

# Nonparametric Depth and Quantile Regression for Functional Data

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## Abstract

We investigate nonparametric regression methods based on spatial depth and quantiles when the response and the covariate are both functions. As in classical quantile regression for finite dimensional data, regression techniques developed here provide insight into the influence of the functional covariate on different parts, like the center as well as the tails, of the conditional distribution of the functional response. Depth and quantile based nonparametric regressions are useful to detect heteroscedasticity in functional regression. We derive the asymptotic behaviour of nonparametric depth and quantile regression estimates, which depend on the small ball probabilities in the covariate space. Our nonparametric regression procedures are used to analyse a dataset about the influence of per capita GDP on saving rates for 125 countries, and another dataset on the effects of per capita net disposable income on the sale of cigarettes in some states in the US.

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# 1 Introduction

Nonparametric regression with functional covariate and real valued response has been extensively studied in the recent literature (see, Masry (2005), Ferraty and Vieu (2006), Ferraty et al. (2007), Rachdi and Vieu (2007), Chagny and Roche (2014), Chagny and Roche (2015), etc.). Since the publication of the seminal paper by Koenker and Bassett Jr (1978), quantile regression has emerged as a powerful statistical tool for investigating the nature of dependence of a response on a covariate. Main advantage of quantile regression is that it provides information about the influence of the covariate on all parts of the conditional distribution of the response unlike the usual mean regression, which focuses only on the center of the conditional distribution. Linear quantile regression, where the response is scalar and the covariate is a function, is studied in Cardot et al. (2005) and Kato (2012). For similar situation, Chen and Müller (2012) considered a semiparametric approach in quantile regression. Nonparametric quantile regression with real valued response and functional covariate is studied by Ferraty and Vieu (2006) and Gardes et al. (2010). The notion of spatial quantiles developed and studied by Breckling and Chambers (1988), Chaudhuri (1996) and Koltchinskii (1997) extends the concept of univariate quantiles to multivariate data. Koltchinskii (1997) showed that spatial quantiles completely characterize a multivariate distribution function as the univariate quantiles characterize a univariate distribution. Spatial quantile regression was considered by Chakraborty (2003) and Cheng and De Gooijer (2007) for problems where both the response and the covariate are finite dimensional. Recently, Chaouch and Laïb (2013) and Chaouch and Laïb (2014) investigated nonparametric spatial quantile regression for finite dimensional multivariate response and functional covariate.

To probe into different parts of the conditional distribution, which are off-center, we investigate nonparametric regression methods based on the conditional spatial distribution and quantiles when both the response and the covariate are functions. The spatial depth is a depth measure based on the spatial distribution. Vardi and Zhang (2000) and Serfling (2002) defined spatial depth for multivariate data, based on the ideas of spatial quantiles in Chaudhuri (1996) and Koltchinskii (1997). Chakraborty and Chaudhuri (2014) extended the concept of spatial depth to infinite dimensional data. In this paper, we employ conditional spatial depth for functional data to investigate the spread of the conditional distribution of the response and detect the presence of heteroscedasticity in functional data. We model functional data as random elements in infinite dimensional spaces. The response is assumed to be an element in a separable Hilbert spaces while the covariate is assumed to be an element in a complete separable metric spaces.

In section 2, the conditional spatial quantiles and conditional maximal depth sets are defined, when the response is a random element in a Hilbert space and the covariate is a random element in a complete separable metric space. We construct measures of conditional spread based on the quantiles and the depth sets. In section 3, kernel based nonparametric estimates for the conditional spatial quantiles, the maximal depth sets and the measures of conditional spread are constructed, which can be used to investigate possible presence of heteroscedasticity in the data. We investigate the asymptotic properties of the estimates in section 4. The estimates of the conditional spatial quantiles and the maximal depth sets are demonstrated using simulated and real data in section 5. The details of the computational procedure of the conditional sample spatial quantiles are given in Appendix A. The proofs of the theorems are

presented in Appendix B.

## 2 Conditional spatial depth and quantiles

Let the response  $\mathbf{Y}$  and the covariate  $\mathbf{X}$  be random elements in a separable Hilbert space  $\mathcal{H}$  and a complete separable metric space  $(\mathcal{M}, d)$ , respectively ( $d$  is the metric on the space  $\mathcal{M}$ ). We adopt the convention of defining  $\|\mathbf{v}\|^{-1}\mathbf{v} = \mathbf{0}$ , when  $\mathbf{v} = \mathbf{0} \in \mathcal{H}$ . Let  $\mathbf{x} \in \mathcal{M}$ . The conditional spatial distribution  $S(\mathbf{y} | \mathbf{x})$  of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  at  $\mathbf{y} \in \mathcal{H}$  is defined as  $S(\mathbf{y} | \mathbf{x}) = E[\|\mathbf{y} - \mathbf{Y}\|^{-1}(\mathbf{y} - \mathbf{Y}) | \mathbf{X} = \mathbf{x}]$ . The conditional spatial depth  $SD(\mathbf{y} | \mathbf{x})$  of a point  $\mathbf{y} \in \mathcal{H}$  given  $\mathbf{X} = \mathbf{x}$  is defined as  $SD(\mathbf{y} | \mathbf{x}) = 1 - \|S(\mathbf{y} | \mathbf{x})\|$  (cf. Chakraborty and Chaudhuri (2014)).

Let  $\boldsymbol{\tau} \in \mathcal{H}$  with  $\|\boldsymbol{\tau}\| < 1$ . Define  $g(\cdot | \mathbf{x}) : \mathcal{H} \rightarrow \mathbb{R}$  by  $g(\mathbf{Q} | \mathbf{x}) = E[\|\mathbf{Q} - \mathbf{Y}\| - \|\mathbf{Y}\| | \mathbf{X} = \mathbf{x}] - \langle \boldsymbol{\tau}, \mathbf{Q} \rangle$ . The conditional  $\boldsymbol{\tau}$ -quantile  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is defined as a minimizer of  $g(\cdot | \mathbf{x})$  (cf. Chakraborty and Chaudhuri (2014)). It is easy to see that  $g(\mathbf{Q} | \mathbf{x})$  is finite for every  $\mathbf{Q}$  and  $\mathbf{x}$ , even when  $E[\|\mathbf{Y}\| | \mathbf{X} = \mathbf{x}] = \infty$ , and if  $E[\|\mathbf{Y}\| | \mathbf{X} = \mathbf{x}] < \infty$ , then a minimizer of  $g(\mathbf{Q} | \mathbf{x})$  is same as a minimizer of  $E[\|\mathbf{Q} - \mathbf{Y}\| | \mathbf{X} = \mathbf{x}] - \langle \boldsymbol{\tau}, \mathbf{Q} \rangle$ . Using the convexity of  $g(\mathbf{Q} | \mathbf{x})$  and the fact that  $g(\mathbf{Q} | \mathbf{x}) \rightarrow \infty$  as  $\|\mathbf{Q}\| \rightarrow \infty$ , one can show that  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  exists in  $\mathcal{H}$ . If the support of the conditional distribution of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is not contained in a straight line in  $\mathcal{H}$ , we get that  $g(\mathbf{Q} | \mathbf{x})$  is strictly convex, and hence  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  becomes the unique minimizer of  $g(\mathbf{Q} | \mathbf{x})$ . If in addition, the conditional distribution of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is non-atomic, it follows that  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  is the unique solution of  $S(\mathbf{y} | \mathbf{x}) = \boldsymbol{\tau}$  for  $\mathbf{y} \in \mathcal{H}$ . Consequently, the conditional spatial median  $\mathbf{Q}(\mathbf{0} | \mathbf{x})$  satisfies  $SD(\mathbf{Q}(\mathbf{0} | \mathbf{x}) | \mathbf{x}) = 1$ .

### 2.1 Conditional maximal depth sets

When the response  $Y$  is a real random variable, the conditional spatial depth  $SD(y | \mathbf{x})$  simplifies to  $SD(y | \mathbf{x}) = 1 - |2F(y | \mathbf{x}) - 1|$ , where  $F(\cdot | \mathbf{x})$  is the conditional distribution function of  $Y$  given  $\mathbf{X} = \mathbf{x}$ . So, the conditional spatial median of  $Y$  given  $\mathbf{X} = \mathbf{x}$  is same as the usual conditional median of  $Y$  given  $\mathbf{X} = \mathbf{x}$ . The conditional inter-quartile interval  $I(\mathbf{x})$  of  $Y$  given  $\mathbf{X} = \mathbf{x}$  is  $[Q(-0.5 | \mathbf{x}), Q(0.5 | \mathbf{x})]$ , where  $Q(-0.5 | \mathbf{x})$  and  $Q(0.5 | \mathbf{x})$  are the conditional first and third quartiles respectively. The conditional inter-quartile range of  $Y$  given  $\mathbf{X} = \mathbf{x}$  is  $Q(0.5 | \mathbf{x}) - Q(-0.5 | \mathbf{x})$ . Denote  $\mathcal{I}$  to be the collection of all intervals  $I$  such that  $P[Y \in I | \mathbf{X} = \mathbf{x}] \geq 0.5$  and  $SD(y_1 | \mathbf{x}) \geq SD(y_2 | \mathbf{x})$  for every  $y_1 \in I$  and  $y_2 \in I^c$ . Then  $I(\mathbf{x}) = \bigcap_{I \in \mathcal{I}} I$ . We can generalize this property of the conditional inter-quartile interval when the response space  $\mathcal{H}$  is a separable Hilbert space.

Given  $\alpha > 0$ , we define the spatial depth based conditional  $\alpha$ -trimmed set  $B(\alpha | \mathbf{x})$  for  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  as  $B(\alpha | \mathbf{x}) = \{\mathbf{y} \in \mathcal{H} | SD(\mathbf{y} | \mathbf{x}) \geq \alpha\}$  (cf. Zuo and Serfling (2000)). Clearly,  $\alpha_1 \geq \alpha_2$  implies that  $B(\alpha_1 | \mathbf{x}) \subseteq B(\alpha_2 | \mathbf{x})$ . Let  $\mathcal{A}_p = \{\alpha > 0 | P[\mathbf{Y} \in B(\alpha | \mathbf{x}) | \mathbf{X} = \mathbf{x}] \geq p\}$ . The set  $\mathcal{A}_p$  is bounded above as  $SD(\mathbf{y} | \mathbf{x}) \leq 1$ . Denote  $\alpha_p = \sup \mathcal{A}_p$ . We define the conditional 100 $p$ % maximal depth set of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  as  $B(\alpha_p | \mathbf{x})$ . So, the set  $B(\alpha_p | \mathbf{x})$  contains 100 $p$ % of the conditional probability mass with its elements having the highest conditional spatial depth. The conditional spatial median of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  belongs to the conditional 100 $p$ % maximal depth set, as it has the highest conditional spatial depth. Note that  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  is the unique solution of  $S(\mathbf{y} | \mathbf{x}) = \boldsymbol{\tau}$  when the conditional distribution of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is non-atomic, and its support is not contained in a straight line in  $\mathcal{H}$ . So, we get  $B(\alpha | \mathbf{x}) = \{\mathbf{y} \in \mathcal{H} | \|S(\mathbf{y} | \mathbf{x})\| \leq 1 - \alpha\} = \{\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) \in \mathcal{H} | \|\boldsymbol{\tau}\| \leq 1 - \alpha\}$ . Hence,

the spatial depth based conditional 100p% maximal depth set of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is  $\{\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) \in \mathcal{H} | \|\boldsymbol{\tau}\| \leq 1 - \alpha_p\}$ . Note that the conditional 50% maximal depth set reduces to the conditional inter-quartile interval when  $\mathcal{H} = \mathbb{R}$ .

The functional box plot defined by Sun and Genton (2011) is a concept closely related to our maximal depth set, which we define in a conditional set up while the concept of functional box plot was introduced in an unconditional set up. The functional box plot was defined using modified band depths of the sample functional observations. Sun and Genton (2011) did not provide any population version of the functional box plot unlike our maximal depth set. If one constructs the functional box plot for the population using their idea of pointwise range, it can be shown using the isolated outliers concept in Section 2.1 of Hubert et al. (2015), that the functional box plot may sometimes become unbounded and hence a non-informative set.

## 2.2 Measures of conditional spread

We define two measures of conditional spread, one based on conditional maximal depth sets and another using conditional spatial quantiles. The measure  $D_1(p | \mathbf{x})$  of conditional spread is defined as the diameter of the conditional 100p% maximal depth set of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$ , i.e.,  $D_1(p | \mathbf{x}) = \sup\{\|\mathbf{y}_1 - \mathbf{y}_2\| | \mathbf{y}_1, \mathbf{y}_2 \in B(\alpha_p | \mathbf{x})\}$ . Equivalently,  $D_1(p | \mathbf{x}) = \sup\{\|\mathbf{Q}(\boldsymbol{\tau}_1 | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau}_2 | \mathbf{x})\| | \|\boldsymbol{\tau}_1\|, \|\boldsymbol{\tau}_2\| \leq 1 - \alpha_p\}$ . Note that  $D_1(0.5 | \mathbf{x})$  generalizes the concept of the conditional interquartile range that we have for a real valued response. We also define a measure of directional spread, based on conditional spatial quantiles. The measure  $D_2(\boldsymbol{\tau} | \mathbf{x})$  of conditional spread is defined as  $D_2(\boldsymbol{\tau} | \mathbf{x}) = \|\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(-\boldsymbol{\tau} | \mathbf{x})\|$ , where  $\|\boldsymbol{\tau}\| < 1$ . Note that  $D_2(\boldsymbol{\tau} | \mathbf{x})$  depends only on the conditional quantiles in the direction of  $\boldsymbol{\tau}$ , while  $D_1(p | \mathbf{x})$  is a ‘global’ measure of the conditional spread in the sense that its definition involves conditional quantiles in all directions. The measure  $D_2(\boldsymbol{\tau} | \mathbf{x})$  also reduces to the conditional inter-quartile range when the response is real valued and  $\boldsymbol{\tau} = 0.5$  as then  $\mathbf{Q}(-\boldsymbol{\tau} | \mathbf{x})$  and  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  become the conditional first and third quartiles respectively.

Both  $D_1(p | \mathbf{x})$  and  $D_2(\boldsymbol{\tau} | \mathbf{x})$  can be used to investigate the presence of heteroscedasticity in regression problems involving functional data, and this will be demonstrated in section 5 using real and simulated data. Investigation of heteroscedasticity using linear regression quantiles for a real valued response and finite dimensional covariate was done as early as in 1982 by Koenker and Bassett Jr. (see Koenker and Bassett Jr (1982)). Subsequently, nonparametric quantile regression was used by Hendricks and Koenker (1992) and Chaudhuri et al. (1997) to study heteroscedasticity in problems involving real valued response and finite dimensional covariate.

## 3 Nonparametric estimates

We describe here the construction of kernel estimators of  $S(\mathbf{y} | \mathbf{x})$ ,  $SD(\mathbf{y} | \mathbf{x})$  and  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$ . Given the sample  $(\mathbf{X}_1, \mathbf{Y}_1), \dots, (\mathbf{X}_n, \mathbf{Y}_n)$ , a Kernel function  $K(\cdot)$  supported on  $[0, 1]$  with a bandwidth  $h_n$ , and for  $\mathbf{y} \in \mathcal{H}$ , the kernel estimator  $\widehat{S}(\mathbf{y} | \mathbf{x})$  of  $S(\mathbf{y} | \mathbf{x})$  is defined as

$$\widehat{S}(\mathbf{y} | \mathbf{x}) = \frac{\sum_{i=1}^n (\|\mathbf{y} - \mathbf{Y}_i\|^{-1} (\mathbf{y} - \mathbf{Y}_i)) K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{\sum_{i=1}^n K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}.$$

The kernel estimator  $\widehat{SD}(\mathbf{y} | \mathbf{x})$  of  $SD(\mathbf{y} | \mathbf{x})$  is defined as  $\widehat{SD}(\mathbf{y} | \mathbf{x}) = 1 - \|\widehat{S}(\mathbf{y} | \mathbf{x})\|$ .

