JOINT CONVERGENCE OF SEVERAL COPIES OF DIFFERENT PATTERNED RANDOM MATRICES

RIDDHIPRATIM BASU, ARUP BOSE, SHIRSHENDU GANGULY, AND RAJAT SUBHRA HAZRA

Abstract. We study the joint convergence of independent copies of several patterned matrices in the noncommutative probability setup. In particular, joint convergence holds for the well known Wigner, Toeplitz, Hankel, reverse circulant and symmetric circulant matrices. We also study some properties of the limits. In particular, we show that copies of Wigner becomes asymptotically free with copies of any of the above other matrices.

1. Introduction

Voiculescu (1991) showed that appropriate deterministic matrices and symmetric random matrices with i.i.d. entries (Wigner matrices) which are Gaussian, converge jointly (see (2.3) for the notion of joint convergence). The noncommutative limit exhibits a certain type of independence, known as *freeness*. This concept of asymptotic freeness has been a key feature in the development of free probability and its various applications. The inclusion of constant matrices in the above result had important implications in the factor theory of von Neumann algebras.

The various aspects of joint convergence of Wigner matrices have been well studied in literature. Some of the works which study properties of joint convergence of Wigner matrices are Capitaine and Casalis (2004), Capitaine and Donati-Martin (2007), Collins (2003), Collins et al. (2009), Dykema (1993), Schultz (2005) and Voiculescu (1998). Joint convergence of other type of matrices has also been considered. For example Ryan and Debbah (2009) studied joint convergence of Vandermonde matrices which also exhibit freeness in the limit. Bose and Sen (2011) has some examples of freenees for finite diagonal random matrices and also Banerjee and Bose (2011) studied some structural properties of a matrix which can result in freeness.

Wigner matrices may be viewed as an example of a patterned random matrices where, along with symmetry, some other assumptions are imposed on the structure of the matrices. Four other examples of pattern matrices are the Toeplitz, Hankel, symmetric circulant and reverse circulant matrices. Bose et al. (2010) showed that under suitable assumptions on the pattern, there is joint convergence of i.i.d. copies of a single pattern matrix, as the dimension goes to infinity. One important consequence is that in the limit other kinds of independence may arise. The symmetric circulant limits are (classically) independent and the reverse circulant limits are half independent.

The Toeplitz and Hankel limits do not exhibit any of the above three independence and any possible independence in their limits have not yet been discovered. We are led to the following natural questions.

Question 1. Does one have joint convergence of two different kind of patterned matrices? For example, is there a joint convergence of multiple copies of Hankel and Toeplitz or Toeplitz and Wigner?

Question 2. Is the Wigner matrix asymptotically free with other patterned matrices?

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We address the above issues in a general framework and derive some interesting properties of the joint convergence of various patterned matrices.

First, we borrow a condition from Bose and Sen (2008) (see *Property B* in (2.4)): the maximum number of times any entry is repeated in a row remains uniformly bounded across all rows as $n \to \infty$.

This property in particular is satisfied for the five well known matrices, namely Wigner, Toepliz, Hankel, reverse circulant and symmetric circulant. Under Property B, and moment condition on the variables, we show in Theorem 3.1 that if a condition (condition (3.1)) holds for one copy each of any subcollection of the matrices, then joint convergence holds for multiple copies. This is an extension of Proposition 1 of Bose et al. (2010). The result holds both in expectation and almost surely.

There is no general way of checking (3.1). However, in Theorem 3.2 we show that this condition holds for the five matrices. One interesting consequence is that any matrix polynomial of multiple copies of the above five matrices that is symmetric, has a limiting spectral distribution (LSD). In particular, let A and B be any two of Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant matrices and are independent satisfying moment Assumption (A2) given later. Then the LSD for $\frac{A+B}{\sqrt{n}}$ exists in the almost sure sense, is symmetric, has unbounded support and does not depend on the underlying distribution of the input sequences of A and B (see Remark 3.1). This LSD has not been identified. It may be mentioned that the LSD is also not known in any explicit form for the Toeplitz and Hankel matrices. Some simulation results on the LSD of sum are given in Figures 1 and 2.

Banerjee and Bose (2011) introduced another property of the matrices (see (3.3)): the total number of entries repeated in the same row remains uniformly bounded over all pairs of columns as $n \to \infty$. We provide additional information on the moments of the LSDs when the patterns satisfy this property. See Theorem 3.3.

This helps in studying the question of asymptotic freeness. It is known that any collection of Wigner and other deterministic matrices which converge, are all free in the limit (under some moment assumptions). See for example Anderson et al. (2010). But in general the moment conditions imply the deterministic matrices have uniform bounds on their norms. As a consequence the Wigner and the other (random) matrix A_n have freeness in the limit if $\sup_n \|A_n\| < \infty$ (see Theorem 22.2.4, Speicher (2010)), where $\|A_n\|$ is the spectral norm of A_n. But it can be noted from the works of Bose et al. (2009), Meckes (2007) that the spectral norm of Toeplitz, Hankel, reverse circulant, symmetric circulant are unbounded and so freeness cannot be established without extra efforts. It is an open question if freeness of Wigner with other suitable random matrices hold in general. In Theorem 3.4, we show that any collection of Wigner matrices is free of the other four matrices under independence.

2. Some basic definitions and notation

2.1. **Noncommutative probability spaces and joint convergence.** A *noncommutative probability space* is a pair $(\mathcal{A}, \varphi)$, where $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$. Elements in a noncommutative probability space will be called *random variables.* A linear functional φ is called a *state* if for $a \geq 0$ we have $\varphi(a) \geq 0$ and it is called *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

The *joint distribution* of a family $(a_i)_{i\in I}$ of random variables in a noncommutative probability space $(\mathcal{A}, \varphi)$ is the collection of *joint moments* $\{\varphi(a_{i_1} \cdots a_{i_k})\},\ k \in \mathbb{N}$ and $i_1, \cdots, i_k \in I$.

We can put random matrices in the setup of noncommutative probability as follows. Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $L(\mu) := \bigcap_{p\geq 1} L^p(X, \mu)$ be the algebra of random variables with finite

moments of all orders. Set
$$\mathcal{A}_n := Mat_n(L(\mu)) \tag{2.1}$$
as the space of $n \times n$ complex random matrices with entries coming from $L(\mu)$. We can consider two functionals on this space. For $A \in \mathcal{A}_n$, we define
$$\varphi_1(A) = \frac{1}{n}\operatorname{Tr}(A) \text{ and } \varphi_2(A) = \frac{1}{n}\operatorname{E}[\operatorname{Tr}(A)]. \tag{2.2}$$

Let $(\mathcal{A}_n, \varphi_n)_{n\geq 1}$ and $(\mathcal{A}, \varphi)$ be noncommutative probability spaces and let $(a_{i,n}; i \in I) \subset \mathcal{A}_n$ for each n, $(a_i; i \in I) \subset \mathcal{A}$. Then $(a_{i,n}; i \in I)$ *converges in law (or in distribution)* to $(a_i; i \in I)$ if for all $p \in \mathbb{C}[X_i, i \in I]$,
$$\lim_n \varphi_n(p(\{a_{i,n}\}_{i\in I})) = \varphi(p(\{a_i\}_{i\in I})). \tag{2.3}$$

A related notion is that of the *limiting spectral distribution* (LSD). For a real symmetric matrix A_n of order $n\times n$, let $\{\lambda_{1n}, \ldots \lambda_{nn}\}$ be its eigenvalues. The (random) *empirical spectral distribution* F_n of A_n is defined as the distribution which puts mass $1/n$ at each of these eigenvalues. If as $n \to \infty$, F_n converges in distribution (almost surely) to F say, then F is called the LSD of $\{A_n\}$.

Finally, we need the well known notion of *free independence*. Suppose $\{\mathcal{A}_i\}_{i\in J} \subset \mathcal{A}$ are unital subalgebras. These subalgebras are called *freely independent* or simply *free* if $\varphi(a_j) = 0,\ a_j \in \mathcal{A}_{i_j}$ and $i_j \neq i_{j+1}$ for all j implies $\varphi(a_1 \ldots a_n) = 0$. The random variables (or elements of an algebra) $a_1, a_2, \ldots, a_k$ are called free if the subalgebras generated by them are free.

It is well known that a sequence of independent Wigner matrices is free in the limit. Further, the marginals are semicircular. These two properties may also be described by the fact that the limit joint moments are precisely the *number of noncrossing colored pair partitions*(see for example Nica and Speicher (2006)). It is also known that any collection of Wigner and any collection of deterministic matrices (which converge), are free in the limit. See for example Anderson et al. (2010), Nica and Speicher (2006).

2.2. **Patterned matrices, link function, trace formula and words.** Patterned matrices are defined via the *link functions*. A link function L is defined as a function $L : \{1, 2, .., n\}^2 \to Z^d_{\geq}, n \geq 1$. For our purposes $d = 1$ or 2. Although L depends on n, to avoid complexity of notation we suppress the n and consider $\mathbb{N}^2$ as the common domain. We also assume that L is symmetric in its arguments, that is, $L(i, j) = L(j, i)$.

Let $\{x(i)\}$ and $\{x(i, j)\}$ be a sequence of real random variables, referred to as the *input sequence*. The sequence of matrices $\{A_n\}$ under consideration will be defined by
$$A_n \equiv ((a_{i,j}))_{1\leq i,j,\leq n} \equiv ((x(L(i, j)))).$$
Some important matrices we shall discuss in this article are:

(W_n) Wigner matrix: $L : \mathbb{N}^2 \to \mathbb{Z}^2$ where $L(i, j) = (\min(i, j), \max(i, j))$.

(T_n) Toeplitz matrix: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = |i - j|$.

(H_n) Hankel matrix: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = i + j$.

(RC_n) Reverse Circulant: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = (i + j) \mod n$.

(SC_n) Symmetric Circulant: $L : \mathbb{N}^2 \to \mathbb{Z}$ where $L(i, j) = n/2 - |n/2 - |i - j||$.

It is now well known that the limiting spectral distribution (LSD) of the above matrices exists. Bose et al. (2010) reviewed the results on LSD of the above matrices. For various results on Wigner matrices we refer to the excellent exposition by Anderson et al. (2010).

The L function for all the five matrices defined above satisfy the following property. This property was introduced by Bose and Sen (2008) and shall be crucial to us. (For any set S, $\#S$ or $|S|$ will denote the number of elements in S).

Property B: We say a link function L satisfies *Property B* if,

$$\Delta(L) = \sup_n \sup_{t \in \mathbb{Z}^d_{\geq}} \sup_{1 \leq k \leq n} \#\{l : 1 \leq l \leq n, L(k,l) = t\} < \infty. \tag{2.4}$$

In particular, $\Delta(L) = 2$ for T_n, SC_n and $\Delta(L) = 1$ for W_n, H_n and RC_n.

Consider h different type of patterned matrices with each type j having p_j independent copies. The different link functions shall be referred to as *colors* and different independent copies of the matrices of any given color shall be referred to as *indices*. Let $\{X^j_{i,n}, 1 \leq i \leq p_j\}$ be j sequences of $n \times n$ symmetric patterned matrices with link functions L_j, $j = 1, 2, \cdots, h$. Let us denote the (p,q)-th entry of $X^j_{i,n}$ by $X^j_i(L_j(p,q))$. We suppress the dependence on n to simplify notation. We note down two natural assumptions on the link function and the input sequence.

