FREE SUBEXPONENTIALITY

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Abstract. In this article, we introduce the notion of *free subexponentiality*, which extends the notion of subexponentiality in the classical probability setup to the noncommutative probability spaces under freeness. We show that distributions with regularly varying tails belong to the class of free subexponential distributions. This also shows that the partial sums of free random elements having distributions with regularly varying tails are tail equivalent to their maximum (in the sense of Ben Arous and Voiculescu, 2006). The analysis is based on the asymptotic relationship between the tail of the distribution and the real and the imaginary parts of the remainder terms in Laurent series expansion of Cauchy transform, as well as, the relationship between the remainder terms in Laurent series expansions of Cauchy and Voiculescu transforms, when the distribution has regularly varying tails.

1. Introduction

A non-commutative probability space is a pair $(\mathcal{A}, \tau)$ where $\mathcal{A}$ is a unital complex algebra and τ is a linear functional on $\mathcal{A}$ satisfying $\phi(1) = 1$. A non-commutative analogue of independence, based on free products, was introduced by Voiculescu (1986). A family of unital subalgebras $\{\mathcal{A}_i\}_{i \in I} \subset \mathcal{A}$ is called *free* if $\tau(a_1 \cdots a_n) = 0$ whenever $\tau(a_j) = 0$, $a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for all j. The above setup is suitable for dealing with bounded random variables. In order to deal with unbounded random variables, we need to consider a tracial W^*-probability space $(\mathcal{A}, \tau)$ with a von Neumann algebra $\mathcal{A}$ and a normal faithful tracial state τ.

A self adjoint operator X is said to be affiliated to a von Neumann algebra $\mathcal{A}$, if $f(X) \in \mathcal{A}$ for any bounded Borel function f on the real line $\mathbb{R}$. A self adjoint operator affiliated with $\mathcal{A}$ will also be called a random element. For an affiliated random element (that is, a self-adjoint operator) X, the algebra generated by X is defined as $\mathcal{A}_X = \{f(X) : \ f \text{ bounded measurable}\}$. The notion of freeness extends to this context easily. A set of random elements $\{X_i\}_{1 \le i \le k}$ affiliated with a von Neumann algebra $\mathcal{A}$, are called freely independent, or simply free, if $\{\mathcal{A}_{X_i}\}_{1 \le i \le k}$ are free.

Given a random element X affiliated with $\mathcal{A}$, the law of X is the unique probability measure μ_X on $\mathbb{R}$ satisfying $\phi(f(X)) = \int_{-\infty}^{\infty} f(t) d\mu_X(t)$ for every bounded Borel function f on $\mathbb{R}$. If e_A denote the projection valued spectral measure associated with X evaluated at the set A, then it is easy to see that $\mu_X(-\infty, x] = \phi(e_{(-\infty,x]}(X))$. The distribution function of X, denoted by F_X, is given by $F_X(x) = \mu_X(-\infty, x]$.

Let $\mathcal{M}$ be the family of probability measures on $\mathbb{R}$. On $\mathcal{M}$, two associative operations $*$ and $\boxplus$ can be defined. The measure $\mu * \nu$ is the classical convolution of

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μ and ν, which also corresponds to the probability law of a random variable $X+Y$, where X and Y are independent and have laws μ and ν respectively. Also, given two measures μ and ν, there exists a unique measure $\mu \boxplus \nu$, called the *free convolution* of μ and ν, such that whenever X and Y are two free random elements on a tracial W^* probability space $(\mathcal{A}, \tau)$ with laws μ and ν respectively, $X+Y$ has the law $\mu \boxplus \nu$. The free convolution was first introduced in Voiculescu (1986) for compactly supported measures, extended by Maassen (1992) to measures with finite variance and by Bercovici and Voiculescu (1993) to measures with unbounded support. The classical and free convolutions of distributions are defined and denoted analogously.

The relationship between $*$ and $\boxplus$ convolution is very striking. They have many similarities like characterizations of infinitely divisible and stable laws (Bercovici and Pata, 1999, 2000a), weak law of large numbers (Bercovici and Pata, 1996) and central limit theorem (Pata, 1996). Analogues of many other classical theories have also been derived. In recent times, links with extreme value theory (Ben Arous and Kargin, 2010, Ben Arous and Voiculescu, 2006) and de Finetti type theorems (Banica et al., 2009) have drawn much attention in the literature. However there are differences too — for example, Cramer's theorem (Bercovici and Voiculescu, 1995) and Raikov's theorem (Benaych-Georges, 2006) fail in the non-commutative setup.

Now we consider an interesting family of distributions in the classical setup called subexponential distributions. The main endeavor of this article is to obtain an analogue of this concept in the non-commutative setup under freeness. A probability measure μ on $[0, \infty)$ is said to be *subexponential*, if for every $n \in \mathbb{N}$,

$$\mu^{*n}(x, \infty) \sim n\mu(x, \infty) \text{ as } x \to \infty.$$

For a random variable X with distribution F and subexponential law μ, X and F are also called subexponential. The above definition can be rephrased in terms of the complementary distribution functions. For a distribution function F, we define its complementary distribution function as $\overline{F} = 1 - F$. Then a subexponential distribution function satisfies, for each natural number n, $\overline{F^{*n}}(x) \sim n\overline{F}(x)$ as $x \to \infty$. The definition can be extended to probability measures μ and equivalently distribution functions F defined on the entire real line. A distribution function F on the real line is called subexponential, if the distribution function F_+, defined as $F_+(x) = F(x)$, for $x \geq 0$ and $F_+(x) = 0$, for $x < 0$, is subexponential. Thus to discuss the subexponential property of the probability measures, it is enough to consider the ones concentrated on $[0, \infty)$. The subexponential random variables satisfy *the principle of one large jump* as well. If $\{X_i\}$ are i.i.d. subexponential random variables, then, for all $n \in \mathbb{N}$,

$$\mathrm{P}[X_1 + \cdots + X_n > x] \sim n\,\mathrm{P}[X_1 > x] = \mathrm{P}[\max_{1 \leq i \leq n} X_i > x] \text{ as } x \to \infty.$$

Such a property makes subexponential distributions an ideal choice for modeling ruin and insurance problems and has caused wide interest in the classical probability literature (cf. Embrechts et al., 1997, Rolski et al., 1999).

The classical definition of subexponential distributions can be easily extended to the non-commutative setup by replacing the classical convolution powers by free convolution powers. We shall define a free subexponential measure on $[0, \infty)$ alone, but the definition can be extended to probability measures on the entire real line, as in the classical case. Formally, we define a free subexponential measure as follows:

Definition 1.1. A probability measure μ on $[0,\infty)$ is said to be free subexponential if for all n,

$$\mu^{\boxplus n}(x,\infty) = \underbrace{(\mu \boxplus \cdots \boxplus \mu)}_{n \text{ times}}(x,\infty) \sim n\mu(x,\infty) \text{ as } x \to \infty.$$

The above definition can be rewritten in terms of distribution functions as well. A distribution function F is called free subexponetial if for all $n \in \mathbb{N}$, $\overline{F^{\boxplus n}}(x) \sim n\overline{F}(x)$ as $x \to \infty$. A random element X affiliated to a tracial W^*-probability space is called free subexponential, if its distribution is so. One immediate consequence of the definition of free subexponentiality is the principle of one large jump.

Ben Arous and Voiculescu (2006) showed that for two distribution functions F and G, there exists a unique measure $F \boxvee G$, such that whenever X and Y are two free random elements on a tracial W^*-probability space, $F \boxvee G$ will become the distribution of $X \vee Y$. Here $X \vee Y$ is the maximum of two self-adjoint operators defined using the spectral calculus via the projection-valued operators, see Ben Arous and Voiculescu (2006) for details. Ben Arous and Voiculescu (2006) showed that $F \boxvee G(x) = \max((F(x) + G(x) - 1), 0)$, and hence $F^{\boxvee n}(x) = \max((nF(x) - (n-1)), 0)$. Then, we have, for each n, $\overline{F^{\boxvee n}}(x) \sim n\overline{F}(x)$ as $x \to \infty$. Thus, by definition of free subexponentiality, we have

Proposition 1.1 (Free one large jump principle)**.** *Free subexponential distributions satisfy the principle of one large jump, namely, if F is freely subexponential, then, for every n,*

$$\overline{F^{\boxplus n}}(x) \sim \overline{F^{\boxvee n}}(x) \text{ as } x \to \infty.$$

In spite of the above important property possessed by the free subexponential distributions, it remains to be checked whether such a class is nonempty. The distributions with regularly varying (right) tails of index $-\alpha$, with $\alpha \geq 0$, form an important class of examples of subexponential distributions in the classical setup. (In further discussions, we shall suppress the qualifier "right".) A (real valued) measurable function f defined on nonnegative real line is called *regularly varying* (at infinity) with index α if, for every $t > 0$, $f(tx)/f(x) \to t^\alpha$ as $x \to \infty$. If $\alpha = 0$, then f is said to be slowly varying (at infinity). Regular variation with index α at zero is defined analogously. In fact, f is regularly varying at zero of index α, if the function $x \mapsto f(1/x)$ is regularly varying at infinity of index $-\alpha$. Unless otherwise mentioned, the regular variation of a function will be considered at infinity. For regular variation at zero, we shall explicitly mention so. A distribution function F on $[0,\infty)$ has regularly varying tail of index $-\alpha$, if $\overline{F}(x)$ is regularly varying of index $-\alpha$. Since $\overline{F}(x) \to 0$ as $x \to \infty$, we must necessarily have $\alpha \geq 0$. As in the case of subexponential distributions, a distribution F on the entire real line is said to have regularly varying tail if F_+ has so. Note that, for $x > 0$, we have $\overline{F_+}(x) = \overline{F}(x)$. A probability measure with regularly varying tail is defined through its distribution function. Equivalently, a measure μ is said to have a regularly varying tail, if $\mu(x,\infty)$ is regularly varying.