### 3.1 Estimation of conditional quantiles

When the response space  $\mathcal{H}$  is finite dimensional, the conditional sample  $\tau$ -quantile  $\widehat{\mathbf{Q}}_n(\tau | \mathbf{x})$  can be defined as a minimizer of the function  $\widehat{g}_n(\mathbf{Q} | \mathbf{x}) = [\sum_{i=1}^n K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))]^{-1} [\sum_{i=1}^n \|\mathbf{Q} - \mathbf{Y}_i\| K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))] - \langle \tau, \mathbf{Q} \rangle$  in  $\mathcal{H}$ .  $\widehat{\mathbf{Q}}_n(\tau | \mathbf{x})$  can be computed using iterative methods (see Chaudhuri (1996)). When  $\mathcal{H}$  is an infinite dimensional separable Hilbert space, let  $\{\mathbf{e}_n\}$  be an orthonormal basis of  $\mathcal{H}$ . For  $\mathbf{v} \in \mathcal{H}$ , let  $\{v_k\}$  satisfy  $\mathbf{v} = \sum_{k=1}^{\infty} v_k \mathbf{e}_k$ . Let  $\{d_n\}$  be a sequence of positive integers increasing to infinity, and let  $\mathcal{Z}_n = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{d_n}\}$ . Define  $\mathbf{v}^{(n)} = \sum_{k=1}^{d_n} v_k \mathbf{e}_k$  for  $\mathbf{v} \in \mathcal{H}$ . We define the function  $\widehat{g}_n(\cdot | \mathbf{x})$  on  $\mathcal{H}$  as

$$\widehat{g}_n(\mathbf{Q} | \mathbf{x}) = \frac{\sum_{i=1}^n \|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{\sum_{i=1}^n K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))} - \langle \tau^{(n)}, \mathbf{Q} \rangle. \quad (3.1)$$

The conditional sample  $\tau$ -quantile  $\widehat{\mathbf{Q}}_n(\tau | \mathbf{x})$  is defined as a minimizer of  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  in  $\mathcal{Z}_n$ . This method of computing conditional spatial quantiles in an infinite dimensional space is similar to the procedure described in Chakraborty and Chaudhuri (2014) for computing unconditional quantiles.

The function  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  is not Fréchet differentiable at  $\mathbf{Q} = \mathbf{Y}_i^{(n)}$  for each  $i$ . So, we cannot compute  $\widehat{\mathbf{Q}}_n(\tau | \mathbf{x})$  by directly solving the equation  $\widehat{g}_n^{(1)}(\mathbf{Q} | \mathbf{x}) = \mathbf{0}$  in  $\mathcal{Z}_n$  using a straightforward Newton-Raphson type iterative method, where  $\widehat{g}_n^{(1)}(\mathbf{Q} | \mathbf{x})$  is the Fréchet derivative of  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  w.r.t.  $\mathbf{Q}$ . Instead, we first check if any of the  $\mathbf{Y}_i^{(n)}$ s minimize  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  in  $\mathcal{Z}_n$ . We apply the Newton-Raphson method if  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  is not minimized at any of the  $\mathbf{Y}_i^{(n)}$ s. Details of this computational procedure are provided in Appendix A. Further details of the estimation procedure and the choices of the bandwidth  $h_n$ , the basis  $\{\mathbf{e}_n\}$  and  $\{d_n\}$  are discussed in section 5.

### 3.2 Estimation of maximal depth sets and conditional spread

We estimate the conditional 100 $p$ % maximal depth set of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  as follows. We order the sample of responses  $\mathbf{Y}_1, \dots, \mathbf{Y}_n$  by their conditional sample spatial depth  $\widehat{SD}(\mathbf{Y}_i | \mathbf{x})$ , and denote the ordered responses as  $\mathbf{Y}_{[1]}, \dots, \mathbf{Y}_{[n]}$ , where  $\widehat{SD}(\mathbf{Y}_{[i]} | \mathbf{x}) \geq \widehat{SD}(\mathbf{Y}_{[i+1]} | \mathbf{x})$  for  $i = 1, \dots, n-1$ . Given  $p \in (0, 1)$ , let  $i_p$  be the smallest integer such that  $\sum_{i=1}^{i_p} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_{[i]})) \geq p$ . The conditional sample 100 $p$ % maximal depth set of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  is the set  $\{\mathbf{Y}_{[1]}, \dots, \mathbf{Y}_{[i_p]}\}$ , which contains 100 $p$ % of the sample observations having the highest conditional sample spatial depth. Define  $\widehat{D}_1(p | \mathbf{x}) = \max\{\|\mathbf{Y}_{[i]} - \mathbf{Y}_{[j]}\| | i, j = 1, \dots, i_p\}$ , which is an estimator of conditional spread  $D_1(p | \mathbf{x})$ . The other measure of conditional spread,  $D_2(\tau | \mathbf{x})$ , is estimated by  $\widehat{D}_2(\tau | \mathbf{x}) = \|\widehat{\mathbf{Q}}_n(\tau | \mathbf{x}) - \widehat{\mathbf{Q}}_n(-\tau | \mathbf{x})\|$ .

## 4 Asymptotic Properties of estimates

We now proceed to derive the asymptotic properties of the estimates of conditional spatial distribution, depth and quantiles. Recall that the response space is a separable Hilbert space  $\mathcal{H}$ , and the kernel function is denoted by  $K(\cdot)$ . Define the small ball probability function  $\phi(\cdot | \mathbf{x})$  as  $\phi(h | \mathbf{x}) = P[d(\mathbf{x}, \mathbf{X}) \leq h]$ . Denote the conditional probability measure of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{z}$  as  $\mu(\cdot | \mathbf{z})$ . We make the following assumptions on  $\phi(\cdot | \mathbf{x})$ ,  $K(\cdot)$  and  $\mu(\cdot | \mathbf{z})$ .

- C(i)  $\phi(h|\mathbf{x}) > 0$  for all  $h > 0$ . Also, for every  $0 \leq s \leq 1$ ,  $[\phi(h|\mathbf{x})]^{-1}\phi(hs|\mathbf{x}) \rightarrow \rho(s|\mathbf{x})$  as  $h \rightarrow 0^+$ .
- C(ii) The kernel function  $K(\cdot)$  is bounded and supported on  $[0, 1]$  with  $K(1) > 0$ , and it has a continuous bounded derivative on  $(0, 1)$  such that  $K'(u) \leq 0$  for all  $0 < u < 1$ .
- C(iii)  $\mu(\cdot|\mathbf{z}) \rightarrow \mu(\cdot|\mathbf{x})$  weakly as  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ .

Assumptions C(i), C(ii) and C(iii) will be considered to be true throughout this section and Appendix B, and we may not always explicitly mention them. It is easy to see that the function  $\rho(s|\mathbf{x})$  in assumption C(i) is non-decreasing in  $s$ , and  $\rho(1|\mathbf{x}) = 1$ . If the covariate  $\mathbf{X}$  is finite dimensional with a positive probability density at  $\mathbf{x}$  or an infinite dimensional fractal process (see (Ferraty and Vieu, 2006, p. 207)), then  $\rho(s|\mathbf{x}) = s^d$  for some  $d > 0$ . On the other hand, if  $\mathbf{X}$  is an infinite dimensional process like a continuous Gaussian Markov process in  $L_p[0, 1]$ , then  $\phi(h|\mathbf{x}) \sim c_1(\mathbf{x}) \exp[-c_2 h^{-2}]$  as  $h \rightarrow 0$ , where  $c_1(\mathbf{x}) > 0$ , and  $c_2 > 0$  does not depend on  $\mathbf{x}$  (see Theorem 3.1 in Li and Shao (2001) and Theorem 1.1 of Li (2001)). In that case, one can show that  $\rho(s|\mathbf{x}) = I(s = 1)$ . See Proposition 1 in Ferraty et al. (2007) for other examples of  $\rho(s|\mathbf{x})$ . Assumption C(iii) states the continuity of the conditional probability measure of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{z}$ , and this holds in many standard models. For example, consider the location-scale model:  $\mathbf{Y} = \mathbf{m}(\mathbf{X}) + f(\mathbf{X})\mathbf{G}$ , where  $\mathbf{X}$  and  $\mathbf{G}$  are independent random elements in  $(\mathcal{M}, d)$  and  $\mathcal{H}$  respectively,  $E[\|\mathbf{G}\|] < \infty$ , and the functions  $\mathbf{m}(\cdot) : \mathcal{M} \rightarrow \mathcal{H}$  and  $f(\cdot) : \mathcal{M} \rightarrow \mathbb{R}$  are both continuous at  $\mathbf{x}$ . It is easy to verify that assumption C(iii) holds in this model.

From assumption C(ii), we get  $0 < l \leq K(u) \leq L < \infty$  for all  $u \in [0, 1]$ , where  $L = K(0)$  and  $l = K(1)$ . Consequently,  $l^j \phi(h|\mathbf{x}) \leq E[K^j(h^{-1}d(\mathbf{x}, \mathbf{X}))] \leq L^j \phi(h|\mathbf{x})$  for any  $h > 0$  and any positive integer  $j$ . Denote  $F_{(j)}(h|\mathbf{x}) = [\phi(h|\mathbf{x})]^{-1} E[K^j(h^{-1}d(\mathbf{x}, \mathbf{X}))]$ . Using assumptions C(i), C(ii) and the arguments similar to those in the proof of Lemma 2 in Ferraty et al. (2007), we get  $F_{(1)}(h|\mathbf{x}) \rightarrow K(1) - \int_0^1 \rho(s|\mathbf{x})K'(s)ds = E_{(1)}(\mathbf{x})$  (say) and  $F_{(2)}(h|\mathbf{x}) \rightarrow K^2(1) - 2 \int_0^1 \rho(s|\mathbf{x})K(s)K'(s)ds = E_{(2)}(\mathbf{x})$  (say) as  $h \rightarrow 0$ . Clearly,  $0 < l^j \leq E_{(j)}(\mathbf{x}) \leq L^j$  for  $j = 1, 2$ .

Denote  $E_n = E[K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))]$ . Define the bilinear operator  $\gamma(\mathbf{y}|\mathbf{z})(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  as  $\gamma(\mathbf{y}|\mathbf{z})(\mathbf{v}, \mathbf{w}) = Cov[\langle \|\mathbf{y} - \mathbf{Y}\|^{-1}(\mathbf{y} - \mathbf{Y}), \mathbf{v} \rangle, \langle \|\mathbf{y} - \mathbf{Y}\|^{-1}(\mathbf{y} - \mathbf{Y}), \mathbf{w} \rangle | \mathbf{X} = \mathbf{z}]$ . Under the following additional assumptions, we get the asymptotic normality of the conditional sample spatial distribution  $\widehat{S}(\mathbf{y}|\mathbf{x})$  and the rate of convergence of the conditional sample spatial depth  $\widehat{SD}(\mathbf{y}|\mathbf{x})$ .

- A-1. The bandwidth  $h_n$  satisfies  $h_n \rightarrow 0$  and  $(n\phi(h_n|\mathbf{x}))^{-1} \log n \rightarrow 0$  as  $n \rightarrow \infty$ .
- A-2. For  $\mathbf{y} \in \mathcal{H}$  and for  $d(\mathbf{z}, \mathbf{x}) \leq C_1$ , we have  $(d(\mathbf{x}, \mathbf{z}))^{-1} \|S(\mathbf{y}|\mathbf{z}) - S(\mathbf{y}|\mathbf{x})\| \leq s_1$ , where  $C_1$  and  $s_1$  are positive constants.

Note that assumption A-2 is a smoothness condition on the conditional spatial distribution  $S(\mathbf{y}|\mathbf{z})$  for  $\mathbf{z}$  lying in a neighbourhood of  $\mathbf{x}$ , which will be required to derive the order of convergence of  $\widehat{SD}(\mathbf{y}|\mathbf{x})$  to  $SD(\mathbf{y}|\mathbf{x})$  in Theorem 4.1. Assumption A-2 holds in many models. As an example, consider the location-scale model considered earlier:  $\mathbf{Y} = \mathbf{m}(\mathbf{X}) + f(\mathbf{X})\mathbf{G}$ , with  $\mathbf{X}$  and  $\mathbf{G}$  being independent random elements in  $(\mathcal{M}, d)$  and  $\mathcal{H}$  respectively. Assume that  $E[\|\mathbf{G}\|] < \infty$ , and  $\mathbf{m}(\cdot) : \mathcal{M} \rightarrow \mathcal{H}$  and  $f(\cdot) : \mathcal{M} \rightarrow \mathbb{R}$  are both Lipschitz continuous at  $\mathbf{x}$ . Also, let assumption B-3, which is

stated and discussed later in this section, holds. Then, one can verify that assumption A-2 holds.

**Theorem 4.1.** Denote  $M_n(\mathbf{y} | \mathbf{x}) = [\sum_{i=1}^n [S(\mathbf{y} | \mathbf{X}_i) - S(\mathbf{y} | \mathbf{x})]K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))] / [\sum_{i=1}^n K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))]$ . Under assumption A-1,  $\sqrt{n\phi(h_n | \mathbf{x})}(\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})) \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ , where  $\mathbf{W}$  is a Gaussian random element in  $\mathcal{H}$  with mean  $\mathbf{0}$  and covariance operator  $[(E_{(1)}(\mathbf{x}))^{-2}E_{(2)}(\mathbf{x})]\gamma_{\mathbf{y}}(\cdot, \cdot | \mathbf{x})$ . Also, under assumptions A-1 and A-2,  $|\widehat{SD}(\mathbf{y} | \mathbf{x}) - SD(\mathbf{y} | \mathbf{x})| = O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1} + h_n)$  as  $n \rightarrow \infty$ .

$M_n(\mathbf{y} | \mathbf{x})$  can be viewed as the bias in the kernel estimate  $\widehat{S}(\mathbf{y} | \mathbf{x})$  of  $S(\mathbf{y} | \mathbf{x})$ . From assumptions A-1 and A-2, it follows that  $\|M_n(\mathbf{y} | \mathbf{x})\| = O(h_n)$  almost surely as  $n \rightarrow \infty$ . So, from Theorem 4.1, we get  $\|\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x})\| = O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1} + h_n)$  as  $n \rightarrow \infty$ .

If  $h_n$  satisfies  $\sqrt{n\phi(h_n | \mathbf{x})}h_n \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\sqrt{n\phi(h_n | \mathbf{x})}M_n(\mathbf{y} | \mathbf{x}) \rightarrow \mathbf{0}$  in probability as  $n \rightarrow \infty$ , and from Theorem 4.1, we get  $\sqrt{n\phi(h_n | \mathbf{x})}(\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x})) \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ . Note that the order of convergence for both  $\widehat{S}(\mathbf{y} | \mathbf{x})$  and  $\widehat{SD}(\mathbf{y} | \mathbf{x})$  is  $O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1} + h_n)$ . The terms  $[\sqrt{n\phi(h_n | \mathbf{x})}]^{-1}$  and  $h_n$  in the order of convergence come from the variance and the bias of the estimate  $\widehat{S}(\mathbf{y} | \mathbf{x})$ , respectively. For a choice of bandwidth  $h_n$ , which balances the bias and the variance,  $\sqrt{n\phi(h_n | \mathbf{x})}h_n$  will be bounded and away from 0 as  $n \rightarrow \infty$ .

Let the covariate  $\mathbf{X}$  be either finite dimensional or a fractal-type process. Then,  $\phi(h_n | \mathbf{x}) = O(h_n^d)$ , where  $d > 0$ . In that case, the choice of the bandwidth  $h_n$ , which balances the asymptotic order of the bias, i.e.,  $O(h_n)$ , and that of the variance, i.e.,  $O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1})$ , is  $c_3n^{-(d+2)^{-1}}$  for some constant  $c_3 > 0$  depending on  $\mathbf{x}$ . It is easy to see that this choice of  $h_n$  satisfies assumption A-1, and for this choice, the optimum rate of convergence of both  $\widehat{SD}(\mathbf{y} | \mathbf{x})$  and  $\widehat{S}(\mathbf{y} | \mathbf{x})$  is  $O_P(n^{-(d+2)^{-1}})$ . On the other hand, if  $\mathbf{X}$  is an infinite dimensional random element like a continuous Gaussian Markov process in  $L_p[0, 1]$ , then  $\phi(h_n | \mathbf{x}) \sim c_1(\mathbf{x})\exp[-c_2h_n^{-2}]$  as  $h_n \rightarrow 0$ , where  $c_1(\mathbf{x}) > 0$  and  $c_2 > 0$ . Here, it is easy to see that if  $\sqrt{n\phi(h_n | \mathbf{x})}h_n$  is bounded above, then  $(n\phi(h_n | \mathbf{x}))^{-1} \log n$  is bounded away from 0 as  $n \rightarrow \infty$ . So, in such a case, there exists no choice of  $h_n$  which simultaneously satisfies assumption A-1 and balances the orders of the bias and the variance. It is easy to verify that assumption A-1 is satisfied if  $h_n = c_4([\sqrt{\log n}]^{-1})$  for a constant  $c_4 > \sqrt{c_2}$ . Then, the rate of convergence of both  $\widehat{SD}(\mathbf{y} | \mathbf{x})$  and  $\widehat{S}(\mathbf{y} | \mathbf{x})$  is  $O_P([\sqrt{\log n}]^{-1})$ . Note that the dimension of the response  $\mathbf{Y}$  does not have any effect on the convergence rate in this case.