(A1) All link functions $\{L_j, j = 1, 2, \cdots, h\}$ satisfy *Property B*, that is,

$$\max_{1 \leq j \leq h} \sup_{n \geq 1} \sup_t \sup_{1 \leq p \leq n} \#\{q : 1 \leq q \leq n, L_j(p,q) = t\} \leq \Delta < \infty.$$

(A2) Input sequences $\{X^j_i(k) : k \in \mathbb{Z} \text{ or } \mathbb{Z}^2\}$ are independent across i, j and k with mean zero and variance 1 and the moments are uniformly bounded, that is,

$$\sup_{1 \leq j \leq h} \sup_{1 \leq i \leq p_j} \sup_{n \geq 1} \sup_t \sup_{1 \leq p,q \leq n} \mathrm{E}\left[|X^j_i(L_j(p,q))|^k\right] \leq c_k < \infty.$$

We consider $\{\frac{1}{\sqrt{n}} X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}$ as elements of $\mathcal{A}_n$ given in (2.1) and investigate the joint convergence with respect to the normalized tracial states φ_1 or φ_2 (as in (2.2)). The sequence of matrices jointly converge if and only if for all monomials q,

$$\varphi_d\left(q\left(\frac{1}{\sqrt{n}}\{X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}\right)\right)$$

converge to a limit as $n \to \infty$ for either $d = 1$ or $d = 2$. For $d = 2$, the convergence is in the almost sure sense. The case of $h = 1$ and $p_1 = 1$ (a single patterned matrix) was dealt in Bose and Sen (2008) and $h = 1$ and $p_1 > 1$ (i.i.d. copies of a single patterned matrix) was dealt in Bose et al. (2010). In particular, convergence holds for i.i.d. copies of any one of the five patterned matrices. The starting point in showing this was the trace formula. The related concepts of circuits, matchings and words will be extended below to multiple copies of several matrices.

Since our primary aim is to show convergence for every monomial, we shall from now on, fix an arbitrary monomial q of length k. Then we may write,

$$q\left(\frac{1}{\sqrt{n}}\{X^j_{i,n}, 1 \leq i \leq p_j\}_{1 \leq j \leq h}\right) = \frac{1}{n^{k/2}} Z_{c_1,t_1} Z_{c_2,t_2} \cdots Z_{c_k,t_k}, \tag{2.5}$$

where $Z_{c_m,t_m} = X^{c_m}_{t_m}$ for $1 \leq m \leq n$.

From (2.5) we get,

$$\begin{aligned}
\widetilde{\mu_n}(q) &:= \frac{1}{n} \mathrm{Tr}\left[\frac{1}{n^{k/2}} Z_{c_1,t_1} Z_{c_2,t_2} \cdots Z_{c_k,t_k}\right] \\
&= \frac{1}{n^{1+k/2}} \sum_{j_1, j_2, \cdots, j_k} [Z_{c_1,t_1}(L_{c_1}(j_1,j_2)) Z_{c_2,t_2}(L_{c_2}(j_2,j_3)) \cdots Z_{c_k,t_k}(L_{c_k}(j_k,j_1))] \\
&= \frac{1}{n^{1+k/2}} \sum_{\substack{\pi:\{1,\cdots,k\} \to \{1,\cdots,n\} \\ \pi(0)=\pi(k)}} \prod_{i=1}^k Z_{c_i,t_i}(L_{c_i}(\pi(i-1), \pi(i)))
\end{aligned}$$

$$= \frac{1}{n^{1+k/2}} \sum_{\substack{\pi:\{1,\cdots,k\}\to\{1,\cdots,n\} \\ \pi(0)=\pi(k)}} \mathbf{Z}_\pi \quad \text{say.} \tag{2.6}$$

Also define,

$$\widehat{\mu_n} = \mathrm{E}[\widetilde{\mu_n}]. \tag{2.7}$$

Keeping in mind that we seek to show the existence of the limits in (2.6) and (2.7) as $n \to \infty$, we now develop some appropriate notions. In particular these help us to show that certain terms in these sums are negligible in the limit.

Any map $\pi : \{1, \cdots, k\} \to \{1, \cdots, n\}$ with $\pi(0) = \pi(k)$ will be called a *circuit*. Its dependence on k and n will be suppressed. Observe that $\widetilde{\mu_n}$ and $\widehat{\mu_n}$ involve sums over circuits. Any value $L_{c_i}(\pi(i-1), \pi(i))$ is called an *L-value* of π. If an L-value is repeated e times in π then π is said to have an *edge* of order e. Due to independence and mean zero of the input sequences,

$$\mathrm{E}[\mathbf{Z}_\pi] = 0 \quad \text{if } \pi \text{ has any edge of order one.} \tag{2.8}$$

If all L-values appear more than once then we say the circuit is *matched* and only these circuits are relevant due to the above.

A circuit is said to be *color matched* if all the L-values are repeated within the same color. A circuit is said to be *color* and *index* matched if in addition, all the L-values are also repeated within the same index.

Denote the colors and indices present in q by $(c_1, c_2, \cdots, c_k)$ and $(t_1, t_2, \cdots, t_k)$ respectively. We can define an equivalence relation on the set of color and index matched circuits, extending the ideas of Bose et al. (2010) and Bose and Sen (2008). We say $\pi_1 \sim \pi_2$ if and only if their matches take place at the same colors and at the same indices. Or,

$$c_i = c_j, t_i = t_j \text{ and } L_{c_i}(\pi_1(i-1), \pi_1(i)) = L_{c_j}(\pi_1(j-1), \pi_1(j))$$
$$\Longleftrightarrow$$
$$c_i = c_j, t_i = t_j \text{ and } L_{c_i}(\pi_2(i-1), \pi_2(i))) = L_{c_j}(\pi_2(j-1), \pi_2(j)).$$

An equivalence class can be expressed as a colored and indexed word w: each word is a string of letters in alphabetic order of their first occurrence with a subscript and a superscript to distinguish the index and the color respectively. The i-th position of w is denoted by $w[i]$. Any i is a *vertex* and it is *generating* (or *independent*) if either $i = 0$ or $w[i]$ is the position of the first occurrence of a letter. By abuse of notation we also use $\pi(i)$ to denote a vertex.

For example, if

$$q = X_1^1 X_2^1 X_1^2 X_1^2 X_2^2 X_2^2 X_2^1 X_1^1 = Z_{1,1} Z_{1,2} Z_{2,1} Z_{2,1} Z_{2,2} Z_{2,2} Z_{1,2} Z_{1,1},$$

then $a_1^1 b_2^1 c_1^2 c_1^2 d_2^2 d_2^2 b_2^1 a_1^1$ is a typical colored and indexed word corresponding to q. Any colored and indexed word uniquely determines the monomial it corresponds to. A colored and indexed (matched) word is *pair matched* if all its letters appear exactly twice. We shall see later that under *Property B*, only such circuits and words survive in the limits of (2.6) and (2.7).

Now we define some useful subsets of the circuits. For a colored and indexed word w, let

$$\Pi_{CI}(w) = \{\pi : w[i] = w[j] \Leftrightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i))) = (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j))\}. \tag{2.9}$$

Also define

$$\Pi^*_{CI}(w) = \{\pi : w[i] = w[j] \Rightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i)) = (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j))\}. \tag{2.10}$$

Every colored and indexed word has a corresponding nonindexed version which is obtained by dropping the indices from the letters (i.e. the subscripts). For example, $a_1^1 b_2^1 c_1^2 c_1^2 d_2^2 d_2^2 b_2^1 a_1^1$ yields

$a^1b^1c^2c^2d^2d^2b^1a^1$. For any monomial q, dropping the indices amounts to replacing, for every j, the independent copies X_i^j by a single X^j with link function L_j. In other words it corresponds to the case where $p_j = 1$ for $1 \leq j \leq h$.

Let $\psi(q)$ be the monomial obtained by dropping the indices from q. For example,

$$\text{if } \; q = Z_{1,1}Z_{1,2}Z_{2,1}Z_{2,1}Z_{2,2}Z_{2,2}Z_{1,2}Z_{1,1} \; \text{ then } \; \psi(q) = Z_1Z_1Z_2Z_2Z_2Z_2Z_1Z_1.$$

Then (2.9) and (2.10) get mapped to the following subsets of nonindexed colored word w' via ψ:

$$\Pi_C(w) = \{\pi : w[i] = w[j] \Leftrightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\},$$

$$\Pi^*_C(w) = \{\pi : w[i] = w[j] \Rightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\}.$$

Since pair matched words are going to be crucial, let us define:

$$CIW(2) = \{w : w \text{ is indexed and colored pair matched corresponding to } q\}$$

$$CW(2) = \{w : w \text{ is nonindexed colored pair matched corresponding to } \psi(q)\}.$$

For $w \in CIW(2)$, let us consider the word obtained by dropping the indices of w. This defines an injective mapping into $CW(2)$ and we continue to denote this mapping by ψ.

For any $w \in CW(2)$ and $w' \in CIW(2)$, we define (*whenever the limits exist*),

$$p_C(w) = \lim_{n\to\infty} \frac{1}{n^{1+k/2}}|\Pi^*_C(w)| \quad \text{and} \quad p_{CI}(w') = \lim_{n\to\infty} \frac{1}{n^{1+k/2}}|\Pi^*_{CI}(w')|.$$

3. Main results

Our first result is on the joint convergence of several patterned random matrices and is analogous to Proposition 1 of Bose et al. (2010) who considered the case $h = 1$.

Theorem 3.1. *Let $\{\frac{1}{\sqrt{n}}X^j_{i,n}, 1 \leq i \leq p_j\}_{1\leq j\leq h}$ be a sequence of real symmetric patterned random matrices satisfying Assumptions* (A1) *and* (A2)*. Fix a monomial q of length k and assume that, for all $w \in CW(2)$*

$$p_C(w) = \lim_{n\to\infty} \frac{1}{n^{1+k/2}}|\Pi^*_C(w)| \text{ exists.} \tag{3.1}$$

Then,

(1) *for all $w \in CIW(2)$, $p_{CI}(w)$ exists and $p_{CI}(w) = p_C(\psi(w))$,*

(2) *we have*

$$\lim_{n\to\infty} \widehat{\mu}_n(q) = \sum_{w\in CIW(2)} p_{CI}(w) = \alpha(q) \text{ (say)} \tag{3.2}$$

with

$$|\alpha(q)| \leq \begin{cases} \frac{k!\Delta^{k/2}}{(k/2)!2^{k/2}} & \text{if } k \text{ is even and each index appears even number of times} \\ 0 & \text{otherwise.} \end{cases}$$

(3) $\lim_{n\to\infty} \widetilde{\mu_n}(q) = \alpha(q)$ *almost surely.*

As a consequence if (3.1) *holds for every q then $\{\frac{1}{\sqrt{n}}X^j_{i,n}, 1 \leq i \leq p_j\}_{1\leq j\leq h}$ converges jointly in both the states φ_1 and φ_2 and the limit is independent of the input sequence.*

Remark 3.1. (i) Theorem 3.1 asserts that if the joint convergence holds for $p_j = 1, j = 1, 2, \cdots, h$ (that is if condition (3.1) holds), then the joint convergence continues to hold for all other values of p_j. There is no general way of checking (3.1). However, see the next Theorem.

(ii) As a consequence, under the conditions of Theorem 3.1, for any fixed monomial q that yields a symmetric matrix, the corresponding LSD exists. To see this note that parts (3) and (2) respectively imply that (C1) and (C3) hold. Using truncation arguments, it is possible to prove this under the weaker assumption on the input sequence that it is i.i.d. with second moment finite.

In particular, let A and B be two independent patterned matrices satisfying Assumptions (A1) and (A2). Suppose $p_C(w)$ exists for every q and every w. Then LSD for $\frac{A+B}{\sqrt{n}}$ exists in the almost sure sense, is symmetric and does not depend on the underlying distribution of the input sequences of A and B. Moreover, if either LSD of $\frac{A}{\sqrt{n}}$ or LSD of $\frac{B}{\sqrt{n}}$ has unbounded support then LSD of $\frac{A+B}{\sqrt{n}}$ also has unbounded support.

Theorem 3.2. *Suppose Assumption* (A2) *holds. Then* $p_C(w)$ *exists for all monomials* q *and for all* $w \in CW(2)$, *for any two of the following matrices at a time: Wigner, Toeplitz, Hankel, Symmetric circulant and Reverse circulant.*

In particular, all conclusions in Remark 3.1 (ii) hold when A and B are any two of Toeplitz, Hankel, Reverse Circulant and Symmetric Circulant matrices. It does not seem easy to identify the LSD for these sums. Some simulation results are given below

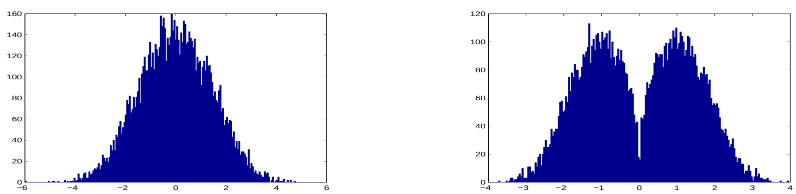

FIGURE 1. (i) (left) Histogram plot of empirical distribution of Reverse Circulant+ Symmetric Circulant ($n = 500$) with entries $N(0,1)$ (ii) (right) Histogram plot of empirical distribution of Reverse Circulant+Hankel ($n = 500$) with $N(0,1)$ entries.