Other than distributions with regularly varying tails, Weibull distributions with shape parameter less than 1 and lognormal distribution are some other well known examples of subexponential distributions in the classical setup. The last two distributions have all moments finite unlike the distributions with regularly varying tails of index $-\alpha$, which have all moments higher than α infinite.

The distributions with regularly varying tails have already attracted attention in the non-commutative probability theory. They play a very crucial role in determining the domains of attraction of stable laws (Bercovici and Pata, 1999, 2000a). In this article, we shall show that the distributions with regularly varying tails form a subclass of the free subexponential distributions.

Theorem 1.1. *If a distribution function F has regularly varying tail of index $-\alpha$ with $\alpha \geq 0$, then F is free subexponential.*

While it need not be assumed that the measure is concentrated on $[0, \infty)$, both the notions of free subexponentiality and regular variation are defined in terms of the measure restricted to $[0, \infty)$. Thus we shall assume the measure to be supported on $[0, \infty)$ except for the definitions of the relevant transforms in the initial part of Subsection 2.2 and in the statement and the proof of Theorem 1.1. Due to the lack of coordinate systems and expressions for joint distributions of non-commutative random elements in terms of probability measures, the proofs of the above results deviate from the classical ones. In absence of the higher moments of the distributions with regularly varying tails, we cannot use the usual moment-based approach used in free probability theory. Instead, Cauchy and Voiculescu transforms become the natural tools to deal with the free convolution of measures. We recall the notions of these transforms in Section 2. We then discuss the relationship between the remainder terms of Laurent expansions of Cauchy and Voiculescu transforms of measures with regularly varying tail of index $-\alpha$. We need to consider two cases separately depending on the maximum number p of integer moments that the measure μ may have. For a nonnegative integer p, let us denote the class of all probability measures μ on $[0, \infty)$ with $\int_0^\infty t^p d\mu(t) < \infty$, but $\int_0^\infty t^{p+1} d\mu(t) = \infty$, by $\mathcal{M}_p$. We shall also denote the class of all probability measures μ in $\mathcal{M}_p$ with regularly varying tail of index $-\alpha$ by $\mathcal{M}_{p,\alpha}$. Note that, we necessarily have $\alpha \in [p, p+1]$. We state Theorem 2.1 for the case $\alpha \in [p, p+1)$ and Theorem 2.2 for the case $\alpha = p+1$ in Section 2. These theorems are the key tools of this article. Section 2 is concluded with two Abel-Tauber type results for Stieltjes transform of measures with regularly varying tail. We then prove Theorems 1.1 in Section 3 using Theorems 2.1 and 2.2. We use the final two sections to prove the Theorems 2.1 and 2.2. In Section 4, we collect some results about the remainder term in Laurent series expansion of Cauchy transform of measures with regularly varying tails. In Section 5, we study the relationship between the remainder terms in Laurent expansions of Cauchy and Voiculescu transforms through a general analysis of the remainder terms of Taylor expansions of a suitable class of functions and their inverses or reciprocals. Combining, the results of Sections 4 and 5, we prove Theorems 2.1 and 2.2.

2. Some transforms and their related properties

In this section, we collect some notations, definitions and results to be used later in the article. In Subsection 2.1, we define the concept of non-tangential limits. Various transforms in non-commutative probability theory, like Cauchy, Voiculescu and R transforms are introduced in Subsection 2.2. Theorems 2.1 and 2.2 regarding the relationship between the remainder terms of Laurent expansions of Cauchy and Voiculescu transforms are given in this subsection as well. Finally, in Subsection 2.3, two results about measures with regularly varying tails are given.

2.1. **Non-tangential limits and notations.** The complex plane will be denoted by $\mathbb{C}$ and for a complex number z, $\Re z$ and $\Im z$ will denote its real and imaginary parts respectively. We say z goes to infinity (zero respectively) *non-tangentially* to $\mathbb{R}$ (n.t.), if z goes to infinity (zero respectively), while $\Re z/\Im z$ stays bounded. We can then define that a function f converges or stays bounded as z goes to infinity (or zero) n.t. To elaborate upon the notion, given positive numbers η, δ and M, let us define the following cones:

(1) $\Gamma_\eta = \{z \in \mathbb{C}^+ : |\Re z| < \eta \Im z\}$ and $\Gamma_{\eta,M} = \{z \in \Gamma_\eta : |z| > M\}$,
(2) $\Delta_\eta = \{z \in \mathbb{C}^- : |\Re z| < -\eta \Im z\}$ and $\Delta_{\eta,\delta} = \{z \in \Delta_\eta : |z| < \delta\}$,

where $\mathbb{C}^+$ and $\mathbb{C}^-$ are the upper and the lower halves of the complex plane respectively, namely, $\mathbb{C}^+ = \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathbb{C}^- = -\mathbb{C}^+$. Then we shall say that $f(z) \to l$ as z goes to ∞ n.t., if for any $\epsilon > 0$ and $\eta > 0$, there exists $M \equiv M(\eta, \epsilon) > 0$, such that $|f(z) - l| < \epsilon$, whenever $z \in \Gamma_{\eta,M}$. This is same as saying that the convergence in $\mathbb{C}^+$ is uniform in each cone Γ_η. The boundedness can be defined analogously.

We shall write $f(z) \approx g(z)$, $f(z) = \mathrm{o}(g(z))$ and $f(z) = \mathrm{O}(g(z))$ as $z \to \infty$ n.t. to mean that $f(z)/g(z)$ converges to a non-zero limit, $f(z)/g(z) \to 0$ and $f(z)/g(z)$ stays bounded as $z \to \infty$ n.t. respectively. If the non-zero limit is 1 in the first case, we write $f(z) \sim g(z)$ as $z \to \infty$ n.t. For $f(z) = \mathrm{o}(g(z))$ as $z \to \infty$ n.t., we shall also use the notations $f(z) \ll g(z)$ and $g(z) \gg f(z)$ as $z \to \infty$ n.t.

The map $z \mapsto 1/z$ maps the set $\Gamma_{\eta,1/\delta}$ onto $\Delta_{\eta,\delta}$ for each positive η and δ. Thus the analogous concepts can be defined for $z \to 0$ n.t. using $\Delta_{\eta,\delta}$.

2.2. **Cauchy and Voiculescu Transform.** For a probability measure $\mu \in \mathcal{M}$, its Cauchy transform is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+.$$

Note that G_μ maps $\mathbb{C}^+$ to $\mathbb{C}^-$. Set $F_\mu = 1/G_\mu$, which maps $\mathbb{C}^+$ to $\mathbb{C}^+$. We shall be also interested in the function $H_\mu(z) = G_\mu(1/z)$ which maps $\mathbb{C}^-$ to $\mathbb{C}^-$.

By Proposition 5.4 and Corollary 5.5 of Bercovici and Voiculescu (1993), for all $\eta > 0$ and for all $\epsilon \in (0, \eta \wedge 1)$, there exists $\delta \equiv \delta(\eta)$ small enough, such that H_μ is a conformal bijection from $\Delta_{\eta,\delta}$ onto an open set $\mathcal{D}_{\eta,\delta}$, where the range sets satisfy

$$\Delta_{\eta-\epsilon,(1-\epsilon)\delta} \subset \mathcal{D}_{\eta,\delta} \subset \Delta_{\eta+\epsilon,(1+\epsilon)\delta}.$$

If we define $\mathcal{D} = \cup_{\eta>0}\mathcal{D}_{\eta,\delta(\eta)}$, then we can obtain an analytic function L_μ with domain $\mathcal{D}$ by patching up the inverses of H_μ on $\mathcal{D}_{\eta,\delta(\eta)}$ for each $\eta > 0$. In this case L_μ becomes the right inverse of H_μ on $\mathcal{D}$. Also it was shown that the sets of type $\Delta_{\eta,\delta}$ were contained in the unique connected component of the set $H_\mu^{-1}(\mathcal{D})$. It follows that H_μ is the right inverse of L_μ on $\Delta_{\eta,\delta}$ and hence on the whole connected component by analytic continuation.