We next turn our attention to the conditional sample spatial quantiles. Denote the conditional probability measures of  $\mathbf{Y}$  and  $\mathbf{Y}^{(n)}$  given  $\mathbf{X} = \mathbf{z}$  as  $\mu(\cdot | \mathbf{z})$  and  $\mu^{(n)}(\cdot | \mathbf{z})$ , respectively. The following assumptions are required for the subsequent results.

- B-1. The bandwidth  $h_n$  satisfies  $h_n \rightarrow 0$  and  $(n\phi(h_n | \mathbf{x}))^{(1/2)-2\alpha} \log n \rightarrow 0$  as  $n \rightarrow \infty$ , where  $1/4 < \alpha \leq 1/2$  is a constant.
- B-2. There exists a constant  $C_2 > 0$  and a positive integer  $N_1$  such that whenever  $d(\mathbf{x}, \mathbf{z}) \leq C_2$ ,  $\mu^{(N_1)}(\cdot | \mathbf{z})$  is non-atomic and its support is not contained in a straight line in  $\mathcal{H}$ , i.e., there exist no  $\mathbf{a}, \mathbf{b} \in \mathcal{H}$  such that  $\mu^{(N_1)}(\{\mathbf{v} \in \mathcal{H} : \mathbf{v} = \mathbf{a} + t\mathbf{b}, t \in (-\infty, \infty)\} | \mathbf{z}) = 1$ .
- B-3. There exists a constant  $C_3 > 0$  and a positive integer  $N_2$  such that whenever  $d(\mathbf{x}, \mathbf{z}) \leq C_3$  and  $n \geq N_2$ , we have for each  $C > 0$ ,  $E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-2} | \mathbf{X} =$

$\mathbf{z}] \leq s_2(C)$  for all  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ . Here  $s_2(C)$  is a positive constant depending on  $C$ .

Note that  $\mu^{(N_1)}(\cdot | \mathbf{z})$  is non-atomic implies that both  $\mu(\cdot | \mathbf{z})$  and  $\mu^{(n)}(\cdot | \mathbf{z})$  are non-atomic for all  $n \geq N_1$ . Also, if the support of  $\mu^{(N_1)}(\cdot | \mathbf{z})$  is not contained in a straight line in  $\mathcal{H}$ , then the support of both  $\mu(\cdot | \mathbf{z})$  and  $\mu^{(n)}(\cdot | \mathbf{z})$  are not contained in any straight line in  $\mathcal{H}$  for all  $n \geq N_1$ . Note that assumption B-3 holds when some conditional trivariate marginal distribution of  $(\langle \mathbf{Y}, \mathbf{e}_1 \rangle, \langle \mathbf{Y}, \mathbf{e}_2 \rangle, \dots)$  given  $\mathbf{X} = \mathbf{z}$  has a density that is uniformly bounded on bounded subsets for  $\mathbf{z}$  satisfying  $d(\mathbf{x}, \mathbf{z}) \leq C_3$ .

Define  $g_n(\mathbf{Q} | \mathbf{z}) = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| | \mathbf{X} = \mathbf{z}] - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle$ , and  $\tilde{g}_n(\mathbf{Q} | \mathbf{x}) = E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))]$ . For  $\mathbf{Q} \in \mathcal{H}$ , define the Fréchet derivatives  $g^{(1)}(\mathbf{Q} | \mathbf{z})$ ,  $g_n^{(1)}(\mathbf{Q} | \mathbf{z})$ ,  $\tilde{g}_n^{(1)}(\mathbf{Q} | \mathbf{x})$ ,  $\hat{g}_n^{(1)}(\mathbf{Q} | \mathbf{x})$  by

$$\begin{aligned} g^{(1)}(\mathbf{Q} | \mathbf{z}) &= \frac{\partial}{\partial \mathbf{Q}} g(\mathbf{Q} | \mathbf{z}) = E[\|\mathbf{Q} - \mathbf{Y}\|^{-1}(\mathbf{Q} - \mathbf{Y}) | \mathbf{X} = \mathbf{z}] - \boldsymbol{\tau}, \\ g_n^{(1)}(\mathbf{Q} | \mathbf{z}) &= \frac{\partial}{\partial \mathbf{Q}} g_n(\mathbf{Q} | \mathbf{z}) = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}^{(n)}) | \mathbf{X} = \mathbf{z}] - \boldsymbol{\tau}^{(n)}, \\ \tilde{g}_n^{(1)}(\mathbf{Q} | \mathbf{x}) &= \frac{\partial}{\partial \mathbf{Q}} \tilde{g}_n(\mathbf{Q} | \mathbf{x}) = E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}^{(n)}) - \boldsymbol{\tau}^{(n)}) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))], \\ \hat{g}_n^{(1)}(\mathbf{Q} | \mathbf{x}) &= \frac{\partial}{\partial \mathbf{Q}} \hat{g}_n(\mathbf{Q} | \mathbf{x}) = \frac{\sum_{i=1}^n (\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}_i^{(n)})) K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{\sum_{i=1}^n K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))} - \boldsymbol{\tau}^{(n)}. \end{aligned}$$

Now, for each  $\mathbf{Q} \in \mathcal{H}$ , define  $g^{(2)}(\mathbf{Q} | \mathbf{z})(\cdot)$ ,  $g_n^{(2)}(\mathbf{Q} | \mathbf{z})(\cdot)$ ,  $\tilde{g}_n^{(2)}(\mathbf{Q} | \mathbf{x})(\cdot) : \mathcal{H} \rightarrow \mathcal{H}$  by

$$\begin{aligned} (g^{(2)}(\mathbf{Q} | \mathbf{z}))(\mathbf{h}) &= \left( \frac{\partial^2}{\partial Q^2} g(\mathbf{Q} | \mathbf{z}) \right) (\mathbf{h}) = E \left[ \frac{\mathbf{h}}{\|\mathbf{Q} - \mathbf{Y}\|} - \frac{\langle \mathbf{h}, \mathbf{Q} - \mathbf{Y} \rangle (\mathbf{Q} - \mathbf{Y})}{\|\mathbf{Q} - \mathbf{Y}\|^3} \mid \mathbf{X} = \mathbf{z} \right], \\ (g_n^{(2)}(\mathbf{Q} | \mathbf{z}))(\mathbf{h}) &= \left( \frac{\partial^2}{\partial Q^2} g_n(\mathbf{Q} | \mathbf{z}) \right) (\mathbf{h}) \\ &= E \left[ \frac{\mathbf{h}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \frac{\langle \mathbf{h}, \mathbf{Q} - \mathbf{Y}^{(n)} \rangle (\mathbf{Q} - \mathbf{Y}^{(n)})}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^3} \mid \mathbf{X} = \mathbf{z} \right], \\ (\tilde{g}_n^{(2)}(\mathbf{Q} | \mathbf{x}))(\mathbf{h}) &= \left( \frac{\partial^2}{\partial Q^2} \tilde{g}_n(\mathbf{Q} | \mathbf{x}) \right) (\mathbf{h}) \\ &= E \left[ \left[ \frac{\mathbf{h}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \frac{\langle \mathbf{h}, \mathbf{Q} - \mathbf{Y}^{(n)} \rangle (\mathbf{Q} - \mathbf{Y}^{(n)})}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^3} \right] \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))}{E_n} \right]. \end{aligned}$$

From assumptions B-1 and B-2, it follows that for  $\mathbf{z}$  lying in a neighbourhood of  $\mathbf{x}$  and for all sufficiently large  $n$ ,  $g_n(\mathbf{Q} | \mathbf{z})$  and  $\tilde{g}_n(\mathbf{Q} | \mathbf{x})$  have unique minimizers  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{z})$  and  $\tilde{\hat{\mathbf{Q}}}_n(\boldsymbol{\tau} | \mathbf{x})$  respectively in  $\mathcal{Z}_n$ , and  $g_n^{(1)}(\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{z}) = \tilde{g}_n^{(1)}(\tilde{\hat{\mathbf{Q}}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) = 0$ . From assumptions B-1, B-2 and B-3, we get that for all sufficiently large  $n$  and for  $\mathbf{z}$  lying in a neighbourhood of  $\mathbf{x}$ ,  $g^{(2)}(\mathbf{Q} | \mathbf{z})(\cdot)$ ,  $g_n^{(2)}(\mathbf{Q} | \mathbf{z})(\cdot)$  and  $\tilde{g}_n^{(2)}(\mathbf{Q} | \mathbf{x})(\cdot)$  are continuous linear operators on  $\mathcal{H}$  for all  $\mathbf{Q}$ . We now state a Bahadur type asymptotic linearisation of  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$ .

**Theorem 4.2.** *Let the assumptions B-1 through B-3 hold, and  $(n\phi(h_n | \mathbf{x}))^{-(1-2\alpha)} d_n \rightarrow$*

$c_5 > 0$  as  $n \rightarrow \infty$ , where  $\alpha$  is as described in assumption B-1. Then,

$$\begin{aligned} & \widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \widetilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) \\ &= -[\widetilde{g}_n^{(2)}(\widetilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} \left[ \frac{n^{-1} \sum_{i=1}^n \left[ \frac{\widetilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\widetilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))} \right] \\ & \quad + R_n(\mathbf{x}), \end{aligned}$$

where  $\|R_n(\mathbf{x})\| = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ , and  $\epsilon_n = (n\phi(h_n | \mathbf{x}))^{-\alpha} \sqrt{\log n}$ .

Define  $\mathbf{B}_n(\boldsymbol{\tau} | \mathbf{x}) = \widetilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})$ . We view it as a kind of bias in the estimate  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$ . Then, one can show that under the assumptions of Theorem 4.2,  $\|\mathbf{B}_n(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . In addition, suppose that there exist a constant  $C_4 > 0$  and a positive integer  $N_3$  such that whenever  $d(\mathbf{x}, \mathbf{z}) \leq C_4$  and  $n \geq N_3$ , we have, for each  $C > 0$ ,  $(d(\mathbf{x}, \mathbf{z}))^{-1} \|g_n^{(1)}(\mathbf{Q} | \mathbf{z}) - g_n^{(1)}(\mathbf{Q} | \mathbf{x})\| \leq s_3(C)$  for all  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ , where  $s_3(C)$  is a positive constant depending on  $C$ . Then, it can be shown that  $\|\mathbf{B}_n(\boldsymbol{\tau} | \mathbf{x})\| = O(h_n)$  as  $n \rightarrow \infty$  (see Lemma B.11). It is easy to show that all these hold when the response  $\mathbf{Y}$  satisfies a location-scale model as discussed in connection with assumption A-2 earlier.

We can show that  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$  (see Lemma B.4). As a consequence, we get  $\|\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  almost surely as  $n \rightarrow \infty$  for each  $\boldsymbol{\tau}$  with  $\|\boldsymbol{\tau}\| < 1$ . Recall that we defined an estimator of the conditional spread as  $\widehat{D}_2(\boldsymbol{\tau} | \mathbf{x}) = \|\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \widehat{\mathbf{Q}}_n(-\boldsymbol{\tau} | \mathbf{x})\|$ . Hence, we get  $\widehat{D}_2(\boldsymbol{\tau} | \mathbf{x}) \rightarrow D_2(\boldsymbol{\tau} | \mathbf{x})$  almost surely as  $n \rightarrow \infty$ .

We next state a result on the asymptotic normality of the conditional sample spatial quantile. Recall  $\gamma(\mathbf{y} | \mathbf{z})(\cdot, \cdot)$  defined before Theorem 4.1. Let  $\gamma_0(\mathbf{Q} | \mathbf{x})(\cdot) : \mathcal{H} \rightarrow \mathcal{H}$  be the continuous linear operator obtained from  $\gamma(\mathbf{Q} | \mathbf{x})(\cdot, \cdot)$ , i.e.,  $\langle \gamma_0(\mathbf{Q} | \mathbf{x})(\mathbf{v}), \mathbf{w} \rangle = \gamma(\mathbf{Q} | \mathbf{x})(\mathbf{v}, \mathbf{w})$  for all  $\mathbf{v}, \mathbf{w} \in \mathcal{H}$ .

**Theorem 4.3.** *Suppose that  $\mathbf{Y}$  has the conditional Karhunen-Loeve expansion given by  $\mathbf{Y} = \mathbf{m}(\mathbf{x}) + \sum_{k=1}^{\infty} \lambda_k Z_k \boldsymbol{\psi}_k$ , where the  $Z_k$ 's are conditionally uncorrelated random variables with conditional mean 0 and conditional variance 1 given  $\mathbf{X} = \mathbf{x}$ , and the  $\lambda_k^2$ 's and the  $\boldsymbol{\psi}_k$ 's are the eigenvalues and the corresponding eigenfunctions of the conditional covariance operator of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$ . Also, let  $\sqrt{n\phi(h_n | \mathbf{x})} \|\boldsymbol{\tau} - \boldsymbol{\tau}^{(n)}\| \rightarrow 0$ ,  $\sqrt{n\phi(h_n | \mathbf{x})} \|\mathbf{m}(\mathbf{x}) - (\mathbf{m}(\mathbf{x}))^{(n)}\| \rightarrow 0$  and  $(n\phi(h_n | \mathbf{x}) \sum_{k>d_n} \lambda_k^2) \rightarrow 0$  as  $n \rightarrow \infty$ . Suppose the assumptions B-1 through B-3 hold, and  $(n\phi(h_n | \mathbf{x}))^{-(1-2\alpha)} d_n \rightarrow c_5 > 0$  as  $n \rightarrow \infty$ , where  $\alpha$  is as in assumption B-1. Then,  $\sqrt{n\phi(h_n | \mathbf{x})} [\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{B}_n(\boldsymbol{\tau} | \mathbf{x})] \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ , where  $\mathbf{W}$  is a Gaussian random element in  $\mathcal{H}$  with mean  $\mathbf{0}$  and covariance operator  $[(E_{(1)}(\mathbf{x}))^{-2} E_{(2)}(\mathbf{x})][g^{(2)}(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} \gamma_0(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})[g^{(2)}(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1}$ .*

The assumption concerning the conditional Karhunen-Loeve expansion of  $\mathbf{Y}$  and the conditions  $\sqrt{n\phi(h_n | \mathbf{x})} \|\boldsymbol{\tau} - \boldsymbol{\tau}^{(n)}\| \rightarrow 0$ ,  $\sqrt{n\phi(h_n | \mathbf{x})} \|\mathbf{m}(\mathbf{x}) - (\mathbf{m}(\mathbf{x}))^{(n)}\| \rightarrow 0$  and  $(n\phi(h_n | \mathbf{x}) \sum_{k>d_n} \lambda_k^2) \rightarrow 0$  as  $n \rightarrow \infty$  are required to ensure  $\sqrt{n\phi(h_n | \mathbf{x})} \|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . Note that  $\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) \in \mathcal{Z}_n$  can be viewed as a finite dimensional approximation of  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$ , and these conditions are necessary to control the asymptotic bias arising from such an approximation. For further insights into these conditions, readers are referred to Chakraborty and Chaudhuri (2014), who

used similar assumptions in their Theorem 3.4 to derive asymptotic normality of unconditional spatial quantiles in Hilbert spaces. Note that when the response space  $\mathcal{H}$  is finite dimensional, we can take  $\alpha = 1/2$ ,  $\mathcal{Z}_n = \mathcal{H}$  and  $d_n = \text{dimension}(\mathcal{H})$  for all  $n$ . Then, there is no such bias, and our theorems yield the Bahadur representation and the asymptotic normality for conditional spatial quantiles of a finite dimensional response as a special case.