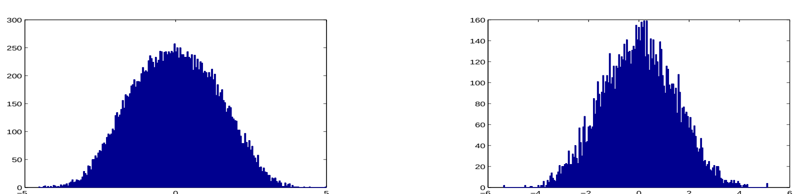

FIGURE 2. (i) (left) Histogram plot of empirical distribution of Toeplitz+Hankel($n = 1000$) with entries $N(0,1)$ (ii) (right) Histogram plot of empirical distribution of Toeplitz+Symmetric circulant ($n = 500$) with $N(0,1)$ entries.

In general the value of $p_C(w)$ cannot be computed for arbitrary pair matched word. In the two tables, we provide some examples.

TABLE 1. $p_C(w)$ for colored words corresponding to monomials $q = q(T, H)$

Monomial	Word	$p_C(w)$
TTHH	aabb	1
THTH	abab	2/3
TTTTHH	aabbcc	1
	abbacc	1
	ababcc	2/3
HHHHTT	aabbcc	1
	abbacc	1
	ababcc	0
TTHTTH	aabccb	1
	abcbac	1/2
	abcabc	1/2
HHTHHT	aabccb	1
	abcbac	1/2
	abcabc	0

TABLE 2. $p_C(w)$ for colored words corresponding to monomials $q = q(H, R)$ and $q(H, S)$

Monomial	Word	$p_C(w)$	Monomial	Word	$p_C(w)$
RRHH	aabb	1	SSHH	aabb	1
RHRH	abab	0	SHSH	abab	2/3
RRRRHH	aabbcc	1	SSSSHH	aabbcc	1
	abbacc	1		abbacc	1
	ababcc	0		ababcc	1
HHHHRR	aabbcc	1	HHHHSS	aabbcc	1
	abbacc	1		abbacc	1
	ababcc	0		ababcc	0
RRHRRH	aabccb	1	HHHSHS	aabcbc	1/2
	abcbac	0		abbcac	1/2
	abcabc	2/3		abcabc	0
HHRHHR	aabccb	1	HHSHHS	aabccb	1
	abcbac	0		abcbac	1/2
	abcabc	1/2		abcabc	0

It may be observed that in the two tables, $p_C(w)$ equals one for certain words. We now identify a class of such words. As discussed later this has ramifications in the study of freeness.

If for a $w \in CW(2)$, sequentially deleting all double letters of the same color each time leads to the empty word then we call w a *colored Catalan word*.

In the noncolored and nonindexed situation, Bose and Sen (2008) established that $p(w) = 1$ for the five matrices for all Catalan words w. Banerjee and Bose (2011) raised the question of when this is true for other matrices and introduced the following condition which guarantees this.

Consider the following boundedness property of the number of matches between rows across all pairs of columns.

Property P: A link function L satisfies *Property P* if

$$M^* = \sup_n \sup_{i,j} \#\{1 \le k \le n : L(k,i) = L(k,j)\} < \infty. \tag{3.3}$$

Note that the five matrices satisfy *Property P*.

It is not hard to see that colored Catalan words are in one one correspondence with noncrossing colored pair partitions. Thus freeness and semicircularity may be described for our limits in the language of words: if the limit satisfies $p_C(w) = 0$ for all words which are not colored Catalan, then the limit is free. *In addition*, if $p_C(w) = 1$ for all colored Catalan words, then the limits are also semicircular, which is precisely what happens for Wigner matrices. For the other four matrices, the limit is neither semicircular nor free but $p_C(w) = 1$ for all colored Catalan words as Theorem 3.3 shows. This extends the main result of Banerjee and Bose (2011) to multiple copies of colored matrices.

Theorem 3.3. *(i) Suppose X and Y satisfy Assumption* (A1) *and Assumption* (A2)*. Consider any monomial in X and Y of length $2k$. Then*

$$|\Pi_C^*(w)| \geq n^{1+k} \ \text{ for any colored Catalan word } \ w.$$

As a consequence, $p_C(w) \geq 1$ for any colored Catalan word w.

(ii) Suppose the link functions satisfy Property B and Property P and the input satisfies Assumption (A2)*. Then for any colored Catalan word, $p_C(w) = 1$.*

It is well known that independent Wigner matrices are asymptotically free and also they are asymptotically free of any class of deterministic matrices, provided the deterministic matrices $\{D_{i,n}\}_{1\leq i\leq p}$ satisfy the following uniformity of moment condition (see Theorem 5.4.5 of Anderson et al. (2010)):

$$\sup_k \max_{1\leq i\leq p} \sup_n \left(\frac{1}{n}\operatorname{Tr}(D_{i,n}^k)\right)^{\frac{1}{k}} < \infty. \tag{3.4}$$

Also, in the above result on asymptotic freeness the deterministic matrices can be replaced by random matrices $\{A_n\}$ provided $\sup_n \|A_n\| < \infty$ (see Speicher (2010)).

The result for deterministic matrices cannot be used as it is known that spectral norm of Toeplitz, Hankel, reverse circulant and symmetric circulant is unbounded and grows with n. From the developments of Anderson et al. (2010) it appears that one can relax condition (3.4) to the condition

$$\max_{1\leq i\leq p} \sup_n \left[\frac{1}{n}\operatorname{Tr}(D_{i,n}^k)\right]^{1/k} < c_k(<\infty) \text{ for all } k,$$

to still obtain freeness. However this appears to need formidable technical developments to justify all the steps in a proof. On the other hand, having developed the notions of circuits and words we are able to provide a relatively simple approach to show freeness for the special patterned matrices in the following Theorem.

Theorem 3.4. *Suppose $\{W_{i,n}, 1 \leq i \leq p, A_{i,n}, 1 \leq i \leq p\}$ are independent matrices satisfying assumption* (A2)*, where $W_{i,n}$ are Wigner matrices and $A_{i,n}$ are any of Toeplitz, Hankel, Symmetric circulant or Reverse circulant matrices. Then $\{W_{i,n}, 1 \leq i \leq p\}$ and $\{A_{i,n}, 1 \leq i \leq p\}$ are free in the limit.*

4. Proofs

To simplify the notational aspects in all our proofs we restrict ourselves to $h = 2$.

4.1. **Proof of Theorem 3.1.**

(1) We first show that

$$\Pi_C^*(w) = \Pi_{CI}^*(w) \ \text{ for all } \ w \in CIW(2). \tag{4.1}$$

Let $\pi \in \Pi_{CI}^*(w)$. As q is fixed,

$$\begin{aligned} \psi(w)[i] = \psi(w)[j] \ &\Rightarrow \ w[i] = w[j] \\ \Rightarrow (c_i, t_i, L_{c_i}(\pi(i-1), \pi(i))) \ &= \ (c_j, t_j, L_{c_j}(\pi(j-1), \pi(j))) \text{ (as } \pi \in \Pi_{CI}^*(w)). \end{aligned}$$

This implies $L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))$. Hence $\pi \in \Pi_C^*(\psi(w))$.

Now conversely, let $\pi \in \Pi_C^*(\psi(w))$. Then we have

$$w[i] \ = \ w[j]$$

$$\begin{aligned}\Rightarrow \psi(w)[i] &= \psi(w)[j]\\ \Rightarrow L_{c_i}(\pi(i-1),\pi(i)) &= L_{c_j}(\pi(j-1),\pi(j))\\ \Rightarrow Z_{c_i,t_i}(L_{c_i}(\pi(i-1),\pi(i))) &= Z_{c_j,t_j}(L_{c_j}(\pi(j-1),\pi(j))).\end{aligned}$$

as $w[i]=w[j] \Rightarrow c_i = c_j$ and $t_i = t_j$. Hence $\pi \in \Pi^*_{CI}(w)$.
So (4.1) is established. As a consequence,

$$p_{CI}(w) = \lim_{n\to\infty} \frac{1}{n^{1+k/2}} |\Pi^*_{CI}(w)| = p_C(\psi(w)).$$

Hence by (4.1) $p_{CI}(w)$ exists for all $w \in CIW(2)$ and $p_{CI}(w) = p_C(\psi(w))$, proving (1).

(2) Recall that $\mathbf{Z}_\pi = \prod_{j=1}^k Z_{c_j,t_j}(L_{c_ij}(\pi(j-1),\pi(j))$ and using (2.7) and (2.8)

$$\widehat{\mu}_n(q) = \frac{1}{n^{1+k/2}} \sum_{w:\ w\ matched} \sum_{\pi\in\Pi_{CI}(w)} \mathrm{E}(\mathbf{Z}_\pi). \tag{4.2}$$

By using Assumption (A2)

$$\sup_\pi \mathrm{E}\,|\mathbf{Z}_\pi| < K < \infty. \tag{4.3}$$

By using often used arguments of Bose and Sen (2008) and of Bryc et al. (2006), for any colored and indexed matched word w which is matched but is not pairmatched,

$$\lim_{n\to\infty} \frac{1}{n^{1+k/2}} \Big|\sum_{\pi\in\Pi_{CI}(w)} \mathrm{E}(\mathbf{Z}_\pi)\Big| \le \frac{K}{n^{1+k/2}} |\Pi_{CI}(w)| \to 0. \tag{4.4}$$

By using (4.4), and the fact that $\mathrm{E}(\mathbf{Z}_\pi) = 1$ for every color index pairmatched word (use Assumption (A2)), calculating the limit in (4.2) reduces to calculating $\lim \frac{1}{n^{1+k/2}} \sum_{w:\ w\in CIW(2)} |\Pi_{CI}(w)|$.

Now consider any $w \in CIW(2)$. Observe that any circuit in $\Pi^*_{CI}(w) - \Pi_{CI}(w)$ must have an edge of order four. Hence by (4.4),

$$\lim_{n\to\infty} \frac{|\Pi^*_{CI}(w) - \Pi_{CI}(w)|}{n^{1+k/2}} = 0.$$

As a consequence, since there are finitely many words,

$$\lim_{n\to\infty} \widehat{\mu}_n(q) = \lim_{n\to\infty} \sum_{w\in CIW(2)} \frac{|\Pi_{CI}(w)|}{n^{1+k/2}} = \lim_{n\to\infty} \sum_{w\in CIW(2)} \frac{|\Pi^*_{CI}(w)|}{n^{1+k/2}} = \sum_{w\in CIW(2)} p_{CI}(w) = \alpha(q). \tag{4.5}$$

To complete the proof of (2), we note that, if either k is odd or some index appears an odd number of times in q then for that q, $CIW(2)$ is empty and hence, $\alpha(q) = 0$. If k is even and every index appears an even number of times, then we know

$$|CIW(2)| \le |CW(2)| \le \frac{k!}{(k/2)!2^{k/2}}.$$

Now note that $p_{CI}(w) \le \Delta^{k/2}$. Combining all these, we get $|\alpha(q)| \le \frac{k!\Delta^{k/2}}{(k/2)!2^{k/2}}$.

(3) Now we claim that

$$\mathrm{E}[(\widetilde{\mu_n}(q) - \widehat{\mu}_n(q))^4] = O(n^{-2}).$$

Observe that,

$$\mathrm{E}[(\widetilde{\mu_n}(q) - \widehat{\mu}_n(q))^4] = \frac{1}{n^{2k+4}} \sum_{\pi_1,\pi_2,\pi_3,\pi_4} \mathrm{E}\Big(\prod_{j=1}^4 (\mathbf{Z}_{\pi_j} - E(\mathbf{Z}_{\pi_j}))\Big). \tag{4.6}$$

We say $(\pi_1,\pi_2,\pi_3,\pi_4)$ are "jointly matched" if each L-value occurs at least twice across all circuits (among same color) and they are said to be "cross matched" if each circuit has at least one L^* value which occurs in some other circuit.