We then define R and Voiculescu transforms of the probability measure μ respectively as:

$$R_\mu(z) = \frac{1}{L_\mu(z)} - \frac{1}{z} \quad \text{and} \quad \phi_\mu(z) = R_\mu(1/z). \tag{2.1}$$

Arguing as in the case of $G_\mu(1/z)$, it can be shown that F_μ has a left inverse, denoted by F_μ^{-1} on a suitable domain and, in that case, we have

$$\phi_\mu(z) = F_\mu^{-1}(z) - z.$$

Bercovici and Voiculescu (1993) established the following relation between free convolution and Voiculescu and R transforms. For probability measures μ and ν,

$$\phi_{\mu\boxplus\nu} = \phi_\mu + \phi_\nu \quad \text{and} \quad R_{\mu\boxplus\nu} = R_\mu + R_\nu,$$

wherever all the functions involved are defined.

We shall also need to analyze the power and Taylor series expansions of the above transforms. For compactly supported measure μ, Speicher (1994) showed that, in an appropriate neighborhood of zero, $R_\mu(z) = \sum_{j=0}^{\infty} \kappa_{j+1}(\mu) z^j$, where $\{\kappa_j(\mu)\}$ denotes the free cumulant sequence of the probability measure μ. For probability measures μ with finite p moments, Taylor expansions of R_μ and H_μ are given by Theorems 1.3 and 1.5 of Benaych-Georges (2005):

$$R_\mu(z) = \sum_{j=0}^{p-1} \kappa_{j+1}(\mu) z^j + z^{p-1} r_{R_\mu}(z), \text{ and } H_\mu(z) = \sum_{j=1}^{p+1} m_{j-1}(\mu) z^j + z^{p+1} r_{H_\mu}(z), \tag{2.2}$$

where the remainder terms $r_{R_\mu}(z) \equiv r_R(z) = \mathrm{o}(1)$ and $r_{H_\mu}(z) \equiv r_H(z) = \mathrm{o}(1)$ as $z \to 0$ n.t., where $\{\kappa_j(\mu) : j \le p\}$ denotes the free cumulant sequence of μ as before and $\{m_j(\mu) : j \le p\}$ denotes the moment sequence of the probability measure μ. When there is no possibility of confusion, we shall sometimes suppress the measure involved in the notation for the moment and the cumulant sequences, as well as the remainder terms. In the study of stable laws and the infinitely divisible laws, the following relationship between Cauchy and Voiculescu transforms of a probability measure μ, obtained in Proposition 2.5 of Bercovici and Pata (1999), played a crucial role:

$$\phi_\mu(z) \sim z^2 \left[G_\mu(z) - \frac{1}{z} \right] \text{ as } z \to \infty \text{ n.t.} \tag{2.3}$$

We shall extend this result for probability measures $\mu \in \mathcal{M}_p$. (Recall that for $\mu \in \mathcal{M}_p$, μ is concentrated on $[0, \infty)$.) We need to introduce the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms for this purpose in analogy to (2.2):

$$r_{G_\mu}(z) \equiv r_G(z) = z^{p+1} \left(G_\mu(z) - \sum_{j=1}^{p+1} m_{j-1}(\mu) z^{-j} \right)$$

and

$$r_{\phi_\mu}(z) \equiv r_\phi(z) = z^{p-1} \left(\phi_\mu(z) - \sum_{j=0}^{p-1} \kappa_{j+1}(\mu) z^{-j} \right), \tag{2.4}$$

where we shall again suppress the measure μ in the notation if there is no possibility of confusion. In (2.4), we interpret the sum on the right side as zero, when $p = 0$. We now state the extensions of (2.3). We first consider the case $\alpha \in [p, p+1)$.

Theorem 2.1. *Let μ be a probability measure in the class $\mathcal{M}_p$ and $\alpha \in [p, p+1)$. The following statements are equivalent:*

(i) *$\mu(y, \infty)$ is regularly varying of index $-\alpha$.*
(ii) *$\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$.*
(iii) *$\Im r_\phi(iy)$ is regularly varying of index $-(\alpha - p)$, $\Re r_\phi(iy) \gg y^{-1}$ as $y \to \infty$ and $r_\phi(z) \gg z^{-1}$ as $z \to \infty$ n.t.*

If any of the above statements holds, we also have, as $z \to \infty$ *n.t.,* $r_G(z) \sim r_\phi(z) \gg z^{-1}$*; as* $y \to \infty$*,*

$$\Im r_\phi(iy) \sim \Im r_G(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty) \gg \frac{1}{y} \text{ and } \Re r_\phi(iy) \sim \Re r_G(iy) \gg \frac{1}{y}.$$

If $\alpha > p$ *and any of the statements* (i)-(iii) *holds, we further have, as* $y \to \infty$*,*

$$\Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\frac{\pi(p+2-\alpha)}{2}}{\sin\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty).$$

If $\alpha = p = 0$ *and any of the statements* (i)-(iii) *holds, we further have, as* $y \to \infty$*,*

$$\Re r_\phi(iy) \sim \Re r_G(iy) \sim -\mu(y,\infty).$$

Next we consider the case $\alpha = p + 1$.

Theorem 2.2. *Let* μ *be a probability measure in the class* $\mathcal{M}_p$ *and* $\beta \in (0, 1/2)$*. The following statements are equivalent:*

(i) $\mu(y,\infty)$ *is regularly varying of index* $-(p+1)$*.*
(ii) $\Re r_G(iy)$ *is regularly varying of index* -1*.*
(iii) $\Re r_\phi(iy)$ *is regularly varying of index* -1*,* $y^{-1} \ll \Im r_\phi(iy) \ll y^{-(1-\beta/2)}$ *as* $y \to \infty$ *and* $z^{-1} \ll r_\phi(z) \ll z^{-\beta}$ *as* $z \to \infty$ *n.t.*

If any of the above statements holds, we also have, as $z \to \infty$ *n.t.,* $z^{-1} \ll r_G(z) \sim r_\phi(z) \ll z^{-\beta}$*; as* $y \to \infty$*,*

$$y^{-(1+\beta/2)} \ll \Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \ll y^{-(1-\beta/2)}$$

and

$$y^{-1} \ll \Im r_\phi(iy) \sim \Im r_G(iy) \ll y^{-(1-\beta/2)}.$$

It is easy to obtain the equivalent statements for H_μ and R_μ through the simple observation that $G_\mu(z) = H_\mu(1/z)$ and $\phi_\mu(z) = R_\mu(1/z)$. For $p = 0$, Theorems 2.1 and 2.2 together give a special case of (2.3) for the probability measures with regularly varying tail and infinite mean. However, Theorems 2.1 and 2.2 give more detailed asymptotic behavior of the real and imaginary parts separately, which is required for our analysis.

2.3. **Karamata type results.**

We provide here two results for regularly varying functions, which we shall be using in the proofs of our results. They are variants of Karamata's Abel-Tauber theorem for Stieltjes transform (cf. Bingham et al., 1987, Section 1.7.5) and explains the regular variation of Cauchy transform of measures with regularly varying tails.

The first result is quoted from Bercovici and Pata (1999).

Proposition 2.1 (Bercovici and Pata, 1999, Corollary 5.4)**.** *Let* ρ *be a positive Borel measure on* $[0,\infty)$ *and fix* $\alpha \in [0,2)$*. Then the following statements are equivalent:*

(i) $y \mapsto \rho[0,y]$ *is regularly varying of index* α*.*
(ii) $y \mapsto \int_0^\infty \frac{1}{t^2+y^2} d\rho(t)$ *is regularly varying of index* $-(2-\alpha)$*.*

If either of the above conditions is satisfied, then

$$\int_0^\infty \frac{1}{t^2+y^2} d\rho(t) \sim \frac{\frac{\pi\alpha}{2}}{\sin\frac{\pi\alpha}{2}} \frac{\rho[0,y]}{y^2} \text{ as } y \to \infty.$$

The constant pre-factor on the right side is interpreted as 1 *when* $\alpha = 0$.

The second result uses a different integrand.

Proposition 2.2. *Let* ρ *be a finite positive Borel measure on* $[0,\infty)$ *and fix* $\alpha \in [0,2)$. *Then the following statements are equivalent:*

(i) $y \mapsto \rho(y,\infty)$ *is regularly varying of index* $-\alpha$.
(ii) $y \mapsto \int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t)$ *is regularly varying of index* $-\alpha$.

If either of the above conditions is satisfied, then

$$\int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) \sim \frac{\frac{\pi\alpha}{2}}{\sin\frac{\pi\alpha}{2}} \rho(y,\infty) \text{ as } y \to \infty.$$

The constant pre-factor on the right side is interpreted as 1 *when* $\alpha = 0$.