Suppose that the covariate  $\mathbf{X}$  is either finite dimensional or a fractal-type process. Then, like in the case of the conditional spatial distribution, an appropriate choice of  $h_n$ , which balances the asymptotic order of the bias term  $\mathbf{B}_n(\boldsymbol{\tau} | \mathbf{x})$  and the asymptotic order of the variance of the estimate  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$ , is  $c_3 n^{-(d+2)^{-1}}$  for some  $d > 0$ . The rate of convergence of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  is  $O_P(n^{-(d+2)^{-1}})$  in that case. If  $\mathbf{X}$  is a continuous Gaussian Markov process in  $L_p[0, 1]$ , there is no choice of  $h_n$ , which simultaneously satisfies B-1 and  $\sqrt{n\phi(h_n | \mathbf{x})}h_n = O(1)$  as  $n \rightarrow \infty$  as observed in the discussion following Theorem 4.1. However, if one chooses  $h_n = c_4([\sqrt{\log n}]^{-1})$  with an appropriate constant  $c_4 > 0$  in a similar way as done before, then assumption B-1 is satisfied, and the rate of convergence of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  becomes  $O_P([\sqrt{\log n}]^{-1})$  as  $n \rightarrow \infty$ .

## 5 Data Analysis

In this section, quantile regression and conditional maximal depth sets are demonstrated using simulated and real data. We consider three datasets here. The first one is a simulated data generated from a heteroscedastic model. The second dataset is a real data concerning per capita GDP and saving rate in 125 countries over a time period of 26 years. The third dataset is about cigarette sales and net disposable income in 46 states in the USA over a 30-year period.

In all our analysis, we consider the functional response and the functional covariate as random elements in appropriate  $L_2$  spaces. For the sake of simplicity, we choose the indicator function on  $[0, 1]$  as the kernel  $K(\cdot)$ , and the bandwidth  $h$  is chosen by leave-one-out cross validation in the following way. Let  $m_n^{(-i)}(\mathbf{z}, h)$  denote the conditional spatial median estimated at  $\mathbf{X} = \mathbf{z}$  using the bandwidth  $h$  and leaving out the  $i$ -th sample observation  $(\mathbf{X}_i, \mathbf{Y}_i)$ . We choose the bandwidth  $h = h_{opt}$  such that  $h_{opt} = \arg \min_h n^{-1} \sum_{i=1}^n \|m_n^{(-i)}(\mathbf{X}_i, h) - \mathbf{Y}_i\|$ . The results do not vary much for the data analysed below if one uses the conditional mean or conditional pointwise median instead of the conditional spatial median for the cross validation. Let  $C_n(\mathbf{x}) = \{\mathbf{Y}_i | d(\mathbf{x}, \mathbf{X}_i) \leq h_{opt}\}$  ( $d(\cdot, \cdot)$  is the  $L_2$  metric here), and  $\#[C_n(\mathbf{x})]$  be the cardinality of  $C_n(\mathbf{x})$ . While computing  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$ , we take  $d_n = \lfloor \min\{\sqrt{\#[C_n(\mathbf{x})]}, 2\#[C_n(\mathbf{x})]^{(1/3)}\} \rfloor$ , where  $\lfloor r \rfloor$  denotes the largest integer less than or equal to  $r$ . Then,  $\alpha = (1/3)$  and  $c_5 = 2$ , where  $\alpha$  and  $c_5$  are as in Theorem 4.2. To fix the basis  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{d_n}\}$ , we first estimate the conditional covariance operator of  $\mathbf{Y}$  given  $\mathbf{X} = \mathbf{x}$  using the same kernel function  $K(\cdot)$  and the bandwidth  $h$ . Then, the eigenfunction corresponding to the  $k$ -th largest eigenvalue of the estimated conditional covariance operator is taken as  $\mathbf{e}_k$  for  $k = 1, \dots, d_n$ .

### 5.1 Simulated Data

We consider a regression model, where the covariate  $\mathbf{X}(t) = \text{Uexp}(t)$  for  $t \in [0, 1]$  with  $U \sim \text{Uniform}[0, 1]$ , and the response  $\mathbf{Y}(t) = \|\mathbf{X}\|\mathbf{B}(t)$ , where  $\mathbf{B}(t)$  is the standard

Brownian Motion on  $[0, 1]$ , and  $\|\cdot\|$  is the  $L_2$  norm. We simulate 100 observations from this model, and construct the quantiles regression estimates and conditional maximal depth sets. The sample size is 100.

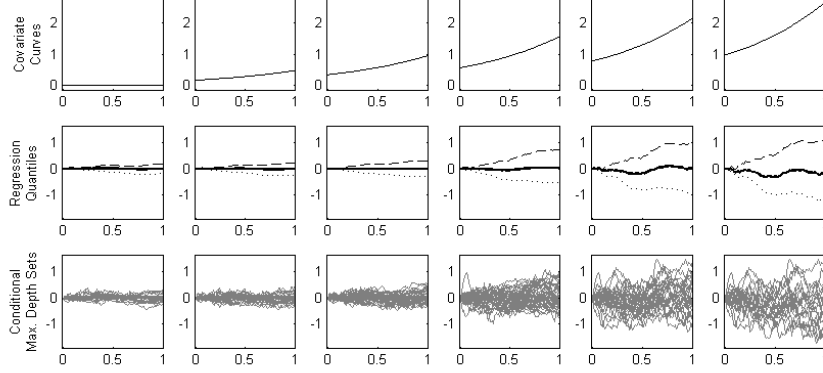


Figure 1: Plots of the selected covariate curves (1<sup>st</sup> row) and the corresponding conditional spatial quantiles (2<sup>nd</sup> row) and conditional maximal depth sets (3<sup>rd</sup> row) for the simulated data. The dashed, the solid and the dotted curves in the 2<sup>nd</sup> row are  $\hat{Q}_n(0.5\mathbf{u} | \mathbf{x})$ ,  $\hat{Q}_n(\mathbf{0} | \mathbf{x})$  and  $\hat{Q}_n(-0.5\mathbf{u} | \mathbf{x})$ , respectively.

The value of the bandwidth  $h$  obtained through cross validation is 0.68. We compute the 50% conditional maximal depth sets, the conditional spatial median and conditional spatial quantiles corresponding to  $\tau = \pm 0.5\mathbf{u}$ , where  $\mathbf{u}$  is the first principal component of the estimated conditional covariance operator of the response. To demonstrate the conditional quantile curves and depth sets, we order the covariate curves by their  $L_2$  norms, and choose 6 covariate curves whose ranks are equidistant in this ordering. The conditional spatial quantiles and the conditional maximal depth sets for these 6 covariate curves are plotted in Figure 1. We see that the spatial quantiles and the maximal depth sets clearly capture the change of the conditional

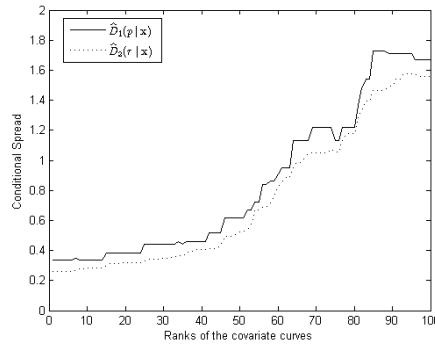


Figure 2: Plots of  $\hat{D}_1(p | \mathbf{x})$  and  $\hat{D}_2(\tau | \mathbf{x})$  against the ranks of the covariate curves in the ordering by their  $L_2$  norms, in the heteroscedastic model.

distributions of the response given the selected covariate curves, and the heteroscedasticity present in the data. The conditional spread measures  $\hat{D}_1(p|\mathbf{x})$  and  $\hat{D}_2(\tau|\mathbf{x})$  for all the covariate curves are plotted in Figure 2 against the ranks of all the covariate curves in the ordering by their  $L_2$  norms. The heteroscedasticity of the model is also evident from these plots.

## 5.2 The Penn Table Data

We consider now a real data, which is heteroscedastic in nature. The Penn Table dataset is a panel of 125 observations for the period 1960–1985. It includes real GDP per capita (in 1985 dollars) and saving rate (in percent) of 125 countries for those 26 years. This dataset is available in the R package ‘Ecdat’ (named as ‘SumHes’ in the ‘Ecdat’ package). We take the saving rate curve as the response and the curve of per capita GDP as covariate and investigate the effect of the covariate on the response.

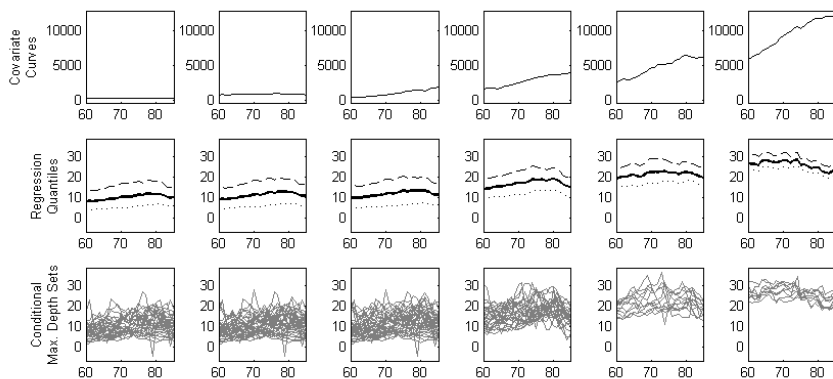


Figure 3: Plots of the selected covariate curves (1<sup>st</sup> row) and the corresponding conditional spatial quantiles (2<sup>nd</sup> row) and conditional maximal depth sets (3<sup>rd</sup> row) for the Penn Table data. The dotted, the dashed and the solid curves in the 2<sup>nd</sup> row are  $\hat{Q}_n(0.5\mathbf{u}|\mathbf{x})$ ,  $\hat{Q}_n(\mathbf{0}|\mathbf{x})$  and  $\hat{Q}_n(-0.5\mathbf{u}|\mathbf{x})$ , respectively.

The curve of per capita GDP indicates the productivity of an average citizen of the nation. The bandwidth  $h$  obtained through cross validation is 9565.71. The resulting conditional quantiles and maximal depth sets are plotted in Figure 3 for 6 selected covariate curves.

We note that saving rate increases gradually with the rise in per capita GDP. We notice that there is a decreasing trend in the saving rate after 1980 in all the plots. This trend is more noticeable in middle and higher GDP levels as indicated by the plots in the 4<sup>th</sup>, the 5<sup>th</sup> and the 6<sup>th</sup> columns of Figure 3. The plots in the 6<sup>th</sup> column correspond to the countries with very high levels of per capita GDP, and the saving rates start decreasing around 1970 in those countries. This indicates an increase in consumption only after 1980 in all the countries except those with very high levels of per capita GDP. The consumption in countries with very high per capita GDP started increasing even earlier, around 1970.

From the plots of  $\hat{D}_1(p|\mathbf{x})$  and  $\hat{D}_2(\tau|\mathbf{x})$  in Figure 4, with the covariate curves arranged by their  $L_2$  norms, we notice that the data is heteroscedastic in nature.

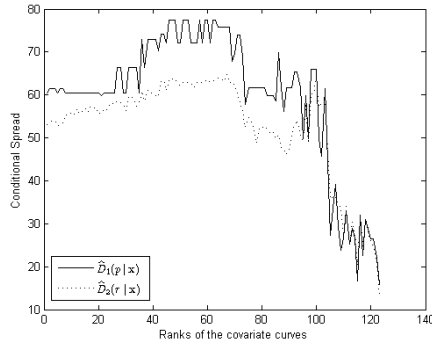


Figure 4: Plots of  $\widehat{D}_1(p|\mathbf{x})$  and  $\widehat{D}_2(\tau|\mathbf{x})$  against the ranks of the covariate curves in the ordering by their  $L_2$  norms, for the Penn Table data.

This observation is also supported by the shrinking difference between the two chosen spatial quantiles and the upper and lower boundaries of the maximal depth sets corresponding to the 4<sup>th</sup>, the 5<sup>th</sup> and the 6<sup>th</sup> chosen covariate curves in Figure 3.

### 5.3 The Cigar Data

Our third example deals with the Cigar Data, which is a panel of 46 observations containing information about the sale of cigarettes in 46 states of the USA for the period from 1963 to 1992. This dataset is available in the ‘plm’ and the ‘Ecdat’ packages in R and was analysed earlier by Baltagi (2008). In addition to information on sales of cigarettes, it also includes per capita net disposable income (NDI) for those 46 states over the 30 year period. We consider the curve of per capita NDI over time as the covariate and the curve of cigarette sales in packs per capita over time as the response. Both the response and the covariate curves are functions of time over a 30 year period from 1963 to 1992, and we view them as random elements in  $L_2[1963, 1992]$ . Choosing the covariate and the response this way, we can investigate the effect of income over consumption of cigarettes in different states. The cross validated choice of the bandwidth is 10061.27. The covariate curves are arranged by their  $L_2$  norms, and the corresponding conditional quantile curves and the maximal depth sets are plotted in Figure 5 for 6 selected covariate curves corresponding to 6 different states.

The plots in Figure 5 illustrate several features of cigarette consumption, both over income and over time, in the states. The difference of the temporal trends in cigarette consumption over different levels of income is noticeable. We can observe from the conditional quantile curves that the sale of cigarettes has a peak around 1980 for each of the selected covariate curves. This peak sale of cigarettes around 1980 is the highest over the time period considered for all the selected covariate curves except the one in the 6<sup>th</sup> column, which corresponds to the state with the highest income level among the selected covariate curves. In the 6<sup>th</sup> column of the plots in Figure 5, the sale of cigarettes around 1980 is slightly lower than the sale around 1963, the beginning of the time period considered here. A small dip is noticeable in the spatial quantile curves and the upper and the lower boundaries of the maximal depth sets around 1970 in all the plots. This means a decrease in cigarette consumption around 1970 in the

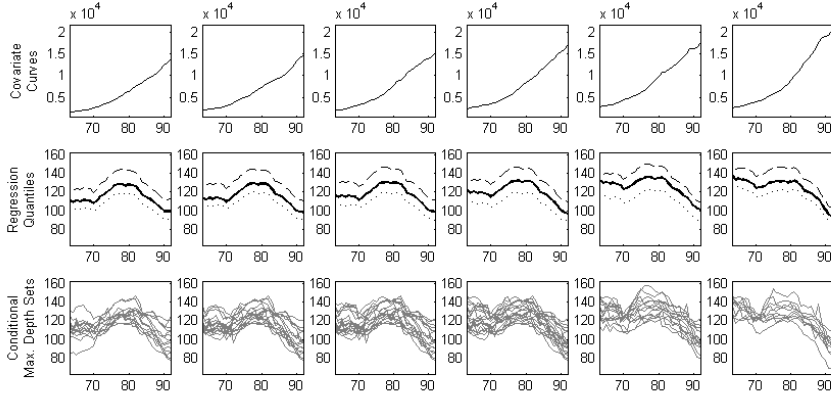


Figure 5: Plots of the selected covariate curves (1<sup>st</sup> row) and the corresponding conditional spatial quantiles (2<sup>nd</sup> row) and conditional maximal depth sets (3<sup>rd</sup> row) for the Cigar Data. The dotted, the dashed and the solid curves in the 2<sup>nd</sup> row are  $\hat{Q}_n(0.5\mathbf{u} | \mathbf{x})$ ,  $\hat{Q}_n(\mathbf{0} | \mathbf{x})$  and  $\hat{Q}_n(-0.5\mathbf{u} | \mathbf{x})$ , respectively.

states. The conditional quantile curves start rising again after 1970 and peak around 1980. After that time, those curves are consistently decreasing. The difference of the two conditional quantiles curves is significantly lower after 1980 in the 6<sup>th</sup> column in Figure 5. This indicates that cigarette consumption decreased more homogeneously in states with very high income levels. In lower income states corresponding to the 1<sup>st</sup> and the 2<sup>nd</sup> columns in Figure 5, the high difference between the two conditional quantile curves after 1980 indicates high variation in the prevalence of smoking in those states. This is further supported by the plots of the corresponding conditional maximal depth sets in the 1<sup>st</sup> and the 2<sup>nd</sup> columns in Figure 5.