If $(\pi_1, \pi_2, \pi_3, \pi_4)$ are not jointly matched then without loss of generality there exist some L-value in π_1 which does not occur anywhere else. Using $\mathrm{E}(\mathbf{Z}_{\pi_1}) = 0$ and independence,

$$\mathrm{E}(\prod_{j=1}^{4}(\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j})) = \mathrm{E}(\mathbf{Z}_{\pi_1}\prod_{j=2}^{4}(\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j})) = 0. \tag{4.7}$$

Again, if $(\pi_1, \pi_2, \pi_3, \pi_4)$ are jointly matched but not cross matched, then without loss of generality, assume π_1 is only self matched. Then by independence,

$$\mathrm{E}(\prod_{j=1}^{4}(\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j})) = \mathrm{E}[\mathbf{Z}_{\pi_1} - \mathrm{E}(\mathbf{Z}_{\pi_1})]\,\mathrm{E}[\prod_{j=2}^{4}(\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j}))] = 0. \tag{4.8}$$

So we are left with circuits that are jointly matched and cross matched with respect to q. Let Q_q be the number of such circuits.

We claim that $Q_q = O(n^{2k+2})$. Since the circuits are jointly matched there are at most $2k$ distinct L values among all the four circuits. Let u be the number of distinct L values (of a single color) in the circuits. Clearly, for a fixed choice of matches among those distinct L values (number of such choices is bounded in n), the number of jointly matched and cross matched circuits are $O(n^{u+4})$, so the number of such circuits with $u \le 2k-2$ is $O(n^{2k+2})$. Hence it suffices to prove that for a fixed choice of matches among $u = 2k-1$ or $u = 2k$ distinct L-values occurring across all four circuits, the number of jointly matched and cross matched circuits is $O(n^{2k+2})$.

We here only consider the case $u = 2k-1$. The other case can be dealt with similarly. Since $u = 2k-1$, it follows that every L-value occurs exactly twice across all four circuits. Since π_1 is not self matched, there must occur an L value in π_1 which does not occur anywhere else in π_1. We consider the following dynamic construction of $(\pi_1, \pi_2, \pi_3, \pi_4)$. Since the circuit is cross matched, there exists an L value which is assigned to a single edge, say $L(\pi_1(i_*-1), \pi(i_*))$. First choose one of the n possible values for the initial value $\pi_1(0)$, and continue filling in the values of $\pi_1(i), i = 1, 2, ..., i_*-1$. Then, starting at $\pi_1(k) = \pi_1(0)$, sequentially choose the values of $\pi_1(k-1), \pi_1(k-2), ..., \pi_1(i_*)$, thus completing the entire circuit π_1. At every stage there are n ways to choose a vertex if there is no L-match of the edge being constructed with the previously constructed edges, otherwise there are at most $\Delta(L_1, L_2)$ choices. So there are $O(n)$ choices for at most $2k-2$ distinct L values and hence the number of jointly matched and cross matched circuits for $u = 2k-1$ is $O(n^{2k-2+4})$, as required.

By Assumption (A2), $\mathrm{E}[\prod_{j=1}^{4}(\mathbf{Z}_{\pi_j} - \mathrm{E}(\mathbf{Z}_{\pi_j}))]$ is uniformly bounded over all $(\pi_1, \pi_2, \pi_3, \pi_4)$ by K, say. By this and (4.6)–(4.8), it follows that

$$\mathrm{E}[(\widetilde{\mu_n}(q) - \widehat{\mu}_n(q))^4] = O(\frac{n^{2k+2}}{n^{2k+4}}) = O(n^{-2}). \tag{4.9}$$

Now using Borel-Cantelli Lemma, $\widetilde{\mu_n}(q) - \widehat{\mu}_n(q) \to 0$ almost surely as $n \to \infty$ and this completes the proof.

4.2. **Proof of Theorem 3.2.** Condition (3.1) which needs to be verified (only for even degree monomials), crucially depends on the type of the link function and hence we need to deal with every example differently. Since we are dealing with only two link functions, we shall simplify the notation. Let X and Y be patterned matrices with link function L_1 and L_2 respectively with independent input sequences satisfying Assumption (A1) and Assumption (A2). Let $q(X,Y)$ be any monomial such that both X and Y occur an even number of times in q. Let $deg(q) = 2k$ and let the number of times X and Y occurs in the monomial be k_1 and k_2 respectively. Note that we have $k = k_1 + k_2$. Then it is enough to show that (3.1) holds for every pair matched colored word w of length $2k$ corresponding to q.

Let X and Y be any of the two following matrices: Wigner(W_n), Toeplitz(T_n), Hankel(H_n), Reverse circulant(RC_n) and Symmetric circulant(SC_n). The case when both X and Y are of the same pattern was dealt in Bose et al. (2010).

Proof of Theorem 3.2 is immediate once we establish the following Lemma.

Lemma 4.1. *Let X and Y be any of the following matrices: W_n, T_n, H_n, RC_n and SC_n satisfying Assumption* (A2). *Let $w \in CW(2)$ corresponding to a monomial q of length $2k$. Then there exists a (finite) index set I independent of n and $\{\Pi^*_{C,l}(w) : l \in I\} \subset \Pi^*_C(w)$ such that*

(1) $\Pi^*_C(w) = \cup_{l\in I}\Pi^*_{C,l}(w)$, *and* $p_{C,l}(w) := \lim_{n\to\infty} \frac{|\Pi^*_{C,l}(w)|}{n^{1+k}}$ *exists for all* $l \in I$,

(2) for $l \neq l'$ *we have,*

$$|\Pi^*_{C,l}(w) \cap \Pi^*_{C,l'}(w)| = o(n^{1+k}). \tag{4.10}$$

Assuming Lemma 4.1, $|\Pi^*_C(w)| = |\cup_{l\in I} \Pi^*_{C,l}(w)|$ for some finite index set I and

$$p_C(w) = \lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_C(w)| = \sum_{l\in I} \lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,l}(w)| = \sum_{l\in I} p_{C,l}(w). \tag{4.11}$$

The proof of this lemma treats each pair of matrices separately. Since the arguments are similar for the different pairs, we do not provide the detailed proof for each case but only a selection of the arguments in most cases.

The set S of all generating vertices of w is split into the three classes $\{0\} \cup S_X \cup S_Y$ where

$$S_X = \{i \wedge j : c_i = c_j = X, \quad w[i] = w[j]\}, \quad S_Y = \{i \wedge j : c_i = c_j = Y, \quad w[i] = w[j]\}.$$

For every $i \in S - \{0\}$, let j_i denote the index such that $w[j_i] = w[i]$. Let $\pi \in \Pi^*_C(w)$.

(i) *Toeplitz and Hankel:* Let X be Toeplitz (T) and Y be the Hankel (H) matrix. Observe that,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)| \text{ for all } i \in S_T$$
$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \text{ for all } i \in S_H.$$

Let I be $\{-1, 1\}^{k_1}$ and $l = (l_1, ..., l_{k_1}) \in I$. Let $\Pi^*_{C,l}(w)$ be the subset of $\Pi^*_C(w)$ such that,

$$\pi(i-1) - \pi(i) = l_i(\pi(j_i - 1) - \pi(j_i)) \qquad \text{for all } i \in S_T,$$
$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \qquad \text{for all } i \in S_H.$$

Now clearly,

$$\Pi^*_C(w) = \bigcup_l \Pi^*_{C,l}(w) \text{ (not a disjoint union)}.$$

Now let us define,

$$v_i = \frac{\pi(i)}{n} \qquad \text{and} \qquad U_n = \{0, \frac{1}{n}, ..., \frac{n-1}{n}\}. \tag{4.12}$$

Then,

$$\begin{aligned} |\Pi^*_{C,l}(w)| \;\; = \;\; & \#\{(v_0, ..., v_{2k}) : v_i \in U_n \; \forall 0 \le i \le 2k, \; v_{i-1} - v_i = l_i(v_{j_i-1} - v_{j_i}) \; \forall i \in S_T \\ & \text{and } v_{i-1} + v_i = v_{j_i-1} + v_{j_i} \; \forall i \in S_H, \; v_0 = v_{2k}\}. \end{aligned}$$

Let us denote $\{v_i : i \in S\}$ by v_S. It can easily be seen from the above equations (other than $v_0 = v_{2k}$) that each of the $\{v_i : i \notin S\}$ can be written uniquely as an integer linear combination $L^l_i(v_S)$. Moreover, $L^l_i(v_S)$ only contains $\{v_j : \; j \in S, \; j < i\}$ with non zero coefficients. Clearly,

$$|\Pi^*_{C,l}(w)| = \#\{(v_0, ..., v_{2k}) : v_i \in U_n \; \forall 0 \le i \le 2k, v_0 = v_{2k}, v_i = L^l_i(v_S) \;\; \forall i \notin S\}. \tag{4.13}$$

Any integer linear combinations of elements of U_n is again in U_n if and only if it is between 0 and 1. Hence,

$$|\Pi^*_{C,l}(w)| = \#\{v_S : v_i \in U_n \; \forall i \in S, v_0 = L^l_{2k}(v_S), 0 \le L^l_i(v_S) < 1 \;\; \forall i \notin S\}. \tag{4.14}$$

From (4.14) it follows that, $\frac{|\Pi^*_{C,l}(w)|}{n^{1+k}}$ is nothing but the Riemann sum for the function $I(0 \le L^l_i(v_S) < 1, i \notin S, v_0 = L^l_{2k}(v_S))$ over $[0,1]^{k+1}$ and converges to the integral and hence

$$p_{C,l}(w) = \lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,l}(w)| = \int_{[0,1]^{k+1}} I\left(0 \le L^l_i(v_S) < 1, i \notin S, v_0 = L^l_{2k}(v_S)\right) dv_S. \qquad (4.15)$$

This shows part (1) of Lemma 4.1. For part (2) let $l \neq l'$. Without loss of generality, let us assume that, $l_{i_1} = -l'_{i_1}$. Let $\pi \in \Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)$. Then $\pi(i_1 - 1) = \pi(i_1)$ and hence $L^l_{i_1-1}(v_S) = v_{i_1}$. It now follows along the lines of the preceding arguments that

$$\lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)| \le \int \cdots \int_{[0,1]^{k+1}} I(v_i = L^l_{i_1-1}(v_S)) dv_S. \qquad (4.16)$$

$L^l_{i_1-1}(v_S)$ contains $\{v_j : \ j \in S, \ j < i_1\}$ and hence $\{L^l_{i_1-1}(v_S) = v_i\}$ is a k-dimensional subspace of $[0,1]^{k+1}$ and hence has Lebesgue measure 0.

(ii) *Hankel and Reverse circulant:* Let X and Y be Hankel (H) and Reverse Circulant (RC) respectively. Then

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \quad \text{for all } i \in S_H, \qquad (4.17)$$

$$(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n \quad \text{for all } i \in S_{RC}. \qquad (4.18)$$

Clearly, as all the $\pi(i)$ are between 1 and n, relation (4.18) implies $(\pi(i-1)+\pi(i)) - (\pi(j_i-1) + \pi(j_i)) = a_i n$ where $a_i \in \{0, 1, -1\}$

Let $a = (a_1, ..., a_{k_2}) \in I = \{-1, 0, 1\}^{k_2}$. Let $\Pi^*_{C,a}(w)$ be the subset of $\Pi^*_C(w)$ such that,

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \ \forall i \in S_H \text{ and}$$

$$(\pi(i-1) + \pi(i)) - (\pi(j_i - 1) + \pi(j_i)) = a_i n \ \forall i \in S_{RC}.$$

Now clearly,

$$\Pi^*_C(w) = \bigcup_a \Pi^*_{C,a}(w) \text{ (a disjoint union).}$$

Now we get that,

$$\begin{aligned} |\Pi^*_{C,a}(w)| \ &= \ \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \le i \le 2k, \ v_{i-1} + v_i = (v_{j_i-1} + v_{j_i}) + a_i \ \forall i \in S_{RC} \\ &\quad \text{and} \ \ v_{i-1} + v_i = v_{j_i-1} + v_{j_i} \ \forall i \in S_H, \ v_0 = v_{2k}\}. \end{aligned}$$

Other than $v_0 = v_{2k}$, each $\{v_i : i \notin S\}$ can be written uniquely as an affine linear combination $L^a_i(v_S) + b^{(a)}_i$ for some integer $b^{(a)}_i$. Moreover, $L^a_i(v_S)$ only contains $\{v_j : \ j \in S, \ j < i\}$ with non zero coefficients. Arguing as in the previous case,

$$|\Pi^*_{C,a}(w)| = \#\{v_S : v_i \in U_n \ \forall i \in S, v_0 = L^a_{2k}(v_S) + b^{(a)}_{2k}, 0 \le L^a_i(v_S) + b^{(a)}_i < 1 \forall i \notin S\}. \qquad (4.19)$$

This is again a Riemann sum and hence as before,

$$p_{C,a}(w) = \lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,a}(w)| = \int_{[0,1]^{k+1}} I\left(0 \le L^a_i(v_S) + b^{(a)}_i < 1, i \notin S, v_0 = L^a_{2k}(v_S) + b^{(a)}_{2k}\right) dv_S$$

and the proof of this case is complete.