Proof. Define $d\widetilde{\rho}(y) = \rho(\sqrt{s},\infty)ds$. By a variant of Karamata's theorem given in Theorem 0.6(a) of Resnick (1987), as $\alpha < 2$, we have

$$\widetilde{\rho}[0,y] \sim \frac{1}{1-\frac{\alpha}{2}} y\rho(\sqrt{y},\infty) \tag{2.5}$$

is regularly varying of index $1-\alpha/2$. Then, we have,

$$\begin{aligned}\int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) &= y^2 \int_0^\infty \int_0^t \frac{2s ds}{(s^2+y^2)^2} d\rho(t) \\ &= y^2 \int_0^\infty \frac{2s\rho(s,\infty)}{(s^2+y^2)^2} ds = y^2 \int_0^\infty \frac{d\widetilde{\rho}(s)}{(s+y^2)^2}.\end{aligned}$$

Now, first applying Theorem 1.7.4 of Bingham et al. (1987), as $\widetilde{\rho}[0,y]$ is regularly varying of index $1-\alpha/2 \in (0,2]$ and then (2.5), we have

$$\int_0^\infty \frac{t^2}{t^2+y^2} d\rho(t) \sim \frac{\left(1-\frac{\alpha}{2}\right)\frac{\pi\alpha}{2}}{\sin\frac{\pi\alpha}{2}} y^2 \frac{\widetilde{\rho}[0,y^2]}{y^4} \sim \frac{\frac{\pi\alpha}{2}}{\sin\frac{\pi\alpha}{2}} \rho(y,\infty).$$

□

3. FREE SUBEXPONENTIALITY OF MEASURES WITH REGULARLY VARYING TAILS

We now use Theorems 2.1 and 2.2 to prove Theorem 1.1. We shall first look at the tail behavior of the free convolution of two probability measures with regularly varying tails and which are tail balanced. Theorem 1.1 will be proved by suitable choices of the two measures.

Lemma 3.1. *Suppose* μ *and* ν *are two probability measures on* $[0,\infty)$ *with regularly varying tails, which are tail balanced, that is, for some* $c > 0$, *we have* $\nu(y,\infty) \sim c\,\mu(y,\infty)$. *Then*

$$\mu \boxplus \nu(y,\infty) \sim (1+c)\mu(y,\infty).$$

Proof. We shall now indicate the associated probability measures in the remainder terms, moments and the cumulants to avoid any confusion. Since μ and ν are tail balanced and have regularly varying tails, for some nonnegative integer p and $\alpha \in [p, p+1]$, we have both μ and ν in the same class $\mathcal{M}_{p,\alpha}$. Obviously, for $\alpha \in [p, p+1)$, we apply Theorem 2.1 on the imaginary parts of the remainder terms in Laurent expansion of Voiculescu transforms, while for $\alpha = p+1$, we apply Theorem 2.2 on the real parts of the corresponding objects. We shall only do the first case in details, while the second case will be similar.

For $\alpha \in [p, p+1)$, by Theorem 2.1, we have

$$r_{\phi_\mu}(z) \gg z^{-1} \quad \text{and} \quad r_{\phi_\nu}(z) \gg z^{-1} \tag{3.1}$$

$$\Re r_{\phi_\mu}(-iy) \gg y^{-1} \quad \text{and} \quad \Re r_{\phi_\nu}(-iy) \gg y^{-1} \tag{3.2}$$

$$\Im r_{\phi_\mu}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty) \quad \text{and} \quad \Im r_{\phi_\nu}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \nu(y,\infty). \tag{3.3}$$

We also know that, both Voiculescu transforms and cumulants add up in case of free convolution. Hence,

$$r_{\phi_{\mu\boxplus\nu}}(z) = r_{\phi_\mu}(z) + r_{\phi_\nu}(z). \tag{3.4}$$

Further, we shall have $\kappa_p(\mu \boxplus \nu) < \infty$, but $\kappa_{p+1}(\mu \boxplus \nu) = \infty$ and similar results hold for the moments of $\mu \boxplus \nu$ as well. Then Theorem 2.1 will also apply for $\mu \boxplus \nu$. Thus, applying (3.4) and its real and imaginary parts evaluated at $z = iy$, together with (3.1), (3.2) and (3.3) respectively, we get,

$$r_{\phi_{\mu\boxplus\nu}}(z) \gg z^{-1} \text{ as } z \to \infty \text{ n.t.,}$$

$$\Re r_{\phi_{\mu\boxplus\nu}}(iy) \gg y^{-1} \text{ as } y \to \infty$$

and

$$\Im r_{\phi_{\mu\boxplus\nu}}(iy) \sim -(1+c)\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty) \text{ as } y \to \infty, \tag{3.5}$$

which is regularly varying of index $-(\alpha - p)$. In the final step, we also use the hypothesis that $\nu(y,\infty) \sim c\mu(y,\infty)$ as $y \to \infty$. Thus, again using Theorem 2.1, we have

$$-\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu \boxplus \nu(y,\infty) \sim \Im r_{\phi_{\mu\boxplus\nu}}(iy). \tag{3.6}$$

Combining (3.5) and (3.6), the result follows. □

We are now ready to prove the subexponentiality of a distribution with regularly varying tail.

Proof of Theorem 1.1. Let μ be the probability measure on $[0,\infty)$ associated with the distribution function F_+. Then μ also has regularly varying tail of index $-\alpha$. We prove that

$$\mu^{\boxplus n}(y,\infty) \sim n\mu(y,\infty), \text{ as } y \to \infty \tag{3.7}$$

by induction on n. To prove (3.7), for $n = 2$, apply Lemma 3.1 with both the probability measures as μ and the constant $c = 1$. Next assume (3.7) holds for $n = m$. To prove (3.7), for $n = m+1$, apply Lemma 3.1 again with the probability measures μ and $\mu^{\boxplus m}$ and the constant $c = m$. □

4. Cauchy transform of measures with regularly varying tail

As a first step towards proving Theorems 2.1 and 2.2, we now collect some results about $r_G(z)$, when the probability measure μ has regularly varying tails. These results will be be useful in showing equivalence between the tail of μ and $r_G(iy)$. It is easy to see by induction that

$$\frac{1}{z-t}-\sum_{j=0}^{p}\left(\frac{t}{z}\right)^{j}=\left(\frac{t}{z}\right)^{p+1}\frac{1}{z-t}.$$

Integrating and multiplying by z^{p+1}, we get

$$r_G(z)=\int_0^\infty \frac{t^{p+1}}{z-t}d\mu(t). \tag{4.1}$$

We use (4.1) to obtain asymptotic upper and lower bounds for $r_G(z)$ as $z\to\infty$ n.t. Similar results about r_H can be obtained easily from the fact that $r_G(z)=r_H(1/z)$, but will not be stated separately. We consider the lower bound first.

Proposition 4.1. *Suppose $\mu\in\mathcal{M}_p$ for some nonnegative integer p, then*

$$z^{-1}\ll r_G(z) \text{ as } z\to\infty \text{ n.t.}$$

Proof. We need to show that, for any $\eta>0$, as $|z|\to\infty$ with $z=x+iy\to\infty$ within the cone Γ_η, $|zr_G(z)|\to\infty$. Note that, for $z=x+iy\in\Gamma_\eta$, we have $|x|<\eta y$. Now, as $|z-t|^2=(z-t)(\bar{z}-t)$ and $z(\bar{z}-t)=|z|^2-zt$, using (4.1), we have,

$$zr_G(z)=z\int_0^\infty\frac{t^{p+1}}{z-t}d\mu(t)=|z|^2\int_0^\infty\frac{t^{p+1}}{|z-t|^2}d\mu(t)-z\int_0^\infty\frac{t^{p+2}}{|z-t|^2}d\mu(t),$$

which gives

$$\Re(zr_G(z))=|z|^2\int_0^\infty\frac{t^{p+1}}{|z-t|^2}d\mu(t)-\Re z\int_0^\infty\frac{t^{p+2}}{|z-t|^2}d\mu(t) \tag{4.2}$$

and

$$\Im(zr_G(z))=\Im z\int_0^\infty\frac{t^{p+2}}{|z-t|^2}d\mu(t). \tag{4.3}$$

On Γ_η and for $t\in[0,\eta y]$, $|t-x|\le t+|x|\le 2\eta y$. Thus, we have,

$$\int_0^\infty\frac{|z|^2t^{p+1}}{|z-t|^2}d\mu(t)\ge\int_0^{\eta y}\frac{y^2t^{p+1}}{(t-x)^2+y^2}d\mu(t)\ge\frac{1}{1+4\eta^2}\int_0^{\eta y}t^{p+1}d\mu(t)\to\infty, \tag{4.4}$$

since $\mu\in\mathcal{M}_p$.

Now fix $\eta>0$, and consider a sequence $\{z_n=x_n+iy_n\}$ in Γ_η, such that $|z_n|\to\infty$, that is, $|x_n|\le uy_n$ and $y_n\to\infty$. If possible, suppose that $\{|z_nr_G(z_n)|\}$ is a bounded sequence, then both the real and the imaginary parts of the sequence will be bounded. However, then the boundedness of the real part and (4.2) and (4.4) give

$$\left|\Re z_n\int_0^\infty\frac{t^{p+2}}{|z_n-t|^2}d\mu(t)\right|\to\infty.$$

Then, using (4.3) and the fact that $|\Re z|\le\eta\Im z$ on Γ_η, we have

$$\Im(z_nr_G(z_n))\ge\frac{1}{\eta}\left|\Re z_n\int_0^\infty\frac{t^{p+2}}{|z_n-t|^2}d\mu(t)\right|\to\infty,$$

which contradicts the fact that the imaginary part of the sequence $\{z_n r_G(z_n)\}$ is bounded and completes the proof. $\square$

We now consider the upper bound for $r_G(z)$. The result and the proof of the following proposition are inspired by Lemma 5.2(iii) of Bercovici and Pata (2000b).