Our preceding observations coincide with several important events in the history of smoking in the US in the previous century (see (Burns et al., 1997, p. 16)), and those offer additional insights into the effects of these events. It was observed in

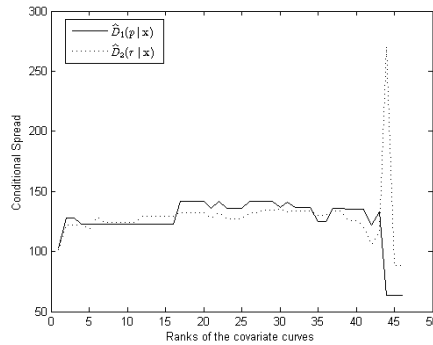


Figure 6: Plots of  $\hat{D}_1(p | \mathbf{x})$  and  $\hat{D}_2(\tau | \mathbf{x})$  against the ranks of the covariate curves in the ordering by their  $L_2$  norms, for the Cigar Data.

(Burns et al., 1997, p. 15) that the per capita cigarette consumption in the US was the highest in 1963. However, we saw in the preceding analysis that the cigarette sales in the low and middle income states did not reach their highest levels until around 1980, while the sales of cigarettes in some very high income states were at their highest levels in 1963. So, that peak of 1963 reported by Burns et al. (1997) was probably due to high consumption in some very high income states. In 1964, the US Surgeon General's report asserted that cigarette consumption is a leading cause of cancer, and counter-advertising on television against smoking was run in the period 1967–1970. These are likely reasons for the small drop in cigarette sales around 1970. We saw that sales of cigarette again started to rise in all the states after 1970. This may be due to the renewed effort to increase sales by the tobacco industry, like introducing special cigarette brands targeted at women, and the end of free time to anti-smoking advertisement in television broadcasting. Nonsmokers' rights movement began after 1970, and gained force by the beginning of the '80s. In 1983, Federal tax on cigarette was increased, and the Surgeon General's report in 1986 linked environmental smoking to lung cancer. The decrease in cigarette sales throughout the '80s and the beginning of the '90s may be explained as the combined effect of these events.

We plot the conditional spread measures  $\widehat{D}_1(p | \mathbf{x})$  and  $\widehat{D}_2(\boldsymbol{\tau} | \mathbf{x})$  in Figure 6, with the covariate curves arranged by their  $L_2$  norms. From these plots, we notice that the data is fairly homoscedastic over the covariate curves except for some extreme covariate curves. The variations in  $\widehat{D}_1(p | \mathbf{x})$  and  $\widehat{D}_2(\boldsymbol{\tau} | \mathbf{x})$  for those extreme covariate curves appear due to the fact that those covariate curves have very few observations in their neighborhoods, and among those observations there seem to be some outliers.

## A Computation of $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$

We describe here the computational procedure of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  in detail, when the response space  $\mathcal{H}$  is infinite dimensional. Observe that  $\widehat{g}_n(\mathbf{Q} | \mathbf{x}) = [\sum_{i=1}^n (\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle) w_i] / [\sum_{i=1}^n w_i]$ , where  $w_i = K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))$  for  $1 \leq i \leq n$ . Since  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  is convex,  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $\widehat{g}_n(\mathbf{Q} | \mathbf{x})$  in  $\mathcal{Z}_n$  if and only if the Gâteaux derivative  $\lim_{t \rightarrow 0^+} t^{-1} [\widehat{g}_n(\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) + t\mathbf{h} | \mathbf{x}) - \widehat{g}_n(\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] \geq 0$  for all  $\mathbf{h} \in \mathcal{Z}_n$ . Denote  $I_n = \{i | \mathbf{Y}_i^{(n)} = \widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\}$ . So, for all  $\mathbf{h} \in \mathcal{Z}_n$ , we need to have that the Gâteaux derivative, which equals  $[\sum_{i \notin I_n} w_i [\|\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|^{-1} (\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}) \cdot \mathbf{h} - \langle \boldsymbol{\tau}^{(n)}, \mathbf{h} \rangle] + \sum_{i \in I_n} w_i [\|\mathbf{h}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{h} \rangle]] \geq 0$ . Replacing  $\mathbf{h}$  by  $-\mathbf{h}$ , we get another version of the above inequality. Since  $|\|\mathbf{h}\| \pm \langle \boldsymbol{\tau}^{(n)}, \mathbf{h} \rangle| \leq (1 + \|\boldsymbol{\tau}^{(n)}\|) \|\mathbf{h}\|$ , we get from the two inequalities that

$$\left\| \sum_{i \notin I_n} w_i [\|\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|^{-1} (\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}) - \boldsymbol{\tau}^{(n)}] \right\| \leq (1 + \|\boldsymbol{\tau}^{(n)}\|) \sum_{i \in I_n} w_i \quad (\text{A.1})$$

if the set  $I_n$  is non-empty. On the other hand, if  $I_n$  is an empty set,

$$\sum_{i=1}^n w_i [\|\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|^{-1} (\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}) - \boldsymbol{\tau}^{(n)}] = 0. \quad (\text{A.2})$$

Now, we state the algorithm for computing  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$ , when  $\mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_n^{(n)}$  do not all lie on a straight line in  $\mathcal{Z}_n$ . For each  $i$ , denote the set  $J_i = \{j | \mathbf{Y}_j^{(n)} = \mathbf{Y}_i^{(n)}\}$ . In

the first step of our computation, we check whether the condition

$$\left\| \sum_{j \notin J_i} w_j [\|\mathbf{Y}_i^{(n)} - \mathbf{Y}_j^{(n)}\|^{-1} (\mathbf{Y}_i^{(n)} - \mathbf{Y}_j^{(n)}) - \boldsymbol{\tau}^{(n)}] \right\| \leq (1 + \|\boldsymbol{\tau}^{(n)}\|) \sum_{j \in J_i} w_j$$

is satisfied for any  $1 \leq i \leq n$ . If it is satisfied for some  $i$ , we take  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) = \mathbf{Y}_i^{(n)}$ . Otherwise, we move to the next step and try to solve Equation (A.2).

To solve Equation (A.2), we take an initial approximation  $\mathbf{Q}_1$  of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  to start the iteration.  $\mathbf{Q}_1$  may be taken as the estimated pointwise conditional median of  $\mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_n^{(n)}$  given  $\mathbf{X} = \mathbf{x}$  if the response is a random function. Let  $\mathbf{Q}_1, \dots, \mathbf{Q}_m$  be the  $m$  successive approximations of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  obtained in the first  $m$  consecutive iterations. Denote  $g_m = \min\{\widehat{g}_n(\mathbf{Q}_j | \mathbf{x}) | 1 \leq j \leq m\}$ . To compute  $\mathbf{Q}_{m+1}$ , we first compute  $\mathbf{V} = \sum_{i=1}^n w_i [\|\mathbf{Q}_m - \mathbf{Y}_i^{(n)}\|^{-1} (\mathbf{Q}_m - \mathbf{Y}_i^{(n)}) - \boldsymbol{\tau}^{(n)}]$  and  $\mathbf{A} = \sum_{i=1}^n w_i [\|\mathbf{Q}_m - \mathbf{Y}_i^{(n)}\|^{-1} Id(\mathcal{Z}_n) - \|\mathbf{Q}_m - \mathbf{Y}_i^{(n)}\|^{-3} (\cdot, \mathbf{Q}_m - \mathbf{Y}_i^{(n)}) \langle \mathbf{Q}_m - \mathbf{Y}_i^{(n)}, \cdot \rangle]$ , where  $Id(\mathcal{Z}_n)$  is the identity map on  $\mathcal{Z}_n$ , and  $(\cdot, \mathbf{y}) \langle \mathbf{z}, \cdot \rangle$  is the outer product of  $\mathbf{y}$  and  $\mathbf{z}$  in  $\mathcal{Z}_n$ . Since  $\mathcal{Z}_n$  is finite dimensional,  $\mathbf{A}$  will be a positive definite operator on  $\mathcal{Z}_n$  if  $\mathbf{Y}_1^{(n)}, \dots, \mathbf{Y}_n^{(n)}$  do not all lie on a straight line. We set  $\mathbf{Q}' = \mathbf{Q}_m - \mathbf{A}^{-1} \mathbf{V}$ . If  $\widehat{g}_n(\mathbf{Q}' | \mathbf{x}) \leq g_m$ , we take  $\mathbf{Q}_{m+1} = \mathbf{Q}'$ . Else, we set  $\mathbf{Q}_{m+1} = f_m \mathbf{Q}_m + (1 - f_m) \mathbf{Q}'$ , where  $f_m = \widehat{g}_n(\mathbf{Q}' | \mathbf{x}) / (\widehat{g}_n(\mathbf{Q}' | \mathbf{x}) + g_m)$ . We stop iteration when two successive approximations of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  are sufficiently close.

## B Mathematical details and proofs

*Proof of Theorem 4.1.* Let  $\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x}) = A_n/B_n$ , where  $A_n = n^{-1} \sum_{i=1}^n [\|\mathbf{y} - \mathbf{Y}_i\|^{-1} (\mathbf{y} - \mathbf{Y}_i) - S(\mathbf{y} | \mathbf{X}_i)] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))$  and  $B_n = n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))$ . Using assumptions C(ii) and A-1, we get  $B_n \rightarrow 1$  almost surely as  $n \rightarrow \infty$ . From assumption C(iii), we get  $\|\gamma(\mathbf{y} | \mathbf{z})(\cdot, \cdot) - \gamma(\mathbf{y} | \mathbf{x})(\cdot, \cdot)\| \rightarrow 0$  as  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$  for any  $\mathbf{y} \in \mathcal{H}$ . So, using assumption A-1 and Theorem 1.1 in Kundu et al. (2000), it follows that  $\sqrt{n\phi(h_n | \mathbf{x})} A_n \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ , where  $\mathbf{W}$  is as described in Theorem 4.1. Hence,  $\sqrt{n\phi(h_n | \mathbf{x})} (\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})) \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ .

Using Taylor expansion of the norm function at  $S(\mathbf{y} | \mathbf{x}) + M_n(\mathbf{y} | \mathbf{x})$ , we get  $[\widehat{SD}(\mathbf{y} | \mathbf{x}) - (1 - \|S(\mathbf{y} | \mathbf{x}) + M_n(\mathbf{y} | \mathbf{x})\|)] = -\|S(\mathbf{y} | \mathbf{x}) + M_n(\mathbf{y} | \mathbf{x})\|^{-1} \langle (S(\mathbf{y} | \mathbf{x}) + M_n(\mathbf{y} | \mathbf{x})), (\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})) \rangle + o(\|\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})\|)$ . So,  $|\widehat{SD}(\mathbf{y} | \mathbf{x}) - SD(\mathbf{y} | \mathbf{x})| \leq |\widehat{SD}(\mathbf{y} | \mathbf{x}) - (1 - \|S(\mathbf{y} | \mathbf{x}) + M_n(\mathbf{y} | \mathbf{x})\|)| + \|M_n(\mathbf{y} | \mathbf{x})\| \leq \|\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})\| + o(\|\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})\|) + \|M_n(\mathbf{y} | \mathbf{x})\|$ . By the asymptotic normality of  $\sqrt{n\phi(h_n | \mathbf{x})} (\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x}))$ , we get  $\|\widehat{S}(\mathbf{y} | \mathbf{x}) - S(\mathbf{y} | \mathbf{x}) - M_n(\mathbf{y} | \mathbf{x})\| = O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1})$  as  $n \rightarrow \infty$ . Under assumptions A-1 and A-2,  $\|M_n(\mathbf{y} | \mathbf{x})\| = O(h_n)$  almost surely as  $n \rightarrow \infty$ . Hence,  $|\widehat{SD}(\mathbf{y} | \mathbf{x}) - SD(\mathbf{y} | \mathbf{x})| = O_P([\sqrt{n\phi(h_n | \mathbf{x})}]^{-1} + h_n)$  as  $n \rightarrow \infty$ .  $\square$

To prove Theorem 4.2, we need the following results.

**Lemma B.1.** *If  $P[\|\mathbf{Y}\| = w | \mathbf{X} = \mathbf{x}] = 0$ , then,  $[P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{z}] - P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{x}]] \rightarrow 0$  as  $n \rightarrow \infty$  and  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ .*

*Proof.* Define  $(r)^+ = rI[r \geq 0]$  for  $r \in \mathbb{R}$ . Note that  $1 - (1 - (\delta^{-1}(w - \|\mathbf{y}^{(n)}\|))^+)^+ \leq I[\|\mathbf{y}^{(n)}\| \leq w] \leq (1 - (\delta^{-1}(\|\mathbf{y}^{(n)}\| - w))^+)^+$  for any positive  $\delta$ . So, we get  $|P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{z}] - P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{x}]| \leq (1 - (\delta^{-1}(\|\mathbf{y}^{(n)}\| - w))^+)^+ - (1 - (\delta^{-1}(w - \|\mathbf{y}^{(n)}\|))^+)^+ \rightarrow 0$  as  $n \rightarrow \infty$  and  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ .

$w | \mathbf{X} = \mathbf{x}] - P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{z}] \leq P[w - \delta < \|\mathbf{Y}^{(n)}\| \leq w + \delta | \mathbf{X} = \mathbf{x}] + \left| \int (1 - (\delta^{-1}(\|\mathbf{y}^{(n)}\| - w))^+)^+ (\mu(d\mathbf{y} | \mathbf{x}) - \mu(d\mathbf{y} | \mathbf{z})) \right|$ . Note that the function  $f(\mathbf{y}) = (1 - (\delta^{-1}(\|\mathbf{y}^{(n)}\| - w))^+)^+$  satisfies  $|f(\mathbf{y}_1) - f(\mathbf{y}_2)| \leq \delta^{-1}\|\mathbf{y}_1 - \mathbf{y}_2\|$  and  $0 \leq f(\mathbf{y}) \leq 1$  for all  $\mathbf{y}$ . Therefore,  $\left| \int (1 - (\delta^{-1}(\|\mathbf{y}^{(n)}\| - w))^+)^+ (\mu(d\mathbf{y} | \mathbf{x}) - \mu(d\mathbf{y} | \mathbf{z})) \right| \leq \delta^{-1}d_{BL}(\mu(\cdot | \mathbf{x}), \mu(\cdot | \mathbf{z}))$  if  $\delta \leq 1$ , where  $d_{BL}(\cdot, \cdot)$  is the Bounded Lipschitz metric (see, e.g., (Pollard, 1984, p. 74)). Given  $\epsilon > 0$ , choose  $\delta > 0$  such that  $\delta \leq 1$ ,  $P[\|\mathbf{Y}\| = w - \delta | \mathbf{X} = \mathbf{x}] = P[\|\mathbf{Y}\| = w + \delta | \mathbf{X} = \mathbf{x}] = 0$  and  $P[w - \delta < \|\mathbf{Y}\| \leq w + \delta | \mathbf{X} = \mathbf{x}] \leq \epsilon/6$ . Choose  $N$  such that for all  $n \geq N$ ,  $|P[\|\mathbf{Y}^{(n)}\| \leq w - \delta | \mathbf{X} = \mathbf{x}] - P[\|\mathbf{Y}\| \leq w - \delta | \mathbf{X} = \mathbf{x}]| \leq \epsilon/6$  and  $|P[\|\mathbf{Y}^{(n)}\| \leq w + \delta | \mathbf{X} = \mathbf{x}] - P[\|\mathbf{Y}\| \leq w + \delta | \mathbf{X} = \mathbf{x}]| \leq \epsilon/6$ . Therefore, for all  $n \geq N$ ,  $P[w - \delta < \|\mathbf{Y}^{(n)}\| \leq w + \delta | \mathbf{X} = \mathbf{x}] \leq \epsilon/2$ . Since  $\mathcal{H}$  is a separable Hilbert space, under assumption C(iii) we get that  $d_{BL}(\mu(\cdot | \mathbf{x}), \mu(\cdot | \mathbf{z})) \rightarrow 0$  as  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ . Choose  $\delta_1 > 0$  such that  $d_{BL}(\mu(\cdot | \mathbf{x}), \mu(\cdot | \mathbf{z})) < (\epsilon/2)\delta$  if  $d(\mathbf{x}, \mathbf{z}) < \delta_1$ . Therefore,  $|P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{x}] - P[\|\mathbf{Y}^{(n)}\| \leq w | \mathbf{X} = \mathbf{z}]| < \epsilon$  if  $n \geq N$  and  $d(\mathbf{x}, \mathbf{z}) < \delta_1$ .  $\square$

**Lemma B.2.** *Under assumption B-1, there exists a constant  $M > 0$  such that  $\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  almost surely for all sufficiently large  $n$ .*

