x + d_n = x + \log(n\alpha_n) + (\alpha_n - 1) \log \log(n\alpha_n). \quad (28)$$

Since  $n\alpha_n \rightarrow \infty$  and  $\alpha_n \rightarrow 0$ , we have

$$d_n \sim x_n \sim \log(n\alpha_n) \rightarrow \infty \quad (29)$$

and thus,  $\alpha_n - 1 = o(x_n)$ , which gives us (17).

Also, using (6), (28), (29) and the fact  $\alpha_n \rightarrow 0$ , we have

$$-\log B_n = x + \log \Gamma(\alpha_n + 1) - (\alpha_n - 1) \log \frac{x_n}{\log(n\alpha_n)} \rightarrow x,$$

Thus we have (16) and the proof is completed using Lemma 2.1.  $\square$

Next we consider  $\alpha_n$ , which goes to 0 at a faster rate. We first look at the case  $n\alpha_n \rightarrow \alpha \in (0, \infty)$ . In this case, the maximum  $M_n$  itself converges to a non-degenerate limiting distribution, which is parametrized by  $\alpha$ . This distribution is not one of the three standard classes of the extreme value distributions.

**Theorem 2.6.** *Assume  $n\alpha_n \rightarrow \alpha \in (0, \infty)$ . Then, for all  $x > 0$ , we have*

$$P[M_n \leq x] \rightarrow F_\alpha(x) := \exp\left(-\alpha \int_x^\infty \frac{e^{-u}}{u} du\right), \quad x \geq 0. \quad (30)$$

*Proof.* The proof follows immediately from (4). Using the dominated convergence theorem, since  $P(Y_{1n} > x) \rightarrow 0$ , we have for all  $x > 0$ ,

$$-\log P[M_n \leq x] \sim nP[Y_{1n} > x] \sim n\alpha_n \int_x^\infty e^{-u} u^{\alpha_n - 1} du \rightarrow \alpha \int_x^\infty \frac{e^{-u}}{u} du.$$

$\square$

When  $n\alpha_n \rightarrow 0$ , there does not exist any non-degenerate limit distribution under linear transformations. However, a power transformation gives Uniform  $(0, 1)$  as the limiting distribution. The idea behind the power scaling is contained in the following lemma. This is used later in Section 3 as well.

**Lemma 2.2.** *Let  $V_n$  be Gamma  $(\delta_n, 1)$  random variables, where  $\delta_n \rightarrow 0$ . Then  $V_n^{\delta_n} \Rightarrow U$ , where  $U$  is a Uniform  $(0, 1)$  random variable. Also, for all  $k > 0$ , we have,  $E[V_n^k] \rightarrow 1/(1+k)$ .*

*Proof.* Observe that for any  $k > 0$ , we have  $E[V_n^{k\delta_n}] = \Gamma(\delta_n(1+k))/\Gamma(\delta_n) \sim 1/(1+k) = E[U^k]$ . Then the result follows easily.  $\square$

Thus,  $Y_n^{\alpha_n}$  is approximately distributed as Uniform  $(0, 1)$ . Since the  $n$ -th power of the maximum of  $n$  i.i.d. Uniform  $(0, 1)$  random variable is again Uniform  $(0, 1)$ , we expect  $M_n^{n\alpha_n}$  to converge to Uniform  $(0, 1)$  distribution.

**Theorem 2.7.** *Assume that  $n\alpha_n \rightarrow 0$ . Then, for all  $0 < x < 1$ ,*

$$P[M_n^{n\alpha_n} \leq x] \rightarrow x.$$

*Proof.* For any  $0 < x < 1$ , we have

$$\begin{aligned} P[M_n^{n\alpha_n} \leq x] &= \left\{ \frac{1}{\Gamma(\alpha_n)} \int_0^{x^{1/(n\alpha_n)}} e^{-u} u^{\alpha_n-1} du \right\}^n \\ &= \left\{ \frac{1}{\alpha_n \Gamma(\alpha_n)} \int_0^{x^{1/n}} e^{-u^{1/\alpha_n}} du \right\}^n \\ &= \left\{ \frac{x^{1/n}}{\Gamma(\alpha_n + 1)} - \frac{1}{\Gamma(\alpha_n + 1)} \int_0^{x^{1/n}} (1 - e^{-u^{1/\alpha_n}}) du \right\}^n \\ &= \frac{x}{\{\Gamma(\alpha_n + 1)\}^n} \left\{ 1 - \frac{1}{n} \int_0^{x^{1/n}} \frac{n(1 - e^{-u^{1/\alpha_n}})}{x^{1/n}} du \right\}^n. \end{aligned} \quad (31)$$