Proposition 4.2. *Let μ be a probability measure in the class $\mathcal{M}_{p,\alpha}$ for some non-negative integer p and $\alpha \in (p, p+1]$. Then, for any $\beta \in [0, (\alpha-p)/(\alpha-p+1))$, we have*

$$r_G(z) = \mathrm{o}(z^{-\beta}) \text{ as } z \to \infty \text{ n.t.} \tag{4.5}$$

Remark 4.1. Note that (4.5) holds also for $p = \alpha$ with $\beta = 0$, which can be readily seen from Theorem 1.5 of Benaych-Georges (2005).

Proof of Proposition 4.2. Define a measure ρ_0 as $d\rho_0(t) = t^p d\mu(t)$. Since $\mu \in \mathcal{M}_p$, ρ_0 is a finite measure. Further, since $p < \alpha$, using Theorem 1.6.5 of Bingham et al. (1987), we have $\rho_0(y, \infty) \sim \frac{\alpha}{\alpha-p} y^p \mu(y, \infty)$, which is regularly varying of index $-(\alpha - p)$.

Now fix $\eta > 0$. It is easy to check that for $t \geq 0$ and $z \in \Gamma_\eta$, $t/|z-t| < \sqrt{1+\eta^2}$. For $z = x + iy$, we have $|z - t| > y$ and hence for $t \in [0, y^{1/(\alpha-p+1)}]$, we have $t/|z-t| < y^{-(\alpha-p)/(\alpha-p+1)}$. Then, using (4.1) and the definition of ρ_0,

$$\begin{aligned} |r_G(z)| &\leq \int_0^{y^{1/(\alpha-p+1)}} \left|\frac{t}{z-t}\right| d\rho_0(t) + \sqrt{1+\eta^2}\rho_0\left(y^{1/(\alpha-p+1)}, \infty\right) \\ &\leq y^{-(\alpha-p)/(\alpha-p+1)} \int_0^\infty t^p d\mu(t) + \sqrt{1+\eta^2}\rho_0\left(y^{1/(\alpha-p+1)}, \infty\right) = \mathrm{o}(y^{-\beta}), \end{aligned}$$

for any $\beta \in [0, (\alpha-p)/(\alpha-p+1))$, as the second term is regularly varying of index $-(\alpha-p)/(\alpha-p+1)$. Further, for $z = x + iy \in \Gamma_\eta$, we have $|z| = \sqrt{x^2+y^2} \leq y\sqrt{1+\eta^2}$, and hence we have the required result. $\square$

Next we specialize to the asymptotic behavior of $r_G(iy)$, as $y \to \infty$. Observe that

$$\Re r_G(iy) = -\int_0^\infty \frac{t^{p+2}}{t^2+y^2} d\mu(t) \quad \text{and} \quad \Im r_G(iy) = -y\int_0^\infty \frac{t^{p+1}}{t^2+y^2} d\mu(t). \tag{4.6}$$

Proposition 4.3. *Let μ be a probability measure in the class $\mathcal{M}_p$.*

If $\alpha \in (p, p+1)$, then the following statements are equivalent:

- (i) *μ has regularly varying tail of index $-\alpha$.*
- (ii) *$\Re r_G(iy)$ is regularly varying of index $-(\alpha - p)$.*
- (iii) *$\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$.*

If any of the above statements holds, then

$$\frac{\sin\frac{\pi(\alpha-p)}{2}}{\frac{\pi(p+2-\alpha)}{2}} \Re r_G(iy) \sim \frac{\cos\frac{\pi(\alpha-p)}{2}}{\frac{\pi(p+1-\alpha)}{2}} \Im r_G(iy) \sim -y^p \mu(y, \infty) \text{ as } y \to \infty.$$

Further, $\Re r_G(iy) \gg y^{-1}$ and $\Im r_G(iy) \gg y^{-1}$ as $y \to \infty$.

If $\alpha = p$, then the statements (i) *and* (iii) *above are equivalent. Also, if either of the statements holds, then*

$$\Im r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y, \infty) \text{ as } y \to \infty. \tag{4.7}$$

Further, $\Im r_G(iy) \gg y^{-1}$ as $y \to \infty$.

If $\alpha = p+1$*, then the statements* (i) *and* (ii) *above are equivalent. Also, if either of the statements holds, then*

$$\Re r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \text{ as } y \to \infty. \tag{4.8}$$

Further, for any $\varepsilon > 0$*,* $\Re r_G(iy) \gg y^{-(1+\varepsilon)}$ *as* $y \to \infty$*.*

Remark 4.2. Note that, for $\alpha = p+1$, $\Re r_G(iy)$ is regularly varying of index -1 and the asymptotic lower bound $\Re r_G(iy) \gg y^{-1}$ need not hold. This causes some difficulty in the proof of Propositions 5.1 and 5.2. The lack of the asymptotic lower bound has to be compensated for by the stronger upper bound obtained in Proposition 4.2, which holds for $\alpha = p+1$. This is reflected in the class $\mathcal{R}_{p,\beta}$ for $\beta > 0$, defined in Section 5. Further note that, the situation reverses for $\alpha = p$, as Proposition 4.2 need not hold. The case, where $\alpha \in (p, p+1)$ is not an integer, is simple, as the asymptotic lower bounds hold for both the real and imaginary parts of $r_G(iy)$ (Proposition 4.3), as well as, the stronger asymptotic upper bound works (Proposition 4.2). However, the case of non-integer $\alpha \in (p, p+1)$ is treated simultaneously with the case $\alpha = p$ in Theorem 2.1 and as the class $\mathcal{R}_{p,0}$ (cf. Section 5) in Propositions 5.1 and 5.2.

Proof of Proposition 4.3. The asymptotic lower bounds for the real and the imaginary parts of $r_G(iy)$ are immediate from (ii) and (iii) respectively. So, we only need to show (4.8) and the equivalence between (i) and (ii) when $\alpha \in (p, p+1]$ and (4.7) and the equivalence between (i) and (iii) when $\alpha \in [p, p+1)$.

Let $d\rho_j(t) = t^{p+j} d\mu(t)$, for $j = 1, 2$. Then, by Theorem 1.6.4 of Bingham et al. (1987), we have, for $\alpha \in [p, p+1)$, $\rho_1[0, y] \sim \alpha/(p+1-\alpha) y^{p+1} \mu(y, \infty)$, which is regularly varying of index $p+1-\alpha \in (0, 1]$, and, for $\alpha \in (p, p+1]$, $\rho_2[0, y] \sim \alpha/(p+2-\alpha) y^{p+2} \mu(y, \infty)$, which is regularly varying of index $p+2-\alpha \in [1, 2)$. Further, from (4.6), we get

$$\Re r_G(iy) = -\int_0^\infty \frac{1}{t^2+y^2} d\rho_2(t) \quad \text{and} \quad \Im r_G(iy) = -y \int_0^\infty \frac{1}{t^2+y^2} d\rho_1(t).$$

Then the results follow immediately from Proposition 2.1. □

While asymptotic equivalences between $\Re r_G(iy)$ and tail of μ for $\alpha = p$ and $\Im r_G(iy)$ and tail of μ for $\alpha = p+1$ are not true in general, we obtain the relevant asymptotic bounds in these cases. We also obtain the exact asymptotic orders when $p = 0$.

Proposition 4.4. *Consider a probability measure* μ *in the class* $\mathcal{M}_p$*.*

If μ *has regularly varying tail of index* $-p$*, then, for any* $\varepsilon > 0$*,* $\Re r_G(iy) \gg y^{-\varepsilon}$ *as* $y \to \infty$*. Further, if* $p = 0$*, then* $\Re r_G(iy) \sim -\mu(y, \infty)$ *as* $y \to \infty$*.*

If μ *has regularly varying tail of index* $-(p+1)$*, then* $\Im r_G(iy)$ *is regularly varying of index* -1 *and* $y^{-1} \ll \Im r_G(iy) \ll y^{-(1-\varepsilon)}$ *as* $y \to \infty$*, for any* $\varepsilon > 0$*.*

Remark 4.3. Note that, in the case $\alpha = p+1$, the lower bound for $\Im r_G(iy)$ is sharper than $\Re r_G(iy)$ and is same as that of $\Im r_G(iy)$ for the case $\alpha \in [p, p+1)$.

Proof of Proposition 4.4. First consider the case where μ has regularly varying tail of index $-p$. Recall from the proof of Proposition 4.2 that $d\rho_0(t) = t^p d\mu(t)$. However, in the current situation Theorem 1.6.4 of Bingham et al. (1987) will not apply.