*Proof.* For a constant  $M > 0$ , we shall show that if  $\|\mathbf{Q}\| > M$ ,  $\hat{g}_n(\mathbf{Q} | \mathbf{x}) > \hat{g}_n(\mathbf{0} | \mathbf{x})$  almost surely for all sufficiently large  $n$ . Then, since  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $\hat{g}_n(\mathbf{Q} | \mathbf{x})$ , we must have  $\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all sufficiently large  $n$  almost surely. Note that  $\hat{g}_n(\mathbf{Q} | \mathbf{x}) - \hat{g}_n(\mathbf{0} | \mathbf{x}) = [n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] / [n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))]$ . By assumption B-1,  $n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) \rightarrow 1$  almost surely as  $n \rightarrow \infty$ . Choose an integer  $m > \sqrt{2}[\sqrt{2} - \sqrt{1 + \|\boldsymbol{\tau}\|}]^{-1}$ . Choose  $M > 0$  such that  $P[\|\mathbf{Y}\| > m^{-1}M | \mathbf{X} = \mathbf{x}] < (1/3)m^{-1}$  and  $P[\|\mathbf{Y}\| = m^{-1}M | \mathbf{X} = \mathbf{x}] = 0$ . We have  $n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] = n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] I[\|\mathbf{Y}_i^{(n)}\| \leq m^{-1}M] + n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] I[\|\mathbf{Y}_i^{(n)}\| > m^{-1}M]$ . Let  $A_n = n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) I[\|\mathbf{Y}_i^{(n)}\| > m^{-1}M]$ . Using assumption B-1, we get  $0 \leq A_n \leq E[A_n] + (1/3)m^{-1}$  almost surely for all sufficiently large  $n$ . Denote  $p_n = P[\|\mathbf{Y}^{(n)}\| > m^{-1}M | \mathbf{X} = \mathbf{x}]$ . From Lemma B.1, it follows that  $E[A_n] - p_n \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\|\mathbf{Y}^{(n)}\| \rightarrow \|\mathbf{Y}\|$  almost surely as  $n \rightarrow \infty$ , we have  $p_n \rightarrow P[\|\mathbf{Y}\| > m^{-1}M | \mathbf{X} = \mathbf{x}] < (1/3)m^{-1}$  as  $n \rightarrow \infty$ . Therefore,  $0 \leq A_n < (2/3)m^{-1}$  almost surely for all sufficiently large  $n$ . So,  $|n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] I[\|\mathbf{Y}_i^{(n)}\| > m^{-1}M]| \leq \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)A_n < \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)m^{-1}$  for all sufficiently large  $n$  almost surely. If  $\|\mathbf{Q}\| > M$ ,  $n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] I[\|\mathbf{Y}_i^{(n)}\| \leq m^{-1}M] \geq \|\mathbf{Q}\|[1 - 2m^{-1} - \|\boldsymbol{\tau}\|][n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) - A_n] > \|\mathbf{Q}\|[1 - 2m^{-1} - \|\boldsymbol{\tau}\|](1 - m^{-1})$  for all sufficiently large  $n$  almost surely, since  $n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) > 1 - (1/3)m^{-1}$  almost surely for all sufficiently large  $n$ . Therefore, when  $\|\mathbf{Q}\| > M$ , we get  $n^{-1} \sum_{i=1}^n [\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| - \|\mathbf{Y}_i^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] > \|\mathbf{Q}\|[(1 - 2m^{-1} - \|\boldsymbol{\tau}\|)(1 - m^{-1}) - (1 + \|\boldsymbol{\tau}\|)m^{-1}] = \|\mathbf{Q}\|[2(1 - m^{-1})^2 - (1 + \|\boldsymbol{\tau}\|)] > 0$  almost surely for all sufficiently large  $n$ , by the choice of  $m$ . Hence,  $\hat{g}_n(\mathbf{Q} | \mathbf{x}) > \hat{g}_n(\mathbf{0} | \mathbf{x})$  almost surely for all sufficiently large  $n$  if  $\|\mathbf{Q}\| > M$ .  $\square$

**Lemma B.3.** *under assumption B-1,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all sufficiently large  $n$ , where  $M$  is the constant defined in Lemma B.2.*

*Proof.* Consider the integer  $m$  and the constant  $M$  defined in Lemma B.2. So, by Lemma B.1, there exists  $\delta_1 > 0$  and an integer  $n_1 \geq 1$  such that whenever  $d(\mathbf{x}, \mathbf{z}) < \delta_1$

and  $n \geq n_1$ ,  $P[\|\mathbf{Y}^{(n)}\| > m^{-1}M | \mathbf{X} = \mathbf{z}] < (2/3)m^{-1}$ . Since  $h_n \rightarrow 0$  by assumption B-1, there exists an integer  $n_2 \geq n_1$  such that whenever  $n \geq n_2$ ,  $P[\|\mathbf{Y}^{(n)}\| > m^{-1}M | \mathbf{X}]I(d(\mathbf{x}, \mathbf{X}) \leq h_n) \leq (2/3)m^{-1}I(d(\mathbf{x}, \mathbf{X}) \leq h_n)$ . Note that  $\tilde{g}_n(\mathbf{Q} | \mathbf{x}) - \tilde{g}_n(\mathbf{0} | \mathbf{x}) = E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| \leq m^{-1}M)E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))] + E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| > m^{-1}M)E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))]$ . Now, if  $n \geq n_2$ ,  $|E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| > m^{-1}M)E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))]| \leq \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)E[P(\|\mathbf{Y}^{(n)}\| > m^{-1}M | \mathbf{X})E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))] < \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)m^{-1}$ . Also, if  $\|\mathbf{Q}\| > M$  and  $n \geq n_2$ ,  $E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| \leq m^{-1}M)E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))] \geq E[(\|\mathbf{Q}\| - 2\|\mathbf{Y}^{(n)}\| - \|\boldsymbol{\tau}^{(n)}\|)\|\mathbf{Q}\|I(\|\mathbf{Y}^{(n)}\| \leq m^{-1}M)E_n^{-1}K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))] > \|\mathbf{Q}\|(1 - 2m^{-1} - \|\boldsymbol{\tau}\|)(1 - m^{-1})$ . So, if  $\|\mathbf{Q}\| > M$  and  $n \geq n_2$ ,  $\tilde{g}_n(\mathbf{Q} | \mathbf{x}) - \tilde{g}_n(\mathbf{0} | \mathbf{x}) > \|\mathbf{Q}\|(1 - 2m^{-1} - \|\boldsymbol{\tau}\|)(1 - m^{-1}) - \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)m^{-1} > 0$  by the choice of  $m$ . Since  $\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $\tilde{g}_n(\mathbf{Q} | \mathbf{x})$ , we must have  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all  $n \geq n_2$ .  $\square$

**Lemma B.4.** *Under assumption B-2,  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* First, we shall show that  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all  $n$ , where  $M > 0$  is the constant defined in Lemma B.2. Consider the integer  $m$  defined in Lemma B.2. Note that  $P[\|\mathbf{Y}\| > m^{-1}M | \mathbf{X} = \mathbf{x}] < m^{-1}$ . Since  $\|\mathbf{Y}^{(n)}\| \leq \|\mathbf{Y}\|$ ,  $P[\|\mathbf{Y}^{(n)}\| > m^{-1}M | \mathbf{X} = \mathbf{x}] < m^{-1}$  for all  $n$ . Note that  $g_n(\mathbf{Q} | \mathbf{x}) - g_n(\mathbf{0} | \mathbf{x}) = g_n(\mathbf{Q} | \mathbf{x}) = E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| \leq m^{-1}M) | \mathbf{X} = \mathbf{x}] + E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| > m^{-1}M) | \mathbf{X} = \mathbf{x}]$ . Now,  $|E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| > m^{-1}M) | \mathbf{X} = \mathbf{x}]| < \|\mathbf{Q}\|(1 + \|\boldsymbol{\tau}\|)m^{-1}$ . If  $\|\mathbf{Q}\| > M$ , then  $E[(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| - \|\mathbf{Y}^{(n)}\| - \langle \boldsymbol{\tau}^{(n)}, \mathbf{Q} \rangle)I(\|\mathbf{Y}^{(n)}\| \leq m^{-1}M) | \mathbf{X} = \mathbf{x}] > \|\mathbf{Q}\|(1 - 2m^{-1} - \|\boldsymbol{\tau}\|)(1 - m^{-1})$ . So, if  $\|\mathbf{Q}\| > M$ ,  $g_n(\mathbf{Q} | \mathbf{x}) > \|\mathbf{Q}\|[(1 - 2m^{-1} - \|\boldsymbol{\tau}\|)(1 - m^{-1}) - (1 + \|\boldsymbol{\tau}\|)m^{-1}] > 0$  by the choice of  $m$ . Since  $\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $g_n(\mathbf{Q} | \mathbf{x})$ , we must have  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all  $n$ .

Note that  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  exists and is unique under assumption B-2. Using Theorem 1 and Theorem 3 in Asplund (1968), it is enough to show  $g(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) \rightarrow g(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})$  as  $n \rightarrow \infty$  to complete the proof. Since  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $g(\mathbf{Q} | \mathbf{x})$ , we have  $0 \leq g(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) = [g(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] + [g_n(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] + [g_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] + [g_n(\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]$ , where  $\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x})$  is the projection of  $\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})$  in  $\mathcal{Z}_n$ . Now,  $|g(\mathbf{Q} | \mathbf{x}) - g_n(\mathbf{Q} | \mathbf{x})| \leq 2E[\|\mathbf{Y} - \mathbf{Y}^{(n)}\| | \mathbf{X} = \mathbf{x}] + \|\boldsymbol{\tau}^{(n)} - \boldsymbol{\tau}\|\|\mathbf{Q}\|$ . So, for every  $C > 0$ ,  $\sup_{\|\mathbf{Q}\| \leq C} |g(\mathbf{Q} | \mathbf{x}) - g_n(\mathbf{Q} | \mathbf{x})| \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, we get  $[g(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] \rightarrow 0$  and  $[g_n(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] \rightarrow 0$  as  $n \rightarrow \infty$  because  $\|\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$  for all  $n$ . Since  $\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})$  minimizes  $g_n(\mathbf{Q} | \mathbf{x})$  in  $\mathcal{Z}_n$  and  $\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) \in \mathcal{Z}_n$ , we have  $[g_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] \leq 0$  for all  $n$ . Also,  $|g_n(\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - g_n(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})| \leq 2\|\mathbf{Q}^{(n)}(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $g(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) \rightarrow g(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})$  as  $n \rightarrow \infty$ .  $\square$

**Lemma B.5.** *Under assumptions B-2 and B-3, there exists  $C_5 > 0$  and an integer  $N_4 > 0$  such that whenever  $d(\mathbf{x}, \mathbf{z}) \leq C_5$  and  $n \geq N_4$ , we have, for every  $C > 0$ ,  $b_C\|\mathbf{h}\| \leq \|g_n^{(2)}(\mathbf{Q} | \mathbf{z})(\mathbf{h})\| \leq B_C\|\mathbf{h}\|$  for any  $\mathbf{Q}, \mathbf{h} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ . Here  $0 < b_C < B_C < \infty$  are constants depending on  $C$ .*

*Proof.* Assumption B-2 ensures that  $g_n^{(2)}(\mathbf{Q} | \mathbf{z})(\cdot)$  is well-defined for  $n \geq N_1$  and  $d(\mathbf{x}, \mathbf{z}) \leq C_2$ . First, we shall show that for every  $C > 0$ , we have  $b_C\|\mathbf{h}\|^2 \leq \langle g_n^{(2)}(\mathbf{Q} | \mathbf{z})(\mathbf{h}), \mathbf{h} \rangle \leq B_C\|\mathbf{h}\|^2$  for any  $\mathbf{Q}, \mathbf{h} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ , whenever  $n \geq N_4$  and  $d(\mathbf{x}, \mathbf{z}) \leq C_5$ . Here  $C_5 > 0$  is a constant and  $N_4 > 0$  is an integer. Without

loss of generality, consider any  $\mathbf{h}$  with  $\|\mathbf{h}\| = 1$ . Define  $P_{\mathbf{h}}(\cdot)$  to be the projection operator on the orthogonal complement of  $\mathbf{h}$ . Then,  $\langle (g_n^{(2)}(\mathbf{Q}|\mathbf{z}))(\mathbf{h}), \mathbf{h} \rangle = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1} [1 - \|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-2} \langle \mathbf{h}, \mathbf{Q} - \mathbf{Y}^{(n)} \rangle^2] | \mathbf{X} = \mathbf{z}] = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-3} \|P_{\mathbf{h}}(\mathbf{Q} - \mathbf{Y}^{(n)})\|^2 | \mathbf{X} = \mathbf{z}]$ . So, from assumption B-3, it follows that whenever  $n \geq \max\{N_1, N_2\}$  and  $d(\mathbf{x}, \mathbf{z}) \leq \min\{C_2, C_3\}$ , we have  $\langle (g_n^{(2)}(\mathbf{Q}|\mathbf{z}))(\mathbf{h}), \mathbf{h} \rangle \leq E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1} | \mathbf{X} = \mathbf{z}] \leq \sqrt{s_2(C)}$  for any  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ . Set  $B_C = \sqrt{s_2(C)}$  and  $N_4 = \max\{N_1, N_2\}$ . For the other inequality, using assumption B-2 and following the arguments in the proof of Proposition 2.1 in Cardot et al. (2013), we get that there exists a subspace  $\mathcal{S} \subset \mathcal{Z}_{N_1}$  of dimension 2 such that  $\langle \mathbf{u}, \mathbf{Y}^{(N_1)} \rangle$  is non-degenerate for every  $\mathbf{u} \in \mathcal{S}$ , which implies that  $\langle \mathbf{u}, \mathbf{Y} \rangle$  is non-degenerate for every  $\mathbf{u} \in \mathcal{S}$ . So, either  $E[\langle \mathbf{u}, \mathbf{Y} \rangle | \mathbf{X} = \mathbf{x}] = \infty$  for every  $\mathbf{u} \in \mathcal{S}$  or  $\inf\{\text{Var}[\langle \mathbf{u}, \mathbf{Y} \rangle | \mathbf{X} = \mathbf{x}] | \mathbf{u} \in \mathcal{S}, \|\mathbf{u}\| = 1 \text{ with } E[\langle \mathbf{u}, \mathbf{Y} \rangle | \mathbf{X} = \mathbf{x}] < \infty\} = b > 0$ . Since  $\mathcal{Z}_n \subset \mathcal{Z}_{n+1}$  for all  $n$ , we have  $\mathcal{S} \subset \mathcal{Z}_n$  for all  $n \geq N_4$ . Also, there is  $\mathbf{v} \in \mathcal{S}$  such that  $\|\mathbf{v}\| = 1$  and  $\langle \mathbf{h}, \mathbf{v} \rangle = 0$ . Then, for any  $\mathbf{y}$ ,  $\|P_{\mathbf{h}}(\mathbf{y})\|^2 \geq \langle \mathbf{y}, \mathbf{v} \rangle^2$ . So, for  $n \geq N_4$ ,  $\|P_{\mathbf{h}}(\mathbf{Q} - \mathbf{Y}^{(n)})\|^2 \geq \langle \mathbf{Q} - \mathbf{Y}^{(n)}, \mathbf{v} \rangle^2 = \langle \mathbf{Q} - \mathbf{Y}, \mathbf{v} \rangle^2$  as  $\mathbf{Q}, \mathbf{Y}^{(n)}, \mathbf{v} \in \mathcal{Z}_n$ . Also, for  $\mathbf{Q} \in \mathcal{Z}_n$ ,  $\|\mathbf{Q} - \mathbf{Y}^{(n)}\| \leq \|\mathbf{Q} - \mathbf{Y}\|$  as  $(\mathbf{Q} - \mathbf{Y}^{(n)}) \in \mathcal{Z}_n$ . Note that for any  $M_1 > 0$ ,  $\langle \mathbf{Q} - \mathbf{Y}, \mathbf{v} \rangle^2 I(\|\mathbf{Y}\| \leq M_1) = \langle \mathbf{Q} - \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle^2 - \langle \mathbf{Q}, \mathbf{v} \rangle^2 I(\|\mathbf{Y}\| > M_1)$ . Therefore, if  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$  and  $n \geq N_4$ , we have  $E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-3} \|P_{\mathbf{h}}(\mathbf{Q} - \mathbf{Y}^{(n)})\|^2 | \mathbf{X} = \mathbf{x}] \geq (C + M_1)^{-3} [E[\langle \mathbf{Q} - \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle^2 | \mathbf{X} = \mathbf{x}] - C^2 P[\|\mathbf{Y}\| > M_1 | \mathbf{X} = \mathbf{x}]]$ . Let  $E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] < \infty$ . We have  $E[\langle \mathbf{Q} - \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle^2 | \mathbf{X} = \mathbf{x}] \geq \text{Var}[\langle \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}]$ . Note that  $\text{Var}[\langle \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] \rightarrow \text{Var}[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] \geq b > 0$  as  $M_1 \rightarrow \infty$ . Choose  $M_1 > 0$  such that  $(C + M_1)^{-3} [\text{Var}[\langle \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] - C^2 P[\|\mathbf{Y}\| > M_1 | \mathbf{X} = \mathbf{x}]] > 0$  and  $P[\|\mathbf{Y}\| = M_1 | \mathbf{X} = \mathbf{x}] = 0$ . Using assumption C(iii), we get  $[\text{Var}[\langle \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle | \mathbf{X} = \mathbf{z}] - C^2 P[\|\mathbf{Y}\| > M_1 | \mathbf{X} = \mathbf{z}]] \rightarrow [\text{Var}[\langle \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] - C^2 P[\|\mathbf{Y}\| > M_1 | \mathbf{X} = \mathbf{x}]]$  as  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ . On the other hand, suppose  $E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] = \infty$ . Then, it is easy to show that  $E[\langle \mathbf{Q} - \mathbf{Y}I(\|\mathbf{Y}\| \leq M_1), \mathbf{v} \rangle^2 | \mathbf{X} = \mathbf{x}] > -C[5C + 2] + E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] I(\|\mathbf{Y}\| \leq M_1, |\langle \mathbf{Y}, \mathbf{v} \rangle| > \max(2C, 1)) | \mathbf{X} = \mathbf{x}] \rightarrow \infty$  as  $M_1 \rightarrow \infty$ . So, we can find  $M_1$  such that  $(C + M_1)^{-3} [-C[5C + 2] + E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}] I(\|\mathbf{Y}\| \leq M_1, |\langle \mathbf{Y}, \mathbf{v} \rangle| > \max(2C, 1)) | \mathbf{X} = \mathbf{x}]] > 0$  and  $P[\|\mathbf{Y}\| = M_1 | \mathbf{X} = \mathbf{x}] = P[|\langle \mathbf{Y}, \mathbf{v} \rangle| = \max(2C, 1) | \mathbf{X} = \mathbf{x}] = 0$ . Again, using assumption C(iii), we get  $E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{z}] \rightarrow E[\langle \mathbf{Y}, \mathbf{v} \rangle | \mathbf{X} = \mathbf{x}]$  as  $d(\mathbf{x}, \mathbf{z}) \rightarrow 0$ . Therefore, there exist  $C_5 > 0$  such that for every  $C$ , we can find  $b_C > 0$  satisfying  $\langle (g_n^{(2)}(\mathbf{Q}|\mathbf{z}))(\mathbf{h}), \mathbf{h} \rangle \geq b_C$  for all  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$  whenever  $n \geq N_4$  and  $d(\mathbf{x}, \mathbf{z}) \leq C_5$ .