(iii) *Hankel and Symmetric circulant:* Let X and Y be Hankel (H) and Symmetric circulant (SC) respectively. Note that

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) \ \forall i \in S_H \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i - 1) - \pi(j_i)|| \ \forall i \in S_S.$$

It can be easily seen from the second equation above that either $|\pi(i-1)-\pi(i)| = |\pi(j_i-1)-\pi(j_i)|$ or $|\pi(i-1)-\pi(i)|+|\pi(j_i-1)-\pi(j_i)| = n$. There are six cases for each Symmetric Circulant match $[i, j_i]$, and with $v_i = \pi(i)/n$, these are:

(1) $v_{i-1} - v_i - v_{j_i-1} + v_{j_i} = 0$.
(2) $v_{i-1} - v_i + v_{j_i-1} - v_{j_i} = 0$.
(3) $v_{i-1} - v_i + v_{j_i-1} - v_{j_i} = 1$.
(4) $v_{i-1} - v_i - v_{j_i-1} + v_{j_i} = 1$.
(5) $v_i - v_{i-1} + v_{j_i-1} - v_{j_i} = 1$.
(6) $v_i - v_{i-1} + v_{j_i} - v_{j_i-1} = 1$.

Now we can write $\Pi^*_C(w)$ as the (not disjoint) union of 6^{k_2} possible $\Pi^*_{C,l}(w)$ where l denotes the combination of cases (1)–(6) above that is satisfied in the k_2 matches of Symmetric Circulant. For each $\pi \in \Pi^*_{C,l}(w)$, each $\{v_i : i \notin S\}$ can be written uniquely as an affine integer combination of v_S. As in the previous two pairs of matrices in (i) and (ii), $\lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,l}(w)|$ exists as an integral.

Now (4.10) can be checked case by case. As a typical case suppose Case 1 and Case 3 hold. Then $\pi(i-1) - \pi(i) = n/2$ and $v_{i-1} - v_i = 1/2$. Since i is generating and v_{i-1} is a linear combination of $\{v_j : j \in S, j < i\}$, this implies a nontrivial linear relation between the independent vertices v_S. This, in turn implies that the number of circuits π satisfying the above conditions is $o(n^{1+k})$.

(iv) *Toeplitz and Symmetric circulant:* Let X and Y be Toeplitz (T) and Symmetric circulant (SC) respectively. Again note that,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)| \ \forall i \in S_T \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i - 1) - \pi(j_i)|| \ \forall i \in S_{SC}. \tag{4.20}$$

Now, (4.20) implies either $|\pi(i-1)-\pi(i)| = |\pi(j_i-1)-\pi(j_i)|$ or $|\pi(i-1)-\pi(i)|+|\pi(j_i-1)-\pi(j_i)| = n$.

There are six cases for each Symmetric Circulant match as in Case (iii) above and two cases for each Toeplitz match.

As before we can write $\Pi^*_C(w)$ as the (not disjoint) union of $2^{k_1} \times 6^{k_2}$ possible $\Pi^*_{C,l}(w)$ where l denotes a combination of cases (1)–(6) for all SC matches (as in Case (iii)) and a combination of cases (1)-(2) for all T matches. As before, for each $\pi \in \Pi^*_{C,l}(w)$, each of the $\{v_i : i \notin S\}$ can be written uniquely as an affine integer combination of v_S. As earlier, $\lim_{n\to\infty} \frac{1}{n^{1+k}}|\Pi^*_{C,l}(w)|$ exists as an integral.

Now, (4.10) is again checked case by case. Suppose $l \neq l'$ and $\pi \in \Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)$. For $l \neq l'$, there must be one Toeplitz or Symmetric Circulant match such that two of the possible cases in (1)–(2) or in (1)–(6) occur simultaneously. Here we just deal with a typical pair Case (1) and Case (2) for the Toeplitz match. Then we have $\pi(i-1) - \pi(i) = 0$ and hence $v_{i-1} - v_i = 0$. Since i is generating and v_{i-1} is a linear combination of $\{v_j : j \in S, j < i\}$, this implies there exist a non trivial relation between the independent vertices v_S. This, in turn implies that the number of circuits π satisfying the above conditions in $o(n^{1+k})$. Now suppose the Symmetric Circulant match happens for both case (1) and case (2). Then again we have $v_i = v_{i-1}$ and we can argue as before to conclude that (4.10) holds.

(v) *Toeplitz and Reverse circulant:* Let X and Y be Toeplitz (T) and Reverse Circulant (RC) respectively. Note,

$$|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)| \quad \text{for all } i \in S_T,$$

$$(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n \quad \text{for all } i \in S_{RC}.$$

Clearly, as all the $\pi(i)$ are between 1 and n, $(\pi(i-1) + \pi(i)) \mod n = (\pi(j_i - 1) + \pi(j_i)) \mod n$ implies $(\pi(i-1) + \pi(i)) - (\pi(j_i - 1) + \pi(j_i)) = a_i n$ where $a_i \in \{0, 1, -1\}$

Let the number of Toeplitz and Reverse circulant matches be k_1, and k_2 respectively and let us denote $S_T = \{i_1, i_2, ..., i_{k_1}\}, S_{RC} = \{i_{k_1+1}, i_{k_1+2}, ..., i_{k_1+k_2}\}$.
Let $l = (c, a) = (c_{i_1}, ..., c_{i_{k_1}}, a_{i_{k_1+1}}, ..., a_{i_{k_1+k_3}}) \in I = \{-1, 1\}^{k_1} \times \{-1, 0, 1\}^{k_3}$.

Let $\Pi^*_{C,l}(w)$ be the subset of $\Pi^*_C(w)$ such that,

$$\pi(i-1) - \pi(i) = c_i(\pi(j_i - 1) - \pi(j_i)) \ \forall i \in S_T$$

$$\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i) + a_i n \ \forall i \in S_{RC}.$$

Now clearly,

$$\Pi^*_C(w) = \bigcup_{l \in I} \Pi^*_{C,l}(w),$$

and translating this in the language of v_i's, we get

$$\begin{aligned} |\Pi^*_{C,l}(w)| \quad = \quad & \#\{(v_0, ..., v_{2k}) : v_i \in U_n \ \forall 0 \le i \le 2k, \ v_{i-1} + v_i = (v_{j_i-1} + v_{j_i}) + a_i \ \forall i \in S_{RC} \\ & \text{and} \quad v_{i-1} - v_i = c_i(v_{j_i-1} - v_{j_i}) \ \forall i \in S_T, \ v_0 = v_{2k}\}. \end{aligned}$$

As in the previous cases, $\lim_{n\to\infty} \frac{|\Pi^*_{C,l}(w)|}{n^{1+k}}$ exists. It remains to show that, $\lim_{n\to\infty} \frac{|\Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)|}{n^{1+k}} = 0$ for $l \neq l'$. If $l = (c, a) \neq l' = (c', a')$, then either $c \neq c'$ or $a \neq a'$. If $c = c'$, then clearly $\Pi^*_{C,l}(w)$ and $\Pi^*_{C,l'}(w)$ are disjoint. Let $c \neq c'$. Without loss of generality, we assume $c_{i_1} = -c_{i_1}$. Then clearly, for every $\pi \in \Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)$ we have $v_{i_1-1} = v_i$, which gives a non trivial relation between $\{v_j : \ j \in S\}$. That in turn implies the required limit is 0.

(vi) *Reverse circulant and symmetric circulant:* Let X and Y be Reverse Circulant (RC) and Symmetric circulant (SC) respectively. Then

$$\pi(i-1) + \pi(i) \mod n = \pi(j_i - 1) + \pi(j_i) \mod n \ \forall i \in S_{RC} \text{ and}$$

$$n/2 - |n/2 - |\pi(i-1) - \pi(i)|| = n/2 - |n/2 - |\pi(j_i - 1) - \pi(j_i)|| \ \forall i \in S_{SC}.$$

As before, the latter equation implies either $|\pi(i-1) - \pi(i)| = |\pi(j_i - 1) - \pi(j_i)|$ or $|\pi(i-1) - \pi(i)| + |\pi(j_i - 1) - \pi(j_i)| = n$.

There are now three cases for each Reverse circulant match:

(1) $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = 0$.
(2) $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = 1$.
(3) $v_{i-1} + v_i - v_{j_i-1} - v_{j_i} = -1$.

Also, there are six cases for each Symmetric Circulant match as in Case (iii).

As before we can write $\Pi^*_C(w)$ as the union of $3^{k_1} \times 6^{k_2}$ possible $\Pi^*_{C,l}(w)$. Hence arguing in a similar manner, $\lim_{n\to\infty} \frac{1}{n^{1+k}} |\Pi^*_{C,l}(w)|$ exists as an integral. Now, to check (4.10), case by case. Suppose $l \neq l'$ and $\pi \in \Pi^*_{C,l}(w) \bigcap \Pi^*_{C,l'}(w)$. Since $l \neq l'$, there must be one Reverse Circulant or Symmetric Circulant match such that two of the possible cases (1)–(3) or (1)–(6) (which appear in Case (iii)) occur simultaneously. It is easily seen that such an occurrence is impossible for a Reverse Circulant match. We deal with one typical symmetric circulant match. Suppose then we have both case (1) and case (2). Then again we have $v_i = v_{i-1}$ and as a consequence (4.10) holds.

(vii) *Wigner and Hankel:* Let X and Y be Wigner (W) and Hankel (H) respectively. Observe that,

$$(\pi(i-1), \pi(i)) = \begin{cases} (\pi(j_i - 1), \pi(j_i)) & \text{(Constraint } C1) \\ (\pi(j_i), \pi(j_i - 1)) & \text{(Constraint } C2, \text{ for all } i \in S_W). \end{cases} \tag{4.21}$$

Also, $\pi(i-1) + \pi(i) = \pi(j_i - 1) + \pi(j_i)$ for all $i \in S_H$. So for each Wigner match there are two constraints and hence there are 2^{k_1} choices. Let λ be a typical choice of k_1 constraints and $\Pi^*_{C,\lambda}(w)$

be the subset of $\Pi^*_C(w)$ where the above relations hold. Hence

$$\Pi^*_C(w) = \bigcup_{\lambda} \Pi^*_{C,\lambda}(w) \quad \text{(not a disjoint union)}.$$

Now using equation (4.12) we have,

$$\begin{aligned}|\Pi^*_{C,\lambda}(w)| &= \#\{(v_0, v_1 \dots v_{2k}) : 0 \le v_i \le 1, v_0 = v_{2k}, v_{i-1} + v_i = v_{j_i-1} + v_{j_i},\ i \in S_H \\ &\qquad v_{i-1} = v_{j_i-1}, v_i = v_{j_i}, (C1), v_{i-1} = v_{j_i}, v_i = v_{j_i-1}(C2), i \in S_W\}.\end{aligned}$$

It can be seen from the above equations that each v_j, $j \notin S$ can be written (not uniquely) as a linear combination L^λ_j of elements in v_S. Hence as before,

$$\begin{aligned}|\Pi^*_{C,\lambda}(w)| &= \#\{v_S : v_i = L^\lambda_i(v_S), v_0 = v_{2k}, \text{ for } i \notin S, , v_{i-1} + v_i = v_{j_i-1} + v_{j_i},\ i \in S_H \\ &\qquad v_{i-1} = v_{j_i-1}, v_i = v_{j_i}, (C1), v_{i-1} = v_{j_i}, v_i = v_{j_i-1}(C2), i \in S_W\}.\end{aligned}$$

So the limit of $|\Pi^*_{C,\lambda}(w)|/n^{1+k}$ exists and can be expressed as an appropriate Riemann sum.