Since  $\Gamma(x)$  is continuously differentiable on  $(0, \infty)$ , we have

$$\{\Gamma(\alpha_n + 1)\}^n = (1 + O(\alpha_n))^n = 1 + O(n\alpha_n) \rightarrow 1. \quad (32)$$

We also have,

$$\int_0^{x^{1/n}} \frac{n(1 - e^{-u^{1/\alpha_n}})}{x^{1/n}} du \leq \frac{n}{x^{1/n}} \int_0^{x^{1/n}} u^{1/\alpha_n} du = n\alpha_n \frac{x^{1/(n\alpha_n)}}{\alpha_n + 1} \rightarrow 0,$$

since  $x^{1/(n\alpha_n)} \rightarrow 0$ , for all  $0 < x < 1$ . Thus, we have,

$$\left\{ 1 - \frac{1}{n} \int_0^{x^{1/n}} \frac{n(1 - e^{-u^{1/\alpha_n}})}{x^{1/n}} du \right\}^n \rightarrow 1.$$

The conclusion follows from (31) and (32).  $\square$

### 3. MAXIMUM OF COORDINATES OF DIRICHLET RANDOM VECTORS

We now extend the results to the maximum of Dirichlet distributions. The discussion is closely related to the Gamma representation of Dirichlet: Recall  $\mathbf{X}_n$  is an  $n$ -dimensional vector having Dirichlet  $(\alpha_1, \dots, \alpha_n; \beta_n)$  distribution. Let  $\{Y_{in} : 1 \leq i \leq n\}$  be i.i.d. Gamma  $(\alpha_n, 1)$  random variables and  $Z_n$  be another independent Gamma  $(\beta_n, 1)$  random variable defined on the same probability space. Then

$$\mathbf{X}_n \stackrel{d}{=} \left( \frac{Y_{1n}}{\sum_{i=1}^n Y_{in} + Z_n}, \dots, \frac{Y_{nn}}{\sum_{i=1}^n Y_{in} + Z_n} \right). \quad (33)$$

This allows us to obtain the limiting distribution corresponding to each case of i.i.d. Gamma random variables. For further calculations, it helps to define  $T_n = (\sum_{i=1}^n Y_{in} + Z_n) / (n\alpha_n + \beta_n)$ . Further, if  $M_n = \vee_{i=1}^n Y_{in}$ , as before, then  $(n\alpha_n + \beta_n) \widetilde{M}_n \stackrel{d}{=} M_n / T_n$ . So for some centering  $d_n$  and scaling  $c_n$ , we shall have

$$\frac{(n\alpha_n + \beta_n) \widetilde{M}_n - d_n}{c_n} \stackrel{d}{=} \frac{M_n / T_n - d_n}{c_n} = \frac{1}{T_n} \left( \frac{M_n - d_n}{c_n} - \frac{d_n}{c_n} (T_n - 1) \right).$$

Note that, if  $n\alpha_n + \beta_n \rightarrow \infty$ , we have  $T_n \xrightarrow{P} 1$ . Thus, we have the following result as a simple application of Slutsky's theorem, which we use repeatedly to obtain the results in Dirichlet case.

**Proposition 3.1.** *Assume  $n\alpha_n + \beta_n \rightarrow \infty$ . Further assume that  $(M_n - d_n)/c_n \Rightarrow F$ . Then  $[(n\alpha_n + \beta_n)\widetilde{M}_n - d_n]/c_n \Rightarrow F$ , whenever*

$$\frac{d_n}{c_n}(T_n - 1) \xrightarrow{P} 1. \quad (34)$$

holds.