Finally, from the arguments in the proofs of Theorem 4.3.13 and Theorem 4.3.16 in Debnath and Mikusiński (2005), it follows that whenever  $n \geq N_4$  and  $d(\mathbf{x}, \mathbf{z}) \leq C_5$ , for every  $C > 0$ , we have  $b_C \|\mathbf{h}\| \leq \|g_n^{(2)}(\mathbf{Q}|\mathbf{z})(\mathbf{h})\| \leq B_C \|\mathbf{h}\|$  for any  $\mathbf{Q}, \mathbf{h} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ .  $\square$

**Lemma B.6.** *Under assumptions B-2 and B-3,  $g_n^{(2)}(\mathbf{Q}_n(\boldsymbol{\tau}|\mathbf{x})|\mathbf{x})(\cdot)$  is invertible for all sufficiently large  $n$ . In addition, if assumption B-1 holds, then  $\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau}|\mathbf{x})|\mathbf{x})(\cdot)$  is invertible for all sufficiently large  $n$ .*

*Proof.* We have  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau}|\mathbf{x})\| \leq M$  and  $\|\mathbf{Q}_n(\boldsymbol{\tau}|\mathbf{x})\| \leq M$  for all sufficiently large  $n$  from Lemma B.3 and Lemma B.4, respectively. It follows from Lemma B.5 that  $b_M \|\mathbf{h}\| \leq \|g_n^{(2)}(\mathbf{Q}_n(\boldsymbol{\tau}|\mathbf{x})|\mathbf{x})(\mathbf{h})\|$  and  $b_M \|\mathbf{h}\| \leq \|\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau}|\mathbf{x})|\mathbf{x})(\mathbf{h})\|$  for all  $\mathbf{h} \in \mathcal{Z}_n$

and all sufficiently large  $n$ . Therefore,  $g_n^{(2)}(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot)$  and  $\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot)$  are invertible for all sufficiently large  $n$ , by Inverse Mapping Theorem.  $\square$

**Lemma B.7.** *Under assumptions B-1, B-2 and B-3,  $\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| = O(\epsilon_n)$  almost surely as  $n \rightarrow \infty$ , where  $\epsilon_n = (n\phi(h_n | \mathbf{x}))^{-\alpha} \sqrt{\log n}$ .*

*Proof.* From Lemma B.2 and Lemma B.3,  $\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq 2M = M_2$  (say) for all sufficiently large  $n$  almost surely. Define  $G_n = \{\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) + \sum_{j=1}^{d_n} \beta_j \mathbf{e}_j \mid \beta_j \in [-M_2, M_2]\}$  such that  $n^4 \beta_j$  is an integer and  $\|\sum_{j=1}^{d_n} \beta_j \mathbf{e}_j\| \leq M_2$ . Denote a point in  $G_n$  that is nearest to  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})$  as  $\bar{\mathbf{Q}}_n$ . So, by the choice of  $d_n$  in Theorem 4.2,  $\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n\| \leq d_n n^{-4}$  for all sufficiently large  $n$  almost surely. We now have

$$\begin{aligned} & \left\| n^{-1} \sum_{i=1}^n \left[ \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n} \right\| \\ & \leq \left\| n^{-1} \sum_{i=1}^n \left[ \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n} \right\| \\ & \quad + n^{-1} \sum_{i=1}^n \left\| \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} \right\| \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n}. \end{aligned}$$

Note that when  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) \neq \mathbf{Y}_i^{(n)}$  and  $\bar{\mathbf{Q}}_n \neq \mathbf{Y}_i^{(n)}$ , we have

$$\begin{aligned} & \left\| \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} - \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} \right\| \\ & \leq \left\| \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} - \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} \right\| + \left\| \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} \right\| \\ & \leq 2 \frac{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n\|}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|}. \end{aligned}$$

Also, when  $\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\| > n^{-2}$ , we have  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) \neq \mathbf{Y}_i^{(n)}$ . So,

$$\begin{aligned} & n^{-1} \sum_{i=1}^n \left\| \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} \right\| \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n} \\ & \leq 2n^{-1} \sum_{i=1}^n \left[ \frac{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n\|}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} I(\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\| > n^{-2}) + I(\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\| \leq n^{-2}) \right] \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n} \\ & \leq \frac{2d_n}{n^2} n^{-1} \sum_{i=1}^n \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n} + 2n^{-1} \sum_{i=1}^n I(\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\| \leq n^{-2}) \frac{K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}{E_n}. \end{aligned}$$

Denote  $p_n(\mathbf{Q}) = E[P(\|\mathbf{Q} - \mathbf{Y}^{(n)}\| \leq n^{-2} | \mathbf{X}) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))]$ . Note that for  $\mathbf{Q} \in G_n$ ,  $\|\mathbf{Q}\| \leq 2M_2$  for all  $n$ . Since  $G_n \subset \mathcal{Z}_n$ , using Markov inequality and assumption B-3, we get that for all sufficiently large  $n$ ,  $\max_{\mathbf{Q} \in G_n} p_n(\mathbf{Q}) \leq n^{-4} s_2(2M_2)$ . Therefore, using assumption C(ii) and the Bernstein inequality, we have  $P[\max_{\mathbf{Q} \in G_n} n^{-1} \sum_{i=1}^n I(\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| \leq n^{-2}) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) > b_1 \epsilon_n^2] \leq \sum_{\mathbf{Q} \in G_n} P[\sum_{i=1}^n n^{-1} I(\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| \leq n^{-2}) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) - p_n(\mathbf{Q}) > 2^{-1} b_1 \epsilon_n^2] \leq (3M_2 n^4)^{d_n} \exp[-b_2 n \phi(h_n | \mathbf{x}) \epsilon_n^2]$  for

all sufficiently large  $n$  and any  $b_1 > 0$ , where  $b_2 = [8L]^{-1}b_1$ . Using assumption B-1, the Borel-Cantelli Lemma, the choice of  $d_n$  and choosing  $b_1$  appropriately, we get  $\max_{\mathbf{Q} \in G_n} n^{-1} \sum_{i=1}^n I(\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\| \leq n^{-2}) E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i)) \leq b_1 \epsilon_n^2$  for all sufficiently large  $n$  *almost surely*. Also, note that  $n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i)) < 2$  for all sufficiently large  $n$  *almost surely*. Hence, we get

$$n^{-1} \sum_{i=1}^n \left\| \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} \right\| \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} \leq \frac{4d_n}{n^2} + 2b_1 \epsilon_n^2 < 3b_1 \epsilon_n^2$$

for all sufficiently large  $n$  *almost surely*. Also, assumption B-2 implies that the  $\mathbf{Y}_i^{(n)}$ s for which  $d(\mathbf{x}, \mathbf{X}_i) \leq h_n$ , are distinct *almost surely* for all sufficiently large  $n$ . So, putting  $w_i = n^{-1} E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))$  in Equation (A.1), it follows that

$$\left\| n^{-1} \sum_{i=1}^n \left[ \frac{\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\mathbf{Y}_i^{(n)} - \hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} \right\| \leq (\ln \phi(h_n | \mathbf{x}))^{-1} 3L < b_1 \epsilon_n^2$$

*almost surely* for all sufficiently large  $n$ . Hence, we get

$$\left\| n^{-1} \sum_{i=1}^n \left[ \frac{\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} \right\| < 4b_1 \epsilon_n^2 \quad (\text{B.1})$$

*almost surely* for all sufficiently large  $n$ . Next, denote

$$\mathbf{V}_{i,n}(\mathbf{Q}) = \left[ \frac{\mathbf{Q} - \mathbf{Y}_i^{(n)}}{\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} - E \left[ \left[ \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))}{E_n} \right].$$

In view of assumption C(ii),  $\|\mathbf{V}_{i,n}(\mathbf{Q})\| \leq (l\phi(h_n | \mathbf{x}))^{-1} 4L$  for all  $i$ . Also,  $E[\|\mathbf{V}_{i,n}(\mathbf{Q})\|^2] \leq [l^2 \phi(h_n | \mathbf{x})]^{-1} (16L^2)$  for all  $i$  and  $n$ . So,  $E[\|\mathbf{V}_{i,n}(\mathbf{Q})\|^m] \leq [(l\phi(h_n | \mathbf{x}))^{-1} 4L]^{m-2} (l^2 \phi(h_n | \mathbf{x}))^{-1} (16L^2)$  for all sufficiently large  $n$  and for all  $m \geq 2$ . Using the Bernstein inequality for random elements in a separable Hilbert space (see (Yurinskiĭ, 1976, p. 491)), we get  $P[\max_{\mathbf{Q} \in G_n} \|n^{-1} \sum_{i=1}^n \mathbf{V}_{i,n}(\mathbf{Q})\| > b_3 \epsilon_n] \leq \sum_{\mathbf{Q} \in G_n} P[\|\sum_{i=1}^n \mathbf{V}_{i,n}(\mathbf{Q})\| > b_3 n \epsilon_n] \leq 2(3M_2 n^4)^{d_n} \exp[-b_4 n \phi(h_n | \mathbf{x}) \epsilon_n^2]$  for all sufficiently large  $n$  and any  $b_3 > 0$ , where  $b_4 = (20L^2)^{-1} l^2 b_3^2$ . Therefore, choosing an appropriate  $b_3$  and using assumption B-1, the choice of  $d_n$  and the Borel-Cantelli Lemma, we get

$$\begin{aligned} & \max_{\mathbf{Q} \in G_n} \left\| n^{-1} \sum_{i=1}^n \left[ \frac{\mathbf{Q} - \mathbf{Y}_i^{(n)}}{\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} - E \left[ \left[ \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))}{E_n} \right] \right\| \\ & \leq b_3 \epsilon_n \end{aligned} \quad (\text{B.2})$$

for all sufficiently large  $n$  *almost surely*. Let  $\mathbf{Q} \in G_n$  and  $\|\mathbf{Q} - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| > b_5 \epsilon_n$ , where  $b_5 > 0$ . From a Taylor expansion and Lemma B.5, we have

$$\left\| E \left[ \left[ \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))}{E_n} \right] \right\| \geq b_{2M_2} \|\mathbf{Q} - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| > b_{2M_2} b_5 \epsilon_n$$

for all sufficiently large  $n$ . Then, using Equation (B.2), we get

$$\left\| n^{-1} \sum_{i=1}^n \left[ \frac{\mathbf{Q} - \mathbf{Y}_i^{(n)}}{\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\|} - \boldsymbol{\tau}^{(n)} \right] \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{E_n} \right\| > (b_{2M_2} b_5 - b_3) \epsilon_n$$

for all sufficiently large  $n$  *almost surely*. Choosing  $b_5$  such that  $(b_{2M_2}b_5 - b_3) > 5b_1$  and from Equation (B.1), we see that we must have  $\|\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq b_5\epsilon_n$  for all sufficiently large  $n$  *almost surely*. Therefore, we have  $\|\hat{\bar{\mathbf{Q}}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq \|\hat{\bar{\mathbf{Q}}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n\| + \|\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq (1 + b_5)\epsilon_n$  for all sufficiently large  $n$  *almost surely*.  $\square$

**Lemma B.8.** *Recall  $\bar{\mathbf{Q}}_n$  from Lemma B.7. Define  $G_n(\mathbf{Q} | \mathbf{x})$  as*

$$G_n(\mathbf{Q} | \mathbf{x}) = \frac{E \left[ \left[ \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \frac{\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}^{(n)}}{\|\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}^{(n)}\|} \right] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X})) \right]}{n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))}.$$

*Under assumptions B-1, B-2 and B-3, we have  $\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ , where  $\epsilon_n$  is as in Lemma B.7.*

*Proof.* Denote  $H_n = \{\mathbf{Q} \in G_n \mid \|\mathbf{Q} - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq b_6\epsilon_n\}$ , where  $b_6 = (1 + b_5)$  and  $b_5$  is the constant defined in Lemma B.7. From the proof of Lemma B.7, we see that  $\bar{\mathbf{Q}}_n \in H_n$ . Now,  $\hat{g}_n^{(1)}(\mathbf{Q} | \mathbf{x}) - \hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - G_n(\mathbf{Q} | \mathbf{x}) = [n^{-1} \sum_{i=1}^n \{A_{i,n}(\mathbf{Q}) - E[A_{i,n}(\mathbf{Q})]\}] / [n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))]$ , where

$$A_{i,n}(\mathbf{Q}) = \left[ \frac{\mathbf{Q} - \mathbf{Y}_i^{(n)}}{\|\mathbf{Q} - \mathbf{Y}_i^{(n)}\|} - \frac{\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}}{\|\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|} \right] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)).$$