Now we show (4.10). Without loss of generality assume λ_1 is a C_1 constraint and λ_2 is a C_2 constraint. For any $\pi \in \Pi^*_{C,\lambda_1}(w) \bigcap \Pi^*_{C,\lambda_2}(w)$ we note that for $i \in S$,

$$(\pi(j_i), \pi(j_i - 1)) = (\pi(i-1), \pi(i)) = (\pi(j_i - 1), \pi(j_i)),$$

which implies $\pi(i) = \pi(i-1)$. Now i is a generating vertex. But $\pi(i) = \pi(i-1)$ and hence is fixed, having chosen the first $i-1$ vertices. This lowers the order by a power of n and hence the claim follows.

(vii) Wigner and other matrices: Since the other cases such as Wigner and Toeplitz and Wigner and Reverse circulant follow by similar and repetitive arguments we refrain from presenting a proof for them.

4.3. **Proof of Theorem 3.3.** Let w be a colored word of length $2k$ for a monomial $q = q(X, Y)$. Let w' be obtained from w by a cyclic permutation, that is, there exists l such that $w'[i] = w[(i + l) \bmod 2k]$. Note that w' is a colored word for the monomial q' obtained from q by the same cyclic permutation. We have the following lemma.

Lemma 4.2. *$|\Pi^*_C(w)| = |\Pi^*_C(w')|$ and $p_C(w) = p_C(w')$.*

Proof of Lemma 4.2. Let $\pi \in \Pi^*_C(w)$. Let $\pi'(i) = \pi((i + l) \bmod 2k))$. Clearly, $\pi'(0) = \pi'(2k)$. Also

$$w'[i] = w'[j] \Rightarrow L^*(\pi'(i-1), \pi'(i)) = L^*(\pi'(j-1), \pi'(j))$$

where L^* is equal to L_1 or L_2 according as $w'[i] = w'[j]$ is an X match or a Y match. Hence, $\pi' \in \Pi^*_C(w')$.

As w can also be obtained from w' by another cyclic permutation, it follows that the map $\pi \to \pi'$ is a bijection between $\Pi^*_C(w)$ and $\Pi^*_C(w')$. Hence $|\Pi^*_C(w)| = |\Pi^*_C(w')|$ and $p_C(w) = p_C(w')$. □

Proof of Theorem 3.3. (i) We use induction on the length of the word.

If $k = 1$ then $q = XX$ or $q = YY$. The only colored Catalan word is aa (drop superscript for ease). In either case, $\pi(0) = i, \pi(1) = j, \pi(2) = i$ is a circuit in $\Pi^*_C(w)$ for$1 \le i \le n, 1 \le j \le n$. Hence, $|\Pi^*_C(w)| \ge n^2$ and the result is true for $k = 1$.

Now let us assume that the claim holds for all monomials q of length less than $2k$ and all Catalan words corresponding to q. By Lemma 4.2, without loss of generality we assume that $w = aaw_1$ where w_1 is a Catalan word of length $(2k-2)$. Now let $\pi' \in \Pi^*_C(w_1)$. For fixed j, $1 \le j \le n$, define π by

$$\pi(0) = \pi'(0) \tag{4.22}$$

$$\pi(1) = j \tag{4.23}$$

$$\pi(j) = \pi'(j-2), \qquad j \geq 2. \tag{4.24}$$

Clearly π is a circuit and $\pi(0) = \pi(2)$ implies $L(\pi(0), \pi(1)) = L(\pi(1), \pi(2))$. Hence $\pi \in |\Pi_C^*(w)|$ and so, $|\Pi_C^*(w)| \geq n|\Pi_C^*(w_1)| \geq n^{k+1}$ and hence (i) is proved.

(ii) We shall now show that $p_C(w) \leq 1$ for matrices whose link functions satisfy *Property B* and *Property P*. The proof is same as the proof of Theorem 2(ii) of Banerjee and Bose (2011), with apporpiate changes to add color and index. We indicate the changes while keeping the notation similar to theirs for easy comparison. The proof uses $(2k+1)$-tuple π which are not necessarily circuit, that is, $\pi(0) = \pi(2k)$ is not assumed. Let w be a colored Catalan word. Define

$$\begin{aligned} C'(w) &= \{\pi :\ w[i] = w[j] \Rightarrow c_i = c_j \text{ and } L_{c_i}(\pi(i-1), \pi(i)) = L_{c_j}(\pi(j-1), \pi(j))\} \\ \Gamma_{i,j}(w) &= \{\pi \in C(w) : \pi(0) = i, \pi(2k) = j\},\ (1 \leq i, j \leq n), \quad \gamma_{i,j}(w) = |\Gamma_{i,j}(w)|. \end{aligned}$$

Clearly, $|\Pi_C^*(w)| = \sum_{i=1}^{n} \gamma_{i,i}(w)$. Now consider the following statement $\mathbf{S}'_k$ for all $k \geq 1$:

$\mathbf{S}'_k$: For any colored Catalan w of length $(2k)$, there exists $M_k > 0$ such that

$$\gamma_{i,j}(w) \leq M_k n^{k-1} \text{ for all } i \neq j \text{ and } \frac{1}{n}\sum_{i=1}^{n}\left|\frac{\gamma_{i,i}(w)}{n^k} - 1\right| = O(1/n).$$

The proof of $\mathbf{S}'_k$ easily follows by repeating the steps of the proof of Theorem 2(ii) of Banerjee and Bose (2011) and changing the set $C(w)$ there by $C'(w)$ and using *Property B* and *Property P*. To avoid repetitive arguments we skip the details. Once the validity of $\mathbf{S}'_k$ is asserted, one gets $p_C(w) \leq 1$ and the result now follows using part(i). □

4.4. **Proof of Remark 3.1(ii).** Note that the assumptions imply that LSD for $\frac{A}{\sqrt{n}}$ and $\frac{B}{\sqrt{n}}$ exists. By Theorem 3.1 it is clear that $\{\frac{A}{\sqrt{n}}, \frac{B}{\sqrt{n}}\}$ converge jointly and hence $\lim_{n\to\infty} \frac{1}{n^{k/2+1}} E(\text{Tr}(A + B)^k = \beta_k$ exists for all $k > 0$. Now let us fix k. Let Q_k be the set of monomials such that $(A+B)^k = \sum_{q \in Q_k} q(A, B)$. Hence

$$\frac{1}{n}\text{Tr}(\frac{A+B}{\sqrt{n}})^k = \frac{1}{n^{1+k/2}}\sum_{q \in Q_k}\text{Tr}(q(A,B)) = \sum_{q \in Q_k}\widehat{\mu_n}(q)$$

where $\widehat{\mu_n}(q)$ is as in Section 1. Now by (3) of Theorem 3.1, $\widehat{\mu_n}(q) \to \alpha(q)$, almost surely and hence,

$$\beta_k = \lim_{n\to\infty}\frac{1}{n}\text{Tr}(\frac{A+B}{\sqrt{n}})^k = \sum_{q \in Q_k}\alpha(q) \textit{ almost surely.}$$

Using (2) of Theorem 3.1, we have

$$\beta_{2k} = \sum_{q \in Q_{2k}}\alpha(q) \leq |Q_{2k}|\frac{(2k)!}{k!2^k}\Delta(L_1, L_2)^k = 2^{2k}\frac{(2k)!}{k!2^k}\Delta(L_1, L_2)^k.$$

Now by using Stirling's formula, $\beta_{2k} \leq (Ck)^k$ for some constant C. Hence $\sum_k \beta_{2k}^{-1/2k} = \infty$ and *Carleman's Condition* is satisfied implying that the LSD exists.

To prove symmetry of the limit, let $q \in Q_{2k+1}$. Then from (2) of Theorem 3.1, it follows that $\alpha(q) = 0$. Hence $\beta_{2k+1} = \sum_{q \in Q_{2k+1}}\alpha(q) = 0$ and the distribution is symmetric.

To prove unboundedness, without loss of generality let us assume that LSD $\mathcal{L}_A$ of $\frac{A}{\sqrt{n}}$ has unbounded support. Let us denote by $\beta_{2k}(A)$ the $(2k)-th$ moment of $\mathcal{L}_A$. Since L^p norm converges to essential supremum as $p \to \infty$ it follows that $(\beta_{2k}(A))^{1/2k} \to \infty$ as $k \to \infty$. Also, $\beta_{2k}(A) = \alpha(q_{2k})$ where $q_{2k}(A, B) = A^{2k}$ and $q_{2k} \in Q_{2k}$. Since $\alpha(q)$ is nonnegative for all q, it implies $\beta_{2k} \geq \beta_{2k}(A)$. So $\lim_{k\to\infty}(\beta_{2k})^{1/2k} = \infty$ and hence the LSD of $\frac{A+B}{\sqrt{n}}$ has unbounded support.

4.5. **Proof of Theorem 3.4.** We need the following development for describing freeness.

Let S_n be the group of permutations of $(1, 2, \dots n)$.

Definition 4.1. Let $\mathcal{A}$ be an algebra. Let $\psi_k : \mathcal{A}^k \longrightarrow C \;\; k > 0$ be multi linear functions. For $\alpha \in S_n$, let $c_1, c_2, \dots c_r$ be the cycles of α. Then define

$$\psi_\alpha[A_1, A_2, \dots, A_n] = \psi_{c_1}[A_1, A_2, \dots, A_n]\psi_{c_2}[A_1, A_2, \dots, A_n] \dots \psi_{c_r}[A_1, A_2, \dots, A_n]$$

where

$$\psi_c[A_1, A_2, \dots, A_n] = \psi_p\left(A_{i_1}A_{i_2}\dots A_{i_p}\right) \;\text{ if }\; c = (i_1, i_2 \dots i_p).$$

Freeness is intimately tied to noncrossing partitions. We describe the relevant portion of this relation in brief below. See Theorem 14.4 of Nica and Speicher (2006) for more details. Let $NC_2(m)$ be the set of non-crossing pair partitions of $\{1, 2, \dots, m\}$. A typical pair partition π will be written in the form $\{(r, \pi(r)),\; r = 1, 2, \dots, m\}$. For $p = (p(1), p(2), \dots, p(m))$ integers (also can be referred to as colors), let

$$NC_2^{(p)}(m) = \{\pi \in NC_2(m) : p(\pi(r)) = p(r) \text{ for all } r = 1, \dots, m\}.$$

Suppose $d_1, d_2, \dots, d_m, s_1, s_2, \dots s_m$ are elements in some noncommutative probability space $(\mathcal{B}, \varphi)$. Suppose $\{s_1, s_2, \dots s_m\}$ are free and each s_i follows the semicircular law. Then the collections $\{s_1, s_2, \dots s_m\}$ and $\{d_1, d_2, \dots d_m\}$ are free if and only if,

$$\begin{aligned} \varphi(s_{p(1)}d_1 \dots s_{p(m)}d_m) &= \sum_{\pi \in NC(m)} k_\pi[s_{p(1)}, \dots s_{p(m)}].\, \varphi_{\pi\gamma}[d_1, \dots, d_m] \\ &= \sum_{\pi \in NC_2^{(p)}(m)} \varphi_{\pi\gamma}[d_1, \dots, d_m], \end{aligned} \tag{4.25}$$

where $\gamma \in S_m$ is the cyclic permutation with one cycle and $\gamma = (1, 2, \dots, m-1, m)$. Here k_n denotes the free cumulants and k_π for a partition π is defined along the same lines as Definition 4.1.

We shall also drop the suffix C from $p_C(w)$, $\Pi_C(w)$, $\Pi_C^*(w)$ etc. for simplicity. Fix a monomial q of Wigner (W) and any other patterned matrix (A) of length $2k$. To prove freeness we show that the limiting variables satisfy the relation (4.25). We have already remarked that freeness is intimately tied to noncrossing partitions but freeness in the limit can also be roughly described in terms of colored words in the following manner.

(1) If for a colored word the pair partitions corresponding to the Wigner matrix cross, then $p(w) = 0$.
(2) If the pair partition corresponding to the letters of matrix A cross with any pair partition of W then also $p(w) = 0$.