We now obtain the results for  $\widetilde{M}_n$  as a corollary to Proposition 3.1 above, whenever  $n\alpha_n \rightarrow \alpha \in (0, \infty]$  and  $n\alpha_n + \beta_n \rightarrow \infty$ .

**Corollary 3.1.** *Assume that  $n\alpha_n \rightarrow \alpha \in (0, \infty]$  and  $n\alpha_n + \beta_n \rightarrow \infty$ . Then*

$$\frac{(n\alpha_n + \beta_n)\widetilde{M}_n - d_n}{c_n} \Rightarrow G.$$

with the centering

$$d_n = \begin{cases} \alpha_n - b_n\sqrt{\alpha_n}, & \text{if } \log n = o(\alpha_n), \\ \zeta_n - \left(1 - \frac{\alpha_n}{\zeta_n}\right) \log\left(1 - \frac{\alpha_n}{\zeta_n}\right), & \text{if } \frac{\alpha_n}{\log n} \text{ is bounded away from both } 0 \text{ and } \infty, \\ \log n + (\alpha_n - 1) \log \log n - \log \Gamma(\alpha_n) + \xi_n, & \text{if } \frac{\alpha_n}{\log n} \rightarrow 0 \text{ and } \alpha_n \rightarrow \infty, \\ \log n + (\alpha_n - 1) \log \log n - \log \Gamma(\alpha_n), & \text{if } \log \alpha_n = o(\log n) \text{ and } \alpha_n \text{ bounded,} \\ \log(n\alpha_n) + (\alpha_n - 1) \log \log(n\alpha_n), & \text{if } \alpha_n \text{ is bounded, } n\alpha_n \rightarrow \infty \text{ and } \frac{\log \alpha_n}{\log n} \\ & \text{is bounded away from } 0, \\ 0, & \text{if } n\alpha_n \rightarrow \alpha \in (0, \infty) \text{ and } \beta_n \rightarrow \infty, \end{cases} \quad (35)$$

and the scaling

$$c_n = \begin{cases} \sqrt{\frac{\alpha_n}{2 \log n}}, & \text{if } \log n = o(\alpha_n), \\ \left(1 - \frac{\alpha_n}{\zeta_n}\right)^{-1}, & \text{if } \frac{\alpha_n}{\log n} \text{ bounded from both } 0 \text{ and } \infty, \\ 1, & \text{if } \frac{\alpha_n}{\log n} \rightarrow 0 \text{ and } n\alpha_n \rightarrow \infty, \\ \frac{\beta_n}{n\alpha_n + \beta_n}, & \text{if } n\alpha_n \rightarrow \alpha \in (0, \infty) \text{ and } \beta_n \rightarrow \infty, \end{cases} \quad (36)$$

where, as in Theorem 2.1,  $b_n$  is the unique root of (2) in the region  $b_n \sim \sqrt{2 \log n}$ , as in Theorem 2.2,  $\zeta_n/\alpha_n$  is the root of (10) which is bigger than 1 and, as in Theorem 2.3,  $\xi_n/\log n$  is the unique positive root of (18).

When  $n\alpha_n \rightarrow \alpha \in (0, \infty)$  and  $\beta_n \rightarrow \infty$ , the statement simplifies to

$$\beta_n \widetilde{M}_n \Rightarrow F_\alpha. \quad (37)$$

*Proof.* We already have  $(M_n - d_n)/c_n$  converges weakly to the appropriate limit,  $G$  or  $F_\alpha$  from the corresponding theorems in Section 2. We only verify (34) one by one by showing

$$\text{Var} \left[ \frac{d_n}{c_n}(T_n - 1) \right] = \left( \frac{d_n}{c_n} \right)^2 \frac{1}{n\alpha_n + \beta_n} \rightarrow 0.$$

First consider  $\log n = o(\alpha_n)$ . From Theorem 2.1, we have  $d_n \sim \alpha_n$ . Hence  $d_n/c_n \sim \sqrt{2\alpha_n \log n}$ . Thus,

$$\left( \frac{d_n}{c_n} \right)^2 \frac{1}{n\alpha_n + \beta_n} \sim \frac{2\alpha_n \log n}{n\alpha_n + \beta_n} \leq \frac{2 \log n}{n} \rightarrow 0.$$