Note that  $\|A_{i,n}(\mathbf{Q})\| \leq 2\|\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|^{-1} \|\mathbf{Q} - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))$  *almost surely* for all  $i$ ,  $\mathbf{Q}$  and for all sufficiently large  $n$ . Let  $U_{i,n}(\mathbf{Q}) = A_{i,n}(\mathbf{Q}) - E[A_{i,n}(\mathbf{Q})]$ . Note that  $\|U_{i,n}(\mathbf{Q})\| \leq (l\phi(h_n | \mathbf{x}))^{-1} 4L$  for all  $i$  and for all sufficiently large  $n$ . Also,  $\|U_{i,n}(\mathbf{Q})\|^2 \leq [\|A_{i,n}(\mathbf{Q})\| + \|E[A_{i,n}(\mathbf{Q})]\|]^2 \leq 2[\|A_{i,n}(\mathbf{Q})\|^2 + \|E[A_{i,n}(\mathbf{Q})]\|^2]$ . So,  $E[\|U_{i,n}(\mathbf{Q})\|^2] \leq 2E[\|A_{i,n}(\mathbf{Q})\|^2 + \|E[A_{i,n}(\mathbf{Q})]\|^2] \leq 4E[\|A_{i,n}(\mathbf{Q})\|^2] \leq [l^2\phi(h_n | \mathbf{x})]^{-1} [16L^2s_2(M)] \|\mathbf{Q} - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\|^2$  for all sufficiently large  $n$  using assumption B-3 and Lemma B.3. Therefore,

$$E[\|U_{i,n}(\mathbf{Q})\|^m] \leq \left[ \frac{4L}{l\phi(h_n | \mathbf{x})} \right]^{m-2} \frac{16L^2s_2(M)}{l^2\phi(h_n | \mathbf{x})} \|\mathbf{Q} - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\|^2.$$

Using assumption B-1, the Bernstein inequality in a separable Hilbert space, the choice of  $d_n$  in Theorem 4.2 and the Borel-Cantelli Lemma, we get  $\max_{\mathbf{Q} \in H_n} \|n^{-1} \sum_{i=1}^n U_{i,n}(\mathbf{Q})\| \leq b_7\epsilon_n^2$  for all sufficiently large  $n$  *almost surely*, for an appropriate positive constant  $b_7$ . Since  $n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) > (1/2)$  *almost surely* for all sufficiently large  $n$ , we have  $\|\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - G_n(\bar{\mathbf{Q}}_n | \mathbf{x})\| \leq [\max_{\mathbf{Q} \in H_n} \|n^{-1} \sum_{i=1}^n U_{i,n}(\mathbf{Q})\|] / [n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))] \leq 2b_7\epsilon_n^2$  *almost surely* for all sufficiently large  $n$ .  $\square$

**Lemma B.9.** *Under assumptions B-2 and B-3, for every  $C > 0$  and for all sufficiently large  $n$ , we have  $\|(\tilde{g}_n^{(2)}(\mathbf{Q}_1 | \mathbf{x}))(\mathbf{h}) - (\tilde{g}_n^{(2)}(\mathbf{Q}_2 | \mathbf{x}))(\mathbf{h})\| \leq 6s_2(C)\|\mathbf{h}\| \|\mathbf{Q}_1 - \mathbf{Q}_2\|$  for any  $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathcal{Z}_n$  with  $\|\mathbf{Q}_1\|, \|\mathbf{Q}_2\| \leq C$  and  $\mathbf{h} \in \mathcal{Z}_n$ .*

*Proof.* Assumption B-2 ensures that for any  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ ,  $\mathbf{Y}^{(n)} \neq \mathbf{Q}_1$  and  $\mathbf{Y}^{(n)} \neq \mathbf{Q}_2$  *almost surely* for all  $n \geq N_1$ , and  $(\tilde{g}_n^{(2)}(\mathbf{Q}_1 | \mathbf{x}))(\mathbf{h}) - (\tilde{g}_n^{(2)}(\mathbf{Q}_2 | \mathbf{x}))(\mathbf{h}) = E[[W_n(\mathbf{Q}_1)(\mathbf{h}) - W_n(\mathbf{Q}_2)(\mathbf{h})] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))]$ , where

$$W_n(\mathbf{Q})(\mathbf{h}) = \left[ \frac{h}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} - \frac{1}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} \left\langle \mathbf{h}, \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} \right\rangle \frac{\mathbf{Q} - \mathbf{Y}^{(n)}}{\|\mathbf{Q} - \mathbf{Y}^{(n)}\|} \right].$$

Note that when  $\mathbf{Q}_1 \neq \mathbf{Y}^{(n)}$  and  $\mathbf{Q}_2 \neq \mathbf{Y}^{(n)}$ , we have

$$\left\| \frac{\mathbf{Q}_2 - \mathbf{Y}^{(n)}}{\|\mathbf{Q}_2 - \mathbf{Y}^{(n)}\|} - \frac{\mathbf{Q}_1 - \mathbf{Y}^{(n)}}{\|\mathbf{Q}_1 - \mathbf{Y}^{(n)}\|} \right\| \leq 2 \frac{\|\mathbf{Q}_2 - \mathbf{Q}_1\|}{\|\mathbf{Q}_2 - \mathbf{Y}^{(n)}\|}. \quad (\text{B.3})$$

In that case, using Equation (B.3) and some straight-forward algebra, we get

$$\begin{aligned} \|W_n(\mathbf{Q}_1)(\mathbf{h}) - W_n(\mathbf{Q}_2)(\mathbf{h})\| &\leq \frac{6\|\mathbf{h}\|\|\mathbf{Q}_2 - \mathbf{Q}_1\|}{\|\mathbf{Q}_1 - \mathbf{Y}^{(n)}\|\|\mathbf{Q}_2 - \mathbf{Y}^{(n)}\|} \\ &\leq 3\|\mathbf{h}\|\|\mathbf{Q}_2 - \mathbf{Q}_1\| \left[ \frac{1}{\|\mathbf{Q}_1 - \mathbf{Y}^{(n)}\|^2} + \frac{1}{\|\mathbf{Q}_2 - \mathbf{Y}^{(n)}\|^2} \right] \end{aligned} \quad (\text{B.4})$$

for any  $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{h} \in \mathcal{H}$ . Using assumption B-3, we get that whenever  $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathcal{Z}_n$  with  $\|\mathbf{Q}_1\|, \|\mathbf{Q}_2\| \leq C$ , we have

$$E \left[ \frac{1}{\|\mathbf{Q}_i - \mathbf{Y}^{(n)}\|^2} \frac{K(h_n^{-1}d(\mathbf{x}, \mathbf{X}))}{E_n} \right] \leq s_2(C) \quad (\text{B.5})$$

for all sufficiently large  $n$ , where  $i = 1, 2$ . Hence, using Equations (B.4) and (B.5), it follows that for all sufficiently large  $n$ , and for any  $\mathbf{Q}_1, \mathbf{Q}_2 \in \mathcal{Z}_n$  with  $\|\mathbf{Q}_1\|, \|\mathbf{Q}_2\| \leq C$ , we have  $\|(\tilde{g}_n^{(2)}(\mathbf{Q}_1 | \mathbf{x}))(\mathbf{h}) - (\tilde{g}_n^{(2)}(\mathbf{Q}_2 | \mathbf{x}))(\mathbf{h})\| \leq 6s_2(C)\|\mathbf{h}\|\|\mathbf{Q}_2 - \mathbf{Q}_1\|$ .  $\square$

**Lemma B.10.** *Let  $G_n(\mathbf{Q} | \mathbf{x})$  be as in Lemma B.8. Under assumptions B-1, B-2 and B-3, we have  $[G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))] = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ , where  $\epsilon_n$  is as in Lemma B.7.*

*Proof.* Note that

$$\begin{aligned} &G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})) \\ &= \frac{[\tilde{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))}{n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))} \\ &\quad + \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})) \left[ \frac{1 - n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))}{n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))} \right]. \end{aligned}$$

Using a Taylor expansion and Lemma B.9, we have  $\|[\tilde{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})] - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))\| \leq 6s_2(2M)\|\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\|^2$  for all sufficiently large  $n$ . From assumption B-1, it follows that  $|1 - n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1}d(\mathbf{x}, \mathbf{X}_i))| = O(\sqrt{(n\phi(h_n | \mathbf{x}))^{-1} \log n})$  almost surely as  $n \rightarrow \infty$ . Lemma B.5 implies  $\|\tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))\| \leq B_M\|\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\|$  almost surely for all sufficiently large  $n$ . From Lemma B.7, we get  $\|\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| = O(\epsilon_n)$  almost surely as  $n \rightarrow \infty$ . Therefore,  $\|G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))\| = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ .  $\square$

*Proof of Theorem 4.2.* See the definitions of  $\bar{\mathbf{Q}}_n$  and  $G_n(\mathbf{Q} | \mathbf{x})$  in Lemma B.7 and Lemma B.8, respectively. Note that  $\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) = [\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - G_n(\bar{\mathbf{Q}}_n | \mathbf{x})] + [G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))] + \tilde{g}_n^{(2)}(\bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \bar{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))$ . From Lemma B.8 and Lemma B.10, it follows

that  $[\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) - \hat{g}_n^{(1)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - G_n(\bar{\mathbf{Q}}_n | \mathbf{x})] = O(\epsilon_n^2)$  and  $[G_n(\bar{\mathbf{Q}}_n | \mathbf{x}) - \tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\bar{\mathbf{Q}}_n - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}))] = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ . From (B.1) in Lemma B.7, we have  $\hat{g}_n^{(1)}(\bar{\mathbf{Q}}_n | \mathbf{x}) = O(\epsilon_n^2)$  for all sufficiently large  $n$  almost surely. From Lemma B.5 and Lemma B.6, we get that for all sufficiently large  $n$ ,  $\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot)$  is invertible and  $\|[\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1}\| \leq b_M^{-1}$ , where  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x})\| \leq M$ . Therefore, we get  $\bar{\mathbf{Q}}_n - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) = -[\tilde{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} (\hat{g}_n^{(1)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})) + r_n(\mathbf{x})$ , where  $\|r_n(\mathbf{x})\| = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ . Since  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \bar{\mathbf{Q}}_n\| = O(\epsilon_n^2)$  almost surely for all sufficiently large  $n$  as shown in Lemma B.7, we finally have  $\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) = -[\hat{g}_n^{(2)}(\hat{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} (\hat{g}_n^{(1)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})) + R_n(\mathbf{x})$ , where  $\|R_n(\mathbf{x})\| = O(\epsilon_n^2)$  almost surely as  $n \rightarrow \infty$ .  $\square$

**Lemma B.11.** *Under assumptions B-1, B-2 and B-3,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . In addition, suppose that there exist a constant  $C_4 > 0$  and an integer  $N_3 > 0$  such that whenever  $d(\mathbf{x}, \mathbf{z}) \leq C_4$  and  $n \geq N_3$ , we have, for each  $C > 0$ ,  $(d(\mathbf{x}, \mathbf{z}))^{-1} \|g_n^{(1)}(\mathbf{Q} | \mathbf{z}) - g_n^{(1)}(\mathbf{Q} | \mathbf{x})\| \leq s_3(C)$  for all  $\mathbf{Q} \in \mathcal{Z}_n$  with  $\|\mathbf{Q}\| \leq C$ , where  $s_3(C)$  is a positive constant depending on  $C$ . Then,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| = O(h_n)$  as  $n \rightarrow \infty$ .*

*Proof.* Define  $S_n(\mathbf{Q} | \mathbf{x}) = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}^{(n)}) | \mathbf{X} = \mathbf{x}]$  and  $\tilde{S}_n(\mathbf{Q} | \mathbf{x}) = E[\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}^{(n)}) E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))]$ . By assumptions B-1, B-2 and using arguments similar to those in the proof of Theorem 3.1 in Chakraborty and Chaudhuri (2014), we get that  $S_n(\mathbf{Q} | \mathbf{x})$  and  $\tilde{S}_n(\mathbf{Q} | \mathbf{x})$  are continuous invertible maps with the entire open unit ball in  $\mathcal{Z}_n$  as their range, and  $S_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) = \tilde{S}_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) = \boldsymbol{\tau}^{(n)}$ . Under assumption C(iii),  $\|S_n(\mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - \boldsymbol{\tau}^{(n)}\| = \|S_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - \tilde{S}_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . Using Lemma B.5, it follows that the Fréchet derivative of  $S_n^{-1}(\cdot | \mathbf{x})$  exists, and for some  $c_6 > 0$ ,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| = \|S_n^{-1}(S_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) | \mathbf{x}) - S_n^{-1}(\boldsymbol{\tau}^{(n)} | \mathbf{x})\| \leq c_6 \|S_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - \boldsymbol{\tau}^{(n)}\|$  for all sufficiently large  $n$ . Therefore,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| \rightarrow 0$  as  $n \rightarrow \infty$ . If the additional condition in Lemma B.11 holds, we have  $\|S_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x}) - \tilde{S}_n(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})\| = O(h_n)$  as  $n \rightarrow \infty$ . So, in that case,  $\|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Q}_n(\boldsymbol{\tau} | \mathbf{x})\| = O(h_n)$  as  $n \rightarrow \infty$ .  $\square$

*Proof of Theorem 4.3.* Define  $T_n = n^{-1} \sum_{i=1}^n [ \|\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}\|^{-1} (\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) - \mathbf{Y}_i^{(n)}) - \boldsymbol{\tau}^{(n)} ] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))$ . Clearly,  $T_n$  is an average of  $n$  iid zero mean random elements in  $\mathcal{Z}_n$ . Define the bilinear operator  $\tilde{\gamma}_{(n)}(\mathbf{Q} | \mathbf{x})(\cdot, \cdot) : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{R}$  as  $\tilde{\gamma}_{(n)}(\mathbf{Q} | \mathbf{x})(\mathbf{v}, \mathbf{w}) = \text{Cov}[\langle \mathbf{V}_n(\mathbf{Q}), \mathbf{v} \rangle, \langle \mathbf{V}_n(\mathbf{Q}), \mathbf{w} \rangle]$ , where  $\mathbf{V}_n(\mathbf{Q}) = [\|\mathbf{Q} - \mathbf{Y}^{(n)}\|^{-1}(\mathbf{Q} - \mathbf{Y}^{(n)}) - \boldsymbol{\tau}^{(n)}] E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}))$ . Using assumption C(iii),  $\|\tilde{\gamma}_{(n)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot, \cdot) - \gamma(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot, \cdot)\| \rightarrow 0$  as  $n \rightarrow \infty$ . One can show that  $g^{(2)}(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})(\cdot)$  is invertible using similar arguments as in the proofs of Lemma B.5 and Lemma B.6. Also, using arguments similar to those in the proof of Theorem 3.4 in Chakraborty and Chaudhuri (2014) and assumptions C(iii) and B-3, one can show that  $\sqrt{n\phi(h_n | \mathbf{x})} \|[\hat{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} T_n - [g^{(2)}(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} T_n\| \rightarrow 0$  in probability as  $n \rightarrow \infty$ . Applying Theorem 1.1 in Kundu et al. (2000), it follows that  $\sqrt{n\phi(h_n | \mathbf{x})} [g^{(2)}(\mathbf{Q}(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} T_n \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ , where  $\mathbf{W}$  is the Gaussian random element described in Theorem 4.3. Therefore,  $\sqrt{n\phi(h_n | \mathbf{x})} [\hat{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} T_n \rightarrow \mathbf{W}$  in distribution as  $n \rightarrow \infty$ . Since  $n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i)) \rightarrow 1$  almost surely as  $n \rightarrow \infty$ , we get  $[n^{-1} \sum_{i=1}^n E_n^{-1} K(h_n^{-1} d(\mathbf{x}, \mathbf{X}_i))]^{-1} \sqrt{n\phi(h_n | \mathbf{x})} [\hat{g}_n^{(2)}(\tilde{\mathbf{Q}}_n(\boldsymbol{\tau} | \mathbf{x}) | \mathbf{x})]^{-1} T_n \rightarrow$

$\mathbf{W}$  in distribution as  $n \rightarrow \infty$ . Using the assumptions stated in Theorem 4.3 and the discussion before Theorem 3.4 in Chakraborty and Chaudhuri (2014), one can show that  $\sqrt{n\phi(h_n|\mathbf{x})}[\mathbf{Q}_n(\boldsymbol{\tau}|\mathbf{x})-\mathbf{Q}(\boldsymbol{\tau}|\mathbf{x})] \rightarrow \mathbf{0}$  in probability as  $n \rightarrow \infty$ . From assumption B-1 and Theorem 4.2, it follows that  $\sqrt{n\phi(h_n|\mathbf{x})}R_n(\mathbf{x}) \rightarrow \mathbf{0}$  almost surely as  $n \rightarrow \infty$ . The proof is complete using the representation of  $\widehat{\mathbf{Q}}_n(\boldsymbol{\tau}|\mathbf{x})$  in Theorem 4.2.  $\square$

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