For example, $p(w_1w_2w_1w_2a_1a_1) = 0$ and $p(w_1a_1w_1a_1) = 0$. This is essentially the main content of Lemma 4.3 given below.

We will discuss in detail the proof of Theorem 3.4 for $p = 1$ and indicate how the results continue to hold for $p \geq 1$.

We need a few preliminary Lemmata to prove the main result. We first use these Lemmata to prove Theorem 3.4 and then provide the proofs of the Lemmata.

We now concentrate only on (colored) pair matched words. For a word w the pair $(i, j)\; 1 \leq i < j \leq 2k$ is said to be a match if $w[i] = w[j]$. A match (i, j) is said to be a W match or an A match according as $w[i] = w[j]$ is Wigner or A letter.

We define $w_{(i,j)}$ to be the word of length $j - i + 1$ as

$$w_{(i,j)}[k] = w[i - 1 + k] \text{ for all } 1 \leq k \leq j - i + 1.$$

Let $w_{(i,j)^c}$ be the word of length $t+i-j-1$ obtained by removing $w_{(i,j)}$ from w, that is,

$$w_{(i,j)^c}[r] = \begin{cases} w[r] & \text{if } r < i, \\ w[r+j-i+1] & \text{if } r \geq i. \end{cases}$$

Note that in general these subwords may not be matched. If (i,j) is a W match, we will call $w_{(i,j)}$ a *Wigner string* of length $(j-i+1)$. For instance, for the monomial $WAAAAWWW$, $w = abbccadd$ is a word and $abbcca$ and dd are Wigner strings of length six and two respectively. For any word w, we define the following two classes:

$$\Pi^*_{(C2)}(w) = \{\pi \in \Pi^*(w) : (i,j)\ W \text{ match} \Rightarrow (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))\}, \quad (4.26)$$

$$\Pi^*_{(i,j)}(w) = \{\pi \in \Pi^*(w) : (\pi(i-1), \pi(i)) = (\pi(j), \pi(j-1))\}. \quad (4.27)$$

Note that the condition appearing above involves $C2$ constraint defined in (4.21) and

$$\Pi^*_{(C2)}(w) = \bigcap_{(i,j):\, W match} \Pi^*_{(i,j)}(w). \quad (4.28)$$

It is well known that if we have a collection of only Wigner matrices then $p(w) \neq 0$ if and only if all the constraints in the word are $C2$ constraints. See for example Bose and Sen (2008). We need the following crucial extension in the present setup.

Lemma 4.3. *For a colored paired matched word w of length $2k$ with $p(w) \neq 0$ we have:*

(a) Every Wigner string is a colored paired matched word;

(b) For any (i,j) which is a W match we have

$$\lim_{n \longrightarrow \infty} \frac{|\Pi^*(w) - \Pi^*_{(i,j)}(w)|}{n^{1+k}} = 0. \quad (4.29)$$

(c)

$$\lim_{n \longrightarrow \infty} \frac{|\Pi^*(w) - \Pi^*_{(C2)}(w)|}{n^{1+k}} = 0. \quad (4.30)$$

Note that (c) and (b) are equivalent by (4.28) and as the number of pairs (i,j) is finite.

Lemma 4.4. *If X_n any of the five patterned matrices in Theorem 3.4 and suppose they satisfy Assumption A2, then for any $l \geq 1$ and integers $(k_1, k_2, \ldots, k_l)$, we have*

$$\mathrm{E}\left[\prod_{i=1}^{l} (\frac{1}{n} \operatorname{Tr}(\left(\frac{X_n}{\sqrt{n}}\right)^{k_i}))\right] - \prod_{i=1}^{l} \mathrm{E}\left[\frac{1}{n} \operatorname{Tr}(\left(\frac{X_n}{\sqrt{n}}\right)^{k_i})\right] \to 0 \text{ as } n \to \infty.$$

Assuming the above lemmas we now prove Theorem 3.4.

Proof of Theorem 3.4. We take a single copy of W and A to show the result but for multiple copies the proof essentially remains same modulo some notations. Let q be a typical monomial, $q = WA^{q(1)}WA^{q(2)} \ldots WA^{q(m)}$ of length $2k$, where the $q(i)$'s may equal 0. So, $k = m/2 + (q(1) + q(2) + \ldots + q(m))/2$. From Theorem 3.2, for every such monomial q, $\frac{1}{n^{k+1}} \operatorname{Tr}(q)$ converges to say $\varphi(sa^{q(1)} \ldots sa^{q(m)})$, where s follows the semicircular law and a is the marginal limit of A, and φ is the appropriate functional defined on the space of noncommutative polynomial algebra generated by a and s. It is enough to prove that φ satisfy (4.25).

Let us expand the expression for

$$\lim_{n \to \infty} \frac{1}{n^{1+k}} \mathrm{E}[\operatorname{Tr}(WA^{q(1)}WA^{q(2)} \ldots WA^{q(m)})]$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{\substack{i(1),i(2),\dots i(m)\\ j(1),j(2),\dots j(m)=1}}^{n} \mathrm{E}[w_{i(1)j(1)} a^{q(1)}_{j(1)i(2)} w_{i(2)j(2)} a^{q(2)}_{j(2)i(3)} \dots w_{i(m)j(m)} a^{q(m)}_{j(m)i(1)}] \tag{4.31}$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{w\in CW(2)} \sum_{\pi\in\Pi^*(w)} \mathrm{E}[\mathbb{X}_\pi]$$

$$= \lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{w\in CW(2)} \sum_{\pi\in\Pi^*_{(C2)}(w)} \mathrm{E}[\mathbb{X}_\pi] \text{ (by Lemma 4.3 (c) and Assumption (A2)).} \tag{4.32}$$

Colored pair matched words of length $2k$ are in bijection with the set of pair partitions on $\{1, 2, \dots, 2k\}$ (denoted by $\mathcal{P}_2(2k)$). Now each such word w induces σ_w a pair partition of $\{1, 2, \dots m\}$ that is induced by only the Wigner matches i.e $(a, b) \in \sigma_w$ iff (a, b) is a Wigner match. So given any pair partition σ of $\{1, 2, \dots, m\}$, we denote by $[\sigma]_W$ the class of all w which induce the partition σ. So the sum in (4.32) can be written as,

$$\lim_{n\to\infty} \frac{1}{n^{1+k}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{w\in[\sigma]_W} \sum_{\pi\in\Pi^*_{(C2)}(w)} E[\mathbb{X}_\pi]. \tag{4.33}$$

By $C2$ constraint imposed on the class $\Pi^*_{(C2)}(w)$, if (r, s) is a W match then $(i(r), j(r)) = (j(s), i(s))$ (or, equivalently in terms of π we have, $(\pi(r-1), \pi(r)) = (\pi(s), \pi(s-1))$).

Therefore, we have the following string of equalities. Let tr be the normalized trace. The equality in (4.34) follows from (4.31) and (4.32). The steps in (4.35), (4.36) and (4.37) follow easily from calculations similar to Proposition 22.32 of Nica and Speicher (2006). The last step follows from the fact that the number of cycles of $\sigma\gamma$ is equal to $1 + m/2$ if and only if $\sigma \in NC_2(m)$. The notation $\mathrm{tr}_{\sigma\gamma}$ is given in Definition 4.1.

$$\lim_{n\to\infty} \frac{1}{n^{k+1}} \mathrm{E}[\mathrm{Tr}(WA^{q(1)}WA^{q(2)} \dots WA^{q(m)})]$$

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\dots i(m)\\ j(1),j(2),\dots j(m)=1}}^{n} \prod_{(r,s)\in\sigma} \delta_{i(r)j(s)}\delta_{i(s)j(r)} \mathrm{E}[a^{q(1)}_{j(1)i(2)} \dots a^{q(m)}_{j(m)i(1)}] \tag{4.34}$$

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\dots i(m)\\ j(1),j(2),\dots j(m)=1}}^{n} \prod_{(r,s)\in\sigma} \delta_{i(r)j(s)}\delta_{i(s)j(r)} \mathrm{E}[a^{q(1)}_{j(1)i(\gamma(1))} \dots a^{q(m)}_{j(m)i(\gamma(m))}] \tag{4.35}$$

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{\substack{i(1),i(2),\dots i(m)\\ j(1),j(2),\dots j(m)=1}}^{n} \prod_{r=1}^{m} \delta_{i(r)j(\sigma(r))} \mathrm{E}[a^{q(1)}_{j(1)i(\gamma(1))} \dots a^{q(m)}_{j(m)i(\gamma(m))}] \tag{4.36}$$

$$= \lim_{n\to\infty} \frac{1}{n^{k+1}} \sum_{\sigma\in\mathcal{P}_2(m)} \sum_{j(1),j(2),\dots j(m)=1}^{n} \mathrm{E}[a^{q(1)}_{j(1)j(\sigma\gamma(1))} \dots a^{q(m)}_{j(m)j(\sigma\gamma(m))}] \tag{4.37}$$

$$= \sum_{\sigma\in NC_2(m)} \lim_{n\to\infty} \mathrm{E}\left(\mathrm{tr}_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \dots, A^{(q_m)})]\right).$$

Now it follows from Lemma 4.4 that,

$$\sum_{\sigma\in NC_2(m)} \lim_{n\to\infty} \mathrm{E}\left(\mathrm{tr}_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \dots, A^{(q_m)})]\right) = \sum_{\sigma\in NC_2(m)} \lim_{n\to\infty} (\mathrm{E}\,\mathrm{tr})_{\sigma\gamma}[A^{(q_1)}, A^{(q_2)}, \dots, A^{(q_m)})]$$

$$= \sum_{\sigma\in NC_2(m)} \varphi_{\sigma\gamma}[a^{(q_1)}, a^{(q_2)}, \dots, a^{(q_m)})].$$

This shows 4.25 and hence freeness in the limit.

The above method can be easily extended to plug in more independent copies of W and A. The following details will be necessary.

(1) The extension of Lemmata 4.3 and 4.4. Note that these extensions can be easily obtained using the injective mapping ψ described in Section 3 and used in Theorem 3.1.

(2) When we consider several independent copies of the Wigner matrix the product in (4.36) gets replaced by

$$\prod_{r=1}^{m} \delta_{i(r)j(\sigma(r))}\delta_{p(r)p(\sigma(r))}.$$

Here $(p(1), p(2), \ldots, p(m))$ denotes the colors corresponding to the independent Wigner matrices. The calculations are similar to Theorem 22.35 of Nica and Speicher (2006).

The rest are some algebraic details, which we skip.

□

Having proved the Theorem we now come back to the proof of Lemma 4.3 and 4.4.

The next Lemma turns out to be the most essential ingredient in proving Lemma 4.3 and it points out the behavior of a colored pair matched word which contains a Wigner string inside it.

Lemma 4.5. *For any colored pair matched word w and a Wigner string $w_{(i,j)}$ which is a pair matched word and satisfies equation* (4.29)

$$p(w) = p(w_{(i,j)})p(w_{(i,j)^c}). \tag{4.38}$$

Further, if $w_{(i+1,j-1)}$ and $w_{(i,j)^c}$ satisfy (4.30) *then so does w.*

Proof. Given any $\pi_1 \in \Pi^*(w_{(i+1,j-1)})$ and $\pi_2 \in \Pi^*(w_{(i,j)^c})$ construct π as:

$$\pi = (\pi_2(0), \ldots \pi_2(i-1), \pi_1(0), \ldots \pi_1(j-i-1) = \pi_1(0), \pi_2(i-1), \ldots (2k-j+i-1)) \in \Pi^*_{(i,j)}(w).$$

Conversely, from any $\pi \in \Pi^*_{(i,j)}(w)$ one can construct π_1 and π_2 by reversing the above construction.

So we have

$$|\Pi^*_{(i,j)}(w)| = |\Pi^*(w_{(i+1,j-1)})||\Pi^*(w_{(i,j)^c})|. \tag{4.39}$$

Let $|w_{(i+1,j-1)}| = 2l_1$ and $|w_{(i,j)^c}| = 2l_2$ and note that $(1+l_1)+(1+l_2) = k+1$.