Next we consider the cases where  $\alpha_n \rightarrow \infty$  and  $\alpha_n/\log n$  is bounded above. We consider the cases where the ratio converges to 0 or stays bounded away from 0 together. In the latter case, we know from Theorem 2.2 that  $\alpha_n/\zeta_n$  is bounded away from 1, and thus, in (35), we have  $d_n \sim \zeta_n =$

$O(\alpha_n) = O(\log n)$ . Also, in (36),  $c_n$  stays bounded away from 0 and 1. Hence  $d_n/c_n = O(\log n)$ . If  $\alpha_n = o(\log n)$ , then, by (23), we have  $d_n \sim \log n$ . Thus, in either case,  $d_n/c_n = O(\log n)$ . Hence,

$$\left(\frac{d_n}{c_n}\right)^2 \frac{1}{n\alpha_n + \beta_n} \sim \frac{\log^2 n}{n\alpha_n + \beta_n} \leq \frac{\log^2 n}{n} \rightarrow 0.$$

Next we consider the cases, where  $\alpha_n$  is bounded,  $n\alpha_n \rightarrow \infty$  and  $\log \alpha_n / \log n$  is bounded above. In these cases,  $c_n = 1$ . Using (27) and (29), we have  $d_n \sim \log(n\alpha_n)$ . Then,

$$\left(\frac{d_n}{c_n}\right)^2 \frac{1}{n\alpha_n + \beta_n} \sim \frac{\log^2(n\alpha_n)}{n\alpha_n + \beta_n} \leq \frac{\log^2(n\alpha_n)}{n\alpha_n} \rightarrow 0.$$

The remaining case, where  $n\alpha_n \rightarrow \alpha \in (0, \infty)$  and  $\beta_n \rightarrow \infty$  is trivial since  $d_n = 0$ . The simplified form (37) follows from Slutsky's theorem, since  $n\alpha_n + \beta_n \sim \beta_n$  in this case.  $\square$

Next we consider the case,  $(n\alpha_n + \beta_n)$  remains bounded and hence Proposition 3.1 does not hold.

**Theorem 3.1.** *Assume that  $n\alpha_n \rightarrow \alpha \in (0, \infty)$  and  $\beta_n \rightarrow \beta \in [0, \infty)$ . Then*

$$\widetilde{M}_n \Rightarrow H,$$

where  $H$  is a distribution supported on  $(0, \infty)$  with  $k$ -th moment given by  $\mu_k/\gamma_k$  where  $\gamma_k$  is the  $k$ th moment of the Gamma  $(\alpha + \beta, 1)$  distribution and  $\mu_k$  is the  $k$ -th moment of the distribution  $F_\alpha$ , given by

$$\mu_k = \alpha \int_0^\infty x^{k-1} \exp\left(-x - \alpha \int_x^\infty \frac{e^{-u}}{u} du\right) dx.$$

*Proof.* From the representation (33)  $\widetilde{M}_n \stackrel{d}{=} M_n/S_n$ , where  $S_n = \sum_{i=1}^n Y_{in} + Z_n$ . Further,  $(Y_{1n}/S_n, \dots, Y_{nn}/S_n)$  is independent of  $S_n$  and hence  $M_n/S_n$  is independent of  $S_n$ .

From Theorem 2.6, we have  $M_n \Rightarrow F_\alpha$  and  $S_n$  converges weakly to Gamma  $(\alpha + \beta)$  distribution. Further since  $\{M_n/S_n\}$  is bounded, it is tight. Hence for any subsequence  $n_k$  there is a further subsequence  $n_{k(l)}$ , such that  $\{M_{n_{k(l)}}/S_{n_{k(l)}}, S_{n_{k(l)}}\}$  converges weakly to say  $(V, W)$  where  $W$ , has Gamma  $(\alpha + \beta)$  distribution. Since  $M_n/S_n$  is independent of  $S_n$ ,  $V$  and  $W$  are independent. Hence  $VW$  has distribution  $F_\alpha$ . This implies that  $E(VW)^k = \mu_k$ , that is  $E(V^k) = \mu_k/\gamma_k$ . Since  $V$  has support  $[0, 1]$ , the moments determine the distribution and the proof of the theorem is complete.  $\square$

Now suppose  $n\alpha_n \rightarrow 0$ . In this case, as for Gamma, no linear transformation of  $\widetilde{M}_n$  will have a limiting distribution, however a power transformation will converge. We use Lemma 2.2 to obtain the following theorem.