If $p = 0$, then $\rho_0 = \mu$ and $\rho_0(y, \infty)$ is slowly varying. If $p > 0$, observe that, as $\int t^p d\mu(t) < \infty$, we have

$$\rho_0(y, \infty) = y^p \mu(y, \infty) + p \int_0^y s^{p-1} \mu(s, \infty) ds \sim p \int_0^y s^{p-1} \mu(s, \infty) ds,$$

which is again slowly varying, where we use Theorem 0.6(a) of Resnick (1987). Thus, in either case, $\rho_0(y, \infty)$ is slowly varying and converges to zero as $y \to \infty$. Now, from (4.6) and Proposition 2.2, we also have

$$\Re r_G(iy) = -\int_0^\infty \frac{t^2}{t^2 + y^2} d\rho_0(t) \sim -\rho_0(y, \infty)$$

as $y \to \infty$. Since $\rho_0(y, \infty)$ is slowly varying, thus, for any $\varepsilon > 0$, we have $|y^\varepsilon \Re r_G(iy)| \to \infty$ as $y \to \infty$. Also, for $p = 0$, we have $\Re r_G(iy) \sim -\rho_0(y, \infty) = -\mu(y, \infty)$.

Next consider the case, where $\mu \in \mathcal{M}_p$ has regularly varying tail of index $-(p+1)$. Define again $d\rho_1(t) = t^{p+1} d\mu(t)$. Then,

$$\rho_1[0, y] = (p + 1) \int_0^y s^p \mu(s, \infty) ds - y^{p+1} \mu(y, \infty) \sim (p + 1) \int_0^y s^p \mu(s, \infty) ds$$

is slowly varying, again by Theorem 0.6(a) of Resnick (1987). Then, by (4.6) and Proposition 2.1, we have

$$\Im r_G(iy) = y \int_0^\infty \frac{d\rho_1(t)}{t^2 + y^2} \sim \frac{1}{y} \rho_1[0, y]$$

is regularly varying of index -1. Further, $\rho_1[0, y] \to \int_0^\infty t^{p+1} d\mu(t) = \infty$ as $y \to \infty$. Then the asymptotic upper and lower bounds follow immediately. □

5. Relationship between Cauchy and Voiculescu transform

The results of the last section relate the tail of a regularly varying probability measure and the behavior of the remainder term in Laurent series expansion of its Cauchy transform. In this section, we shall relate the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms. Finally, we collect the results from Sections 4 and 5 to prove Theorems 2.1 and 2.2.

To study the relation between the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms, we consider a class of functions, which include the functions H_μ for the probability measures μ with regularly varying tails. We then show that for the functions with the inverse and the reciprocal appropriately defined, the inverse and the reciprocal are also in the defined class.

Let $\mathcal{H}$ denote the set of all functions A which are analytic in a domain $\mathcal{D}_A$ such that for all positive η, there exists $\delta > 0$ with $\Delta_{\eta,\delta} \subset \mathcal{D}_A$.

For a nonnegative integer p and $\beta \in [0, 1/2)$, let $\mathcal{R}_{p,\beta}$ denote the set of all functions $A \in \mathcal{H}$ which satisfies the following conditions:

(R1) A has Taylor series expansion with real coefficients of the form

$$A(z) = z + \sum_{j=1}^p a_j z^{j+1} + z^{p+1} r_A(z),$$

where $a_1, \ldots, a_p$ are real numbers. For $p = 0$, we interpret the sum in the middle term as absent.

(R2) $z \ll r_A(z) \ll z^\beta$ as $z \to 0$ n.t.

(R3) $\Re r_A(-iy) \gg y^{1+\beta/2}$ and $\Im r_A(-iy) \gg y$ as $y \to 0+$.
For $p = 0 = \beta$, we further require that $\Re r_A(-iy) \approx \Im r_A(-iy)$ as $y \to 0+$.
For $\beta \in (0, 1/2)$, we further require that

$$\Re r_A(-iy) \ll y^{1-\beta/2} \quad \text{and} \quad \Im r_A(-iy) \ll y^{1-\beta/2} \text{ as } y \to 0+.$$

From Propositions 4.1 and 4.3, this class with $\beta = 0$ will include, in particular, H_μ for all probability measures $\mu \in \mathcal{M}_{p,\alpha}$, where p is a positive integer and $\alpha \in [p, p+1)$. We do not impose the condition $\Re r_A(-iy) \approx \Im r_A(-iy)$ for $p > 0$, as it may fail for some measures in $\mathcal{M}_{p,p}$. However, from Propositions 4.1, 4.3 and 4.4, this class will also include H_μ for all probability measures μ with regularly varying tail of index $-\alpha$ with $\alpha \in [0, 1)$. For the case $\alpha = p + 1$, we need $\beta > 0$. According to Propositions 4.2 and 4.4, for any $\beta \in (0, 1/2)$, H_μ will be in $\mathcal{R}_{p,\beta}$, whenever $\mu \in \mathcal{M}_{p,p+1}$.

The first result deals with the reciprocals. Note that $U(z)$ and $zU(z)$ have the same remainder functions and if one belongs to the class $\mathcal{H}$, so does the other.

Proposition 5.1. *Suppose $zU(z) \in \mathcal{H}$ be a function belonging to $\mathcal{R}_{p,\beta}$ for some nonnegative integer p and $0 \le \beta < 1/2$, such that U does not vanish in a neighborhood of zero. Further assume that $V = 1/U$ also belongs to $\mathcal{H}$. Then $zV(z)$ is also in $\mathcal{R}_{p,\beta}$ respectively. Furthermore, we have, as $z \to 0$ n.t., $r_V(z) \sim -r_U(z)$; and as $y \to 0+$, $\Re r_V(-iy) \sim -\Re r_U(-iy)$ and $\Im r_V(-iy) \sim -\Im r_U(-iy)$.*

The second result shows that for each of the above classes, for a bijective function from the class, its inverse is also in the same class.

Proposition 5.2. *Suppose $U \in \mathcal{H}$ be a bijective function with the inverse in $\mathcal{H}$ as well and $U \in \mathcal{R}_{p,\beta}$ for some nonnegative integer p and $0 \le \beta < 1/2$. Then the inverse V is also in $\mathcal{R}_{p,\beta}$. Furthermore, we have, as $z \to 0$ n.t., $r_V(z) \sim -r_U(z)$ and as $y \to 0+$, $\Re r_V(-iy) \sim -\Re r_U(-iy)$ and $\Im r_V(-iy) \sim -\Im r_U(-iy)$.*

Next we prove Propositions 5.1 and 5.2. In both the proofs, all the limits will be taken as $z \to 0$ n.t. or $y \to 0+$, unless otherwise mentioned and these conventions will not be stated repeatedly. We shall also use that, for any nonnegative integer p and $\beta \in [0, 1/2)$, with $U \in \mathcal{R}_{p,\beta}$, we have

$$|\Re r_U(-iy)| \le |r_U(-iy)| \ll 1 \quad \text{and} \quad |\Im r_U(-iy)| \le |r_U(-iy)| \ll 1. \tag{5.1}$$

We first prove the result regarding the reciprocal.

Proof of Proposition 5.1. We shall consider zero and positive values of p separately. First we shall deal with $p = 0$. Let $zU(z) = z + zr_U(z)$ be a function in this class. Then $V(z) = 1 - r_U(z) + \mathrm{O}(|r_U(z)|^2)$. By uniqueness of Taylor expansion from Lemma A.1 of Benaych-Georges (2005), we have

$$r_V(z) = -r_U(z) + \mathrm{O}(|r_U(z)|^2). \tag{5.2}$$

Since, by (R2), $r_U(z) \ll 1$, we have $r_V(z) \sim -r_U(z)$.

Further, evaluating (5.2) at $z = -iy$ and equating the real and the imaginary parts, we have

$$\Re r_V(-iy) = -\Re r_U(-iy) + \mathrm{O}(|r_U(-iy)|^2)$$

and

$$\Im r_V(-iy) = -\Im r_U(-iy) + \mathrm{O}(|r_U(-iy)|^2).$$

Thus, to obtain the equivalences of the real and the imaginary parts, it is enough to show that $|r_U(-iy)|^2 = |\Re r_U(-iy)|^2 + |\Im r_U(-iy)|^2$ is negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. For $\beta = 0$, using (5.1) and $\Re r_U(-iy) \approx \Im r_U(-iy)$ from (R3), we have the required negligibility condition. For $\beta \in (0, 1/2)$, we have, using (R3),

$$\frac{|\Im r_U(-iy)|^2}{|\Re r_U(-iy)|} = \frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \left(\frac{|\Im r_U(-iy)|}{y^{1-\beta/2}} \right)^2 y^{1-3\beta/2} \to 0$$

and

$$\frac{|\Re r_U(-iy)|^2}{|\Im r_U(-iy)|} = \frac{y}{|\Im r_U(-iy)|} \left(\frac{|\Re r_U(-iy)|}{y^{1-\beta/2}} \right)^2 y^{1-\beta} \to 0.$$

They, together with (5.1), give the required negligibility condition.

Next, we consider the case $p \geq 1$. Let $zU(z) = z + \sum_{j=1}^{p} u_j z^{j+1} + z^{p+1} r_U(z)$ be a function in this class. Note that, as $p \geq 1$ and by (R2), as $z \ll r_U(z)$, we have $\sum_{j=1}^{p} u_j z^j + z^p r_U(z) = u_1 z + \mathrm{O}(z r_U(z))$. Thus, we have,

$$V(z) = 1 + \sum_{j=1}^{p} (-1)^j v_j \left(\sum_{m=1}^{p} u_m z^m + z^p r_U(z) \right)^j + c_1 z^{p+1} + \mathrm{O}(z^{p+1} r_U(z)),$$

for some real number c_1.