Now using the fact that $w_{(i,j)}$ satisfies (4.29) and dividing equation (4.39) by n^{k+1} we get in the limit,

$$p(w) = p(w_{(i+1,j-1)})p(w^c_{(i,j)}).$$

Now we claim that

$$|\Pi^*(w_{(i,j)})| = n|\Pi^*(w_{(i+1,j-1)})|. \tag{4.40}$$

Now given $\pi \in \Pi^*(w_{(i,j)})$, one can always get a $\pi' \in \Pi^*(w_{(i+1,j-1)})$, where the $\pi(i-1)$ is arbitrary and hence $\frac{|\Pi^*(w_{(i,j)})|}{n} \le |\Pi^*(w_{(i+1,j-1)})|$. Also given a $\pi' \in \Pi^*(w_{(i+1,j-1)})$ one can choose $\pi(i-1)$ in n ways and also assign $\pi(j) = \pi(i-1)$ or $\pi(i)$, making j a dependent vertex. So we get that, $|\Pi^*(w_{(i,j)})| \ge n|\Pi^*(w_{(i+1,j-1)})|$. This shows (4.40). So from (4.40) it follows that

$$p(w_{(i,j)}) = p(w_{(i+1,j-1)}),$$

whenever $w_{(i,j)}$ is a Wigner string.

Also note that from the first construction,

$$|\Pi^*_{(C2)}(w)| = |\Pi^*_{(C2)}(w_{(i+1,j-1)})||\Pi^*_{(C2)}(w_{(i,j)^c})|.$$

Now suppose $w_{(i+1,j-1)}$ and $w_{(i,j)^c}$ satisfy (4.30). So we have that

$$|\Pi^*(w_{(i+1,j-1)})| = |\Pi^*_{(C2)}(w_{(i+1,j-1)})| + o(n^{l_1+1}) \text{ and } |\Pi^*(w_{(i,j)^c})| = |\Pi^*_{(C2)}(w_{(i,j)^c})| + o(n^{l_2+1}).$$

Multiplying these and using the fact (from (4.38)) $|\Pi^*(w)| = |\Pi^*(w_{(i+1,j-1)})||\Pi^*(w_{(i,j)^c})| + o(n^{k+1})$, the result follows. □

We now give a proof of Lemma 4.3.

Proof of Lemma 4.3. We use induction on the length l of the Wigner string. Let w be a pair matched colored word of length $2k$ with $p(w) \neq 0$. First suppose the Wigner string is of length 2, that is, $l = 2$. We may without loss of generality assume them in the starting position. So we for any $\pi \in \Pi^*(w)$ with above property we have

$$(\pi(0), \pi(1)) = \begin{cases} (\pi(1), \pi(2)) \\ (\pi(2), \pi(1)). \end{cases}$$

In the first case $\pi(0) = \pi(1) = \pi(2)$ and so $\pi(1)$ is not generating vertex and this lowers the number of generating vertices (which is not possible as $p(w) \neq 0$). Hence, the only possibility is $(\pi(0), \pi(1)) = (\pi(2), \pi(1))$ and the circuit is complete for the Wigner string and so it is a pair matched word, proving part (a). Also, as a result of the above arguments only $C2$ constraints survive, which shows (b).

Now suppose the result holds for all Wigner strings of length strictly less than l. Consider a Wigner string of length l, say $w_{(1,l)}$ (we assume it to start from the first position). We break the proof into two cases I and II. In case I, we suppose that the Wigner string has a Wigner string of smaller order and use induction hypothesis and Lemma 4.5 to show the result. In Case II, we assume that there is no Wigner string inside. So there is a string of letters coming from matrix A after a Wigner letter. We show that this string is pair matched and the last Wigner letter before the l-th position is essentially at the first position. This also implies that the string within a Wigner string do not cross a Wigner letter.

Case I: Suppose that $w_{(1,l)}$ contains a Wigner string of length less than l at the position (p, q) with $1 \leq p < q \leq l$. Since $w_{(p,q)}$ is a Wigner string, by Lemma 4.5 we have,

$$p(w) = p(w_{(p,q)})p(w_{(p,q)^c}) \neq 0.$$

So by induction hypothesis and the fact that both $p(w_{(p,q)})$ and $p(w_{(p,q)^c})$ are not equal to zero we have, $w_{(p,q)}$ and $w_{(p,q)^c}$ are pair matched words and they also satisfy (4.29). So $w_{(1,l)}$ is a pair matched word, as it is made up of $w_{(p,q)}$ and $w_{(p,q)^c}$ which are pair matched. Also from second part of Lemma 4.5, we have $w_{(1,l)}$ satisfies part (b) and (c).

Case II: Suppose there is no Wigner string in the first l positions. We look at the last Wigner letter in the first $l-1$ positions. Let this be at position j_0. Now as we have assumed that there is no Wigner string of smaller length, $\pi(j_0)$ is a generating vertex. Also, as j_0 is the last Wigner letter, the positions from j_0 to $l-1$ are all letters coming from the matrix A.

Now we use the structure of the matrix A.

Subcase II(i): Suppose A is a Toeplitz matrix. Let $s_i = (\pi(j_0 + i) - \pi(j_0 + i - 1))$ with $i = 1, 2, \ldots, l - 1 - j_0$. Now consider the following equation

$$s_1 + s_2 \ldots + s_{l-1-j_0} = (\pi(l-1) - \pi(j_0)). \tag{4.41}$$

If for any j, $w[j]$ is the first appearance of that letter, then consider s_j to be an independent variable (can be chosen freely). Then due to the Toeplitz link function, if $w[k] = w[j]$, where $k > j$, then $s_k = \pm s_j$. Since $(1, l)$ is a W match, $\pi(l-1)$ is either $\pi(0)$ or $\pi(1)$ and hence $\pi(l-1)$ is not a generating vertex. Note that (4.41) is a constraint on the independent variables unless $s_1 + \ldots + s_{l-1-j_0} = 0$. If this is non zero, this non-trivial constraint lowers the number of independent variables and hence the limit contribution will be zero, which is not possible as $p(w) \neq 0$. So we must have,

$$\pi(l-1)=\pi(j_0) \qquad \text{and} \qquad j_0=1.$$

This also shows $(\pi(l),\pi(l-1))=(\pi(0),\pi(1))$ and hence $w_{(1,l)}$ is a colored word. As $s_1+\ldots+s_{l-1-j_0}=0$, all the independent variables occur twice with different signs in the left side, since otherwise it would again mean a nontrivial relation among them and thus would lower the order. Hence we conclude that the Toeplitz letters inside the first l positions are also pair matched. Since the $C2$ constraint is satisfied at the position $(1,l)$, part (b) also holds.

Subcase II(ii): Suppose A is a Hankel matrix. We write, $t_i=(\pi(j_0+i)+\pi(j_0+i-1))$ and consider

$$-t_1+t_2-t_3\ldots(-1)^{l-j_0-1}t_{l-j_0-1}=(-1)^{l-j_0-1}(\pi(l-1)-\pi(j_0)). \tag{4.42}$$

Now again as earlier, the t_i's are independent variables, and so this implies that again to avoid a nontrivial constraint which would lower the order, both sides of the equation(4.42) have to vanish, which automatically leads to the conclusion that $\pi(l-1)=\pi(j_0)=\pi(1)$. So $j_0=1$ and again the Wigner paired string of length l is pair matched. Part (b) also follows as the $C2$ constraint holds.

Subcase II(iii): A is symmetric or reverse circulant. Note that they have link functions which are quite similar to Toeplitz and Hankel respectively, the proofs are very similar to the above two cases and hence we skip them. □

Proof of Lemma 4.4. We first show that,

$$\mathrm{E}\left[\prod_{i=1}^{l}\left(\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}-\mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right]=\mathrm{O}(\frac{1}{n}) \text{ as } n\to\infty, \tag{4.43}$$

where tr denotes the normalized trace. To prove (4.43), we see that,

$$\mathrm{E}\left[\prod_{i=1}^{l}\left(\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}-\mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right]=\frac{1}{n^{\sum_{i=1}^{l}k_i/2+l}}\sum_{\pi_1,\pi_2,..\pi_l}\mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i}-\mathrm{E}(X_{\pi_i})))]. \tag{4.44}$$

If the circuit π_i is not jointly matched with the other circuits then $\mathrm{E}\,X_{\pi_i}=0$ and

$$\mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i}-\mathrm{E}(X_{\pi_i})))]=\mathrm{E}[X_{\pi_i}(\prod_{j\neq i}(X_{\pi_i}-\mathrm{E}(X_{\pi_i})))]=0.$$

If any of the circuits is self matched i.e. it has no cross matched edge then

$$\mathrm{E}[(\prod_{j=1}^{l}(X_{\pi_i}-\mathrm{E}(X_{\pi_i})))]=\mathrm{E}[X_{\pi_i}-\mathrm{E}(X_{\pi_i})]\,\mathrm{E}[(\prod_{j\neq i}(X_{\pi_i}-\mathrm{E}(X_{\pi_i})))]=0.$$

Now total number of circuits $\{\pi_1,\pi_2,\ldots\pi_l\}$ where each edge appears at least twice and one edge at least thrice is $\leq Cn^{\sum_{i=1}^{l}k_i/2+l-1}$, by Property B. Hence using Assumption (A2) such terms in (4.44) are of the order $\mathrm{O}(\frac{1}{n})$. Now consider rest of terms where all the edges appear exactly twice. As a consequence $\sum_{i=1}^{l}k_i$ is even. Also number of partitions of $\frac{1}{2}\sum_{i=1}^{l}k_i$ into l circuits is independent of n. We need to consider only $\{\pi_1,\pi_2,\ldots\pi_l\}$ which are jointly matched but not self matched.

If we prove that for such a partition the number of circuits is less than $Cn^{\sum_{i=1}^{l}k_i+l-1}$ we are done since the number of such partitions is independent of n and (4.3).

Since π_1 is not self matched we can without loss of generality assume that the edge value for $(\pi(0),\pi(1))$ occurs exactly once in π_1. So construct π_1 as follows. First choose $\pi_1(0)=\pi_1(k_1)$ and then choose the remaining vertices in the order $\pi_1(k_1),\pi_1(k_1-1)\ldots\pi_1(1)$. One sees that we loose one degree of freedom as in this way the edge $(\pi(0),\pi(1))$ is determined and we cannot choose it arbitrarily.

The result now follows from (4.43) by using induction. For $l = 2$ expanding and using the fact that expected normalized trace of the powers of $X_n/\sqrt{n}$ converges we get,

$$\begin{aligned}
&\mathrm{E}\left[\prod_{i=1}^{2}\left(\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_i}}{n^{k_i/2}}\right]\right)\right] \\
&= \mathrm{E}\left[\left(\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}}\right]\right)\left(\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}} - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}}\right]\right)\right] \\
&= \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1/2}}\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2/2}}\right] - \mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_1}}{n^{k_1}}\right]\mathrm{E}\left[\operatorname{tr}\frac{X_n^{k_2}}{n^{k_2}}\right] \to 0 \text{ as } n \to \infty.
\end{aligned}$$

So the result holds for $l = 2$. Now suppose it is true for all $2 \le m < l$. We expand

$$\lim_{n\to\infty} \mathrm{E}[\prod_{i=1}^{l}(\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i}) - \mathrm{E}(\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})))] = 0$$

to get

$$\lim_{n\to\infty}\sum_{m=1}^{l}(-1)^m \sum_{i_1<i_2\ldots<i_m} \mathrm{E}[\prod_{j=1}^{m}\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_{i_j}})] \prod_{i\notin\{i_1,i_2,\ldots i_m\}} \mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})] = 0.$$

Now using the result for products of smaller order successively,

$$\lim_{n\to\infty}(-1)^l\,\mathrm{E}[\prod_{j=1}^{l}\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_j})] = \lim_{n\to\infty}\sum_{m<l}(-1)^m \sum_{i_1<i_2\ldots<i_m} \mathrm{E}[\prod_{j=1}^{m}\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_{i_j}})] \prod_{i\notin\{i_1,i_2,\ldots i_m\}} \mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})].$$

Now every term in right side is by induction hypothesis $\lim_{n\to\infty}\prod_{i=1}^{l}\mathrm{E}[\operatorname{tr}((\frac{X_n}{\sqrt{n}})^{k_i})]$ and from this the Lemma follows.

□

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