**Theorem 3.2.** *Assume that  $n\alpha_n \rightarrow 0$  and  $n\alpha_n/\beta_n \rightarrow \lambda \in [0, \infty]$ . Then,*

$$\left(\sigma_n \widetilde{M}_n\right)^{n\alpha_n} \Rightarrow U_\lambda$$

where  $U_\lambda$  is the distribution of  $B_\lambda U + (1 - B_\lambda)$ ,  $U$  and  $B_\lambda$  are independent,  $P(B_\lambda = 1) = \frac{1}{1+\lambda} = 1 - P(B_\lambda = 0)$ ,  $U$  is uniform  $(0, 1)$ , and

$$\sigma_n = \begin{cases} \beta_n, & \text{when } \beta_n \rightarrow \infty, \\ 1, & \text{otherwise.} \end{cases}$$

When  $\lambda = \infty$ , we interpret  $B_\infty$  as the random variable degenerate at 0.

*Proof.* We first consider the case  $\lambda = 0$ . Define  $S_n = \sum_{i=1}^n Y_{in} + Z_n$  as before.

When  $\beta_n \rightarrow \infty$ , clearly  $S_n/(n\alpha_n + \beta_n) \xrightarrow{P} 1$  and  $n\alpha_n + \beta_n \sim \beta_n$ . Thus,  $S_n/\beta_n \xrightarrow{P} 1$ , and hence  $(S_n/\beta_n)^{n\alpha_n} \xrightarrow{P} 1$ .

If  $\beta_n$  is bounded away from 0 and  $\infty$ ,  $S_n$ , which has Gamma  $(n\alpha_n + \beta_n)$  distribution, is a tight sequence on  $(0, \infty)$  and hence  $S_n^{n\alpha_n} \xrightarrow{P} 1$ .

If  $\beta_n \rightarrow 0$ , by Lemma 2.2,  $S_n^{n\alpha_n + \beta_n}$  converges to Uniform  $(0, 1)$  distribution weakly. Hence  $S_n^{n\alpha_n} \xrightarrow{P} 1$ .

So whenever  $\lambda = 0$ ,  $S_n^{n\alpha_n} \xrightarrow{P} 1$ . Also, from Theorem 2.7, we have  $M_n \Rightarrow U$ . Thus by Slutsky's theorem and the gamma representation (33), we have the required result when  $\lambda = 0$ .

When  $\lambda \in (0, \infty]$ , the proof is very similar to that of Theorem 3.1. When  $\lambda = \infty$ , we shall interpret  $\lambda/(1 + \lambda) = 1$  and  $1/(1 + \lambda) = 0$ . From Theorem 2.7, we have that  $M_n^{n\alpha_n} \rightarrow U$ . Also, from Lemma 2.2, we have that  $S_n^{n\alpha_n + \beta_n} \Rightarrow U$ , and hence  $S_n^{n\alpha_n} \Rightarrow U^{\lambda/(1+\lambda)}$ . Then arguing as in Theorem 3.1,  $\{M_n^{n\alpha_n}/S_n^{n\alpha_n}\}$  converges weakly and the  $k$ -th moment of the limit is given by

$$\frac{E[U^k]}{E[U^{\lambda k/(1+\lambda)}]} = \frac{1 + \lambda + k}{(1 + k)(1 + \lambda)} = \frac{\lambda}{1 + \lambda} + \frac{1}{1 + \lambda} \frac{1}{1 + k},$$

which is the  $k$ -th moment of the required limiting distribution.  $\square$

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