Now we expand the second term on the right side. As $z \ll r_U(z)$ from (R2), all powers of z with indices greater than $(p+1)$ can be absorbed in the last term on the right side. Then collect the $(p+1)$-th powers of z in the penultimate term to get $c_2 z^{p+1}$ for some real number c_2. The remaining powers of z form a polynomial $P(z)$ of degree at most p with real coefficients. Finally we consider the terms containing some power of $r_U(z)$. It will contain terms of the form $z^{l_1} (z^p r_U(z))^{l_2}$ for integers $l_1 \geq 0$ and $l_2 \geq 1$, with the leading term being $-z^p r_U(z)$. Since $p \geq 1$ and $r_U(z) \ll 1$, the remaining terms can be absorbed in the last term on the right side. Thus, we get,

$$V(z) = 1 + P(z) - z^p r_U(z) + c_2 z^{p+1} + \mathrm{O}(z^{p+1} r_U(z)).$$

By uniqueness of Taylor series expansion from Lemma A.1 of Benaych-Georges (2005), we have

$$r_V(z) = -r_U(z) + c_2 z + \mathrm{O}(z r_U(z)).$$

The form of r_V immediately gives $r_V(z) \sim -r_U(z)$, since $z \ll r_U(z)$, by (R2). Also, using (5.1), $\Im r_V(-iy) = -\Im r_U(-iy) + \mathrm{O}(y)$ and as $y \ll \Im r_U(-iy)$ from (R3), we have $\Im r_V(-iy) \sim -\Im r_U(-iy)$. Further, as c_2 is real, $\Re r_V(-iy) = -\Re r_U(-iy) + \mathrm{O}(y|r_U(-iy)|)$. Thus, to conclude $\Re r_V(-iy) \sim -\Re r_U(-iy)$, it is enough to show that $y|r_U(-iy)| \ll \Re r_U(-iy)$, for which, using (5.1) it is enough to show that $y \Im r_U(-iy) \ll \Re r_U(-iy)$. We then write

$$\frac{y \Im r_U(-iy)}{\Re r_U(-iy)} = \begin{cases} \frac{y}{\Re r_U(-iy)} \cdot \Im r_U(-iy), & \text{for } \beta = 0, \\ \frac{y^{1+\beta/2}}{\Re r_U(-iy)} \cdot \frac{\Im r_U(-iy)}{y^{1-\beta/2}} \cdot y^{1-\beta}, & \text{for } \beta \in (0, 1/2), \end{cases}$$

and the limit is zero in either case. □

Before proving the result regarding the inverse, we provide a result connecting a function in the class $\mathcal{H}$ and its derivative.

Lemma 5.1. *Let $v \in \mathcal{H}$ satisfy $v(z) = \mathrm{o}(z^{\beta})$ as $z \to 0$ n.t., for some real number β. Then $v'(z) = \mathrm{o}(z^{\beta-1})$ as $z \to 0$ n.t.*

Proof. The result for $\beta = 0$ follows from the calculations in the proof of Proposition A.1(ii) of Benaych-Georges (2005). For the general case, define $w(z) = z^{-\beta}v(z)$. Then $w \in \mathcal{H}$ and $w(z) = \mathrm{o}(1)$. So by the case $\beta = 0$, we have $w'(z) = -\beta z^{-\beta-1}v(z) + z^{-\beta}v'(z) = \mathrm{o}(z^{-1})$. Thus, $zw'(z) = -\beta z^{-\beta}v(z) + z^{-(\beta-1)}v'(z)$, where the left side and the first term on the right side are o(1) and hence the second term on the right side is o(1) as well. □

We are now ready to prove the result regarding the inverse.

Proof of Proposition 5.2. Since U is of the form

$$U(z) = z + \sum_{j=1}^{p} u_j z^{j+1} + z^{p+1} r_U(z)$$

and $r_U(z) \ll 1$ for either class, by Proposition A.3 of Benaych-Georges (2005), the inverse function V also has the same form with the remainder term r_V satisfying

$$r_V(z) \ll 1. \tag{5.3}$$

Also note that $V(z) \sim z$. Further, Lemma A.1 of Benaych-Georges (2005) shows that the coefficients are determined by the limits of the derivatives of the function at 0. Hence, the real coefficients of U guarantee that the coefficients of V are real. So we only need to check the asymptotic equivalences of the remainder functions. For that purpose, we define

$$I(z) = r_U(V(z)) - r_U(z) = \int_{\gamma_z} r_U'(\zeta)d\zeta,$$

where γ_z is the closed line segment joining z and $V(z)$. By definition of the class $\mathcal{H}$, given any $\eta > 0$, there exists $\delta > 0$, such that $\Delta_{\eta,\delta} \subset \mathcal{D}_U$. Since $V(z) \sim z$ as $z \to 0$ n.t., given any $\eta > 0$, there exists $\delta > 0$, such that both z and $V(z)$ belong to $\Delta_{\eta,\delta}$ and hence γ_z is contained in $\Delta_{\eta,\delta} \subset \mathcal{D}_U$. (Note that $\Delta_{\eta,\delta}$ is a convex set.) Thus r_U' is defined on the entire line γ_z. We shall need the following estimate that

$$|I(z)| \le |\gamma_z| \sup_{\zeta\in\gamma_z} |r_U'(\zeta)| = |V(z) - z| \sup_{\zeta\in\gamma_z} |r_U'(\zeta)| = |V(z) - z||r_U'(\zeta_0(z))|,$$

for some $\zeta_0(z) \in \gamma_z$, since γ_z is compact. Note that $\zeta_0(z) = z + \theta(z)(V(z) - z)$, for some $\theta(z) \in [0, 1]$ and hence $\zeta_0(z) \sim z$. Now, $r_U(z) = \mathrm{o}(z^{\beta})$ by (R2) and thus, by Lemma 5.1, we have $r_U'(\zeta_0(z)) = \mathrm{o}(\zeta_0(z)^{\beta-1}) = \mathrm{o}(z^{\beta-1})$. So, using $V(z) = z + zr_V(z)$ for $p = 0$ and $V(z) = z + \mathrm{O}(z^2)$ for $p \ge 1$, we have,

$$|I(z)| = \begin{cases} \mathrm{o}(z^{\beta}r_V(z)), & \text{for } p = 0, \\ \mathrm{o}(z^{1+\beta}), & \text{for } p \ge 1, \end{cases} \tag{5.4}$$

Now, consider the case $p = 0$. Then $U(z) = z + zr_U(z)$ and $V(z) = z + zr_V(z)$. Using $U(V(z)) = z$, we have $0 = zr_V(z) + (z + zr_V(z))(r_U(z) + I(z))$, giving, using (5.3)

$$0 = r_U(z) + r_V(z) + r_U(z)r_V(z) + \mathrm{O}(I(z)). \tag{5.5}$$

Using (5.4) for $p = 0$ and $r_U(z) \ll 1$ from (R2), we have $r_V(z) \sim -r_U(z)$. Then, using (R2) and evaluating at $z = -iy$, we have, for $\beta \in [0, 1/2)$,

$$|r_V(-iy)| \ll y^{\beta}. \tag{5.6}$$

Evaluating (5.5) at $z = iy$ and equating the real and the imaginary parts, we have

$$0 = \Re r_U(-iy) + \Re r_V(-iy) + \mathrm{O}(|r_U(-iy)||r_V(-iy)|) + \mathrm{O}(|I(-iy)|) \tag{5.7}$$

and

$$0 = \Im r_U(-iy) + \Im r_V(-iy) + \mathrm{O}(|r_U(-iy)||r_V(-iy)|) + \mathrm{O}(|I(-iy)|). \tag{5.8}$$

By (R3) and (5.4), for $\beta = 0$, we have, $|I(-iy)| \ll |r_V(-iy)| \sim |r_U(-iy)| \approx |\Re r_U(-iy)| \approx |\Im r_U(-iy)|$. Then, further using $r_U(-iy) \to 0$ from (R2), we have, from (5.7) and (5.8), $\Re r_U(-iy) \sim -\Re r_V(-iy)$ and $\Im r_U(-iy) \sim -\Im r_V(-iy)$ respectively.

For $\beta > 0$, we have, by (R3),

$$y^{\beta}\frac{|\Im r_U(-iy)|}{|\Re r_U(-iy)|} = \frac{|\Im r_U(-iy)|}{y^{1-\beta/2}}\frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \to 0$$

and

$$y^{\beta}\frac{|\Re r_U(-iy)|}{|\Im r_U(-iy)|} = \frac{|\Re r_U(-iy)|}{y^{1-\beta/2}}\frac{y}{|\Im r_U(-iy)|}y^{\beta/2} \to 0,$$

which, together with (5.1), gives $y^{\beta}|r_U(-iy)|$ is negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. Further, using (5.4) and (5.6) respectively, we have $|I(-iy)| \ll y^{\beta}|r_V(-iy)| \sim y^{\beta}|r_U(-iy)|$ and $|r_U(-iy)r_V(-iy)| \ll y^{\beta}|r_U(-iy)|$. Thus, both $|I(-iy)|$ and $|r_U(-iy)r_V(-iy)|$ are negligible with respect to both the real and the imaginary parts of $r_U(-iy)$. Then, from (5.7) and (5.8), we immediately have $\Re r_U(-iy) \sim -\Re r_V(-iy)$ and $\Im r_U(-iy) \sim -\Im r_V(-iy)$.

Next, we consider the case $p \geq 1$. Then $U(z) = z + \sum_{j=1}^{p} u_j z^{j+1} + z^{p+1} r_U(z)$ and $V(z) = z + \sum_{j=1}^{p} v_j z^{j+1} + z^{p+1} r_V(z) = z(1 + v_1 z(1 + \mathrm{o}(1)))$. Using $U(V(z)) = z$, we have

$$\begin{aligned} 0 = \sum_{j=1}^{p} v_j z^{j+1} + z^{p+1} r_V(z) + \sum_{m=1}^{p} u_m \left(z + \sum_{j=1}^{p} v_j z^{j+1} + z^{p+1} r_V(z) \right)^{m+1} \\ + z^{p+1}\Big(r_U(z) + I(z)\Big)\Big(1 + (p+1)v_1 z(1 + \mathrm{o}(1))\Big). \end{aligned} \tag{5.9}$$

Note that all the coefficients on the right side are real. We collect the powers of z till degree $p+1$ on the right side in the polynomial $Q(z)$. Let $c_3 \in \mathbb{R}$ be the coefficient of z^{p+2} on the right side. The remaining powers of z on the right side will be $\mathrm{O}(z^{p+3})$. We next consider the terms with $r_V(z)$ as a factor and observe that $z^{p+1} r_V(z)$ is the leading term and the remaining terms contribute $\mathrm{O}(z^{p+2} r_U(z))$, which absorbs the term $\mathrm{O}(z^{p+3})$, as $z \ll r_U(z)$ by (R2). Finally, the last term on the right side gives $z^{p+1} r_U(z) + \mathrm{O}(z^{p+2} r_U(z)) + \mathrm{O}(z^{p+1} I(z))$. Combining the above facts and dividing (5.9) by z^{p+1}, we get,

$$0 = z^{-(p+1)}Q(z) + \big(r_U(z) + c_3 z + \mathrm{O}(I(z)) + \mathrm{O}(z r_U(z))\big) + \big(r_V(z) + \mathrm{O}(z r_V(z))\big).$$

Now, the first group of terms within the bracket is $r_U(z)(1 + \mathrm{o}(1)) = \mathrm{o}(1)$, as $I(z) \ll z^{1+\beta} \ll z \ll r_U(z)$ by (5.4) and (R2). Also, the second group of terms within the bracket is $r_V(z)(1+\mathrm{o}(1)) = \mathrm{o}(1)$. However, the first term, Q being a polynomial of degree at most p, becomes unbounded unless $Q \equiv 0$. So we must have $Q \equiv 0$ and we have

$$r_U(z) + c_3 z + \mathrm{O}(I(z)) + \mathrm{O}(z r_U(z)) = -r_V(z) + \mathrm{O}(z r_V(z)). \tag{5.10}$$

As observed earlier, the left side is $r_U(z)(1+\mathrm{o}(1))$ and the right side is $-r_V(z)(1+\mathrm{o}(1))$ giving $r_U(z) \sim -r_V(z)$. Then, as in the case $p = 0$, we have (5.6). Further, (5.10) becomes

$$-r_V(z) = r_U(z) + c_3 z + \mathrm{O}(I(z)) + \mathrm{O}(z r_U(z)). \tag{5.11}$$

Evaluating (5.11) at $z = -iy$ and equating the imaginary parts, we have, using (5.4),

$$-\Im r_V(-iy) = \Im r_U(-iy) + \mathrm{O}(y).$$

This gives $-\Im r_V(-iy) \sim \Im r_U(-iy)$, since $y \ll \Im r_U(-iy)$ by (R3).

Evaluating (5.10) at $z = -iy$ again and now equating the real parts, we have, as c_3 is real,

$$-\Re r_V(-iy) = \Re r_U(-iy) + \mathrm{O}(|I(-iy)|) + \mathrm{O}(y|r_U(-iy)|).$$

From (5.4) and (R3), we have $|I(-iy)| \ll y^{1+\beta} \ll \Re r_U(-iy)$. Thus, to obtain $-\Re r_V(-iy) \sim \Re r_U(-iy)$, we only need to show that $y|r_U(-iy)| \ll \Re r_U(-iy)$, which follows using (5.6) and (R3), since

$$\frac{y|r_U(-iy)|}{|\Re r_U(-iy)|} = \frac{y^{1+\beta/2}}{|\Re r_U(-iy)|} \frac{|r_U(-iy)|}{y^\beta} y^{\beta/2}.$$

□

We wrap up the article by collecting the results from Sections 4 and 5 and proving Theorems 2.1 and 2.2.

Proofs of Theorems 2.1 and 2.2. We shall prove both the theorems together, as the proofs are very similar. The statements involving the tail of the probability measure μ and Cauchy transform follow from the results in Section 4. The equivalence of the statements about the tail of the probability measure and Cauchy transform (the imaginary part for $\alpha \in [p, p+1)$ in Theorem 2.1 and the real part for $\alpha = p+1$ in Theorem 2.2) and their asymptotic equivalence are given in Proposition 4.3. The asymptotic equivalence of the tail of the probability measure and the real part of Cauchy transform for $\alpha \in (p, p+1)$ also follows from Proposition 4.3. Similar asymptotic equivalence for $\alpha = p = 0$ follows from Proposition 4.4. All the asymptotic upper and lower bounds are immediate from the regularly variation property of the corresponding functions, except for $\Im r_G(iy)$ for the case $\alpha = p+1$, which follows from Proposition 4.4.

We shall obtain the statements about Cauchy and Voiculescu transforms from Propositions 5.1 and 5.2. We shall actually deal with the functions H_μ and R_μ. We shall also use (2.1):

$$R_\mu(z) = \frac{1}{z}\left(\frac{1}{\frac{H_\mu^{-1}(z)}{z}} - 1\right) \quad \text{and} \quad H_\mu^{-1}(z) = z\frac{1}{1+zR_\mu(z)}.$$

Thus, we can move from the function H_μ to R_μ through inverse and reciprocal and vice versa. These observations set up the stage for Propositions 5.1 and 5.2. We shall use the class $\mathcal{R}_{p,0}$ for Theorem 2.1 and the class $\mathcal{R}_{p,\beta}$ with any $\beta \in (0, 1/2)$ for Theorem 2.2.

Suppose $\mu \in \mathcal{M}_p$ with $\alpha \in [p, p+1)$ as in Theorem 2.1. Then, $H_\mu(z)$ and $zR_\mu(z)$ necessarily have Taylor expansions of the form given in the hypothesis (R1) for the class $\mathcal{R}_{p,0}$ with $r_H(z) \ll 1$ and $r_R(z) \ll 1$ as $z \to \infty$. First assume that $\Im r_G(iy)$ is regularly varying of index $-(\alpha - p)$. Then, from the already proven part of Theorem 2.1, we have the asymptotic lower bounds for $r_G(z)$, $\Re r_G(iy)$ and $\Im r_G(iy)$. They translate to the asymptotic lower bounds for the function H_μ, as required by the hypotheses (R2) and (R3). As H_μ is also invertible with $L = H^{-1} \in \mathcal{H}$, we also have $L \in \mathcal{R}_{p,0}$ and $r_H(z) \sim -r_L(z)$, $\Re r_H(-iy) \sim -\Re r_L(-iy)$ and $\Im r_H(-iy) \sim -\Im r_L(-iy)$. Clearly, then Proposition 5.1 applies to the function $L(z)/z$, which has reciprocal $K(z) = z/L(z)$ with $K \in \mathcal{H}$. Thus, r_K and r_L satisfy the relevant asymptotic equivalences. Furthermore, $R(z) = (K(z) - 1)/z$ and the expansion of K has constant term 1. Thus, $r_R \equiv r_K$. Combining, we have $r_H(z) \sim r_R(z)$, $\Re r_H(-iy) \sim \Re r_R(-iy)$ and $\Im r_H(-iy) \sim \Im r_R(-iy)$. Then R_μ inherits the appropriate properties from H_μ and passes them on to ϕ_μ. Conversely, if we make the assumptions on r_ϕ, they imply all the properties for r_R, so that $z(1 + zR_\mu(z)) = zK(z)$ belongs to the class $\mathcal{R}_{p,0}$. Note that, $r_R \equiv r_K$. Then apply Proposition 5.1 on K and then Proposition 5.2 on $z/K(z)$ to obtain $H_\mu(z)$. Arguing, by checking the asymptotic equivalences as in the direct case, we obtain the required conclusions about r_H and hence r_G.

The argument is same in the case $\alpha = p+1$ with the observation that the stronger upper bounds required in the hypotheses (R2) and (R3) with $\beta > 0$ is assumed for r_ϕ and hence for r_R and is proved for r_G and hence for r_H in Proposition 4.2 and 4.4. □

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