PRODUCT OF EXPONENTIALS
AND
SPECTRAL RADIUS OF RANDOM k CIRCULANTS

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Abstract

Even though the distribution of the g fold product of independent and identically distributed (i.i.d.) random variables have been studied in the literature, there does not seem to be much work on its tail behaviour, when they are in Gumbel domain of attraction. We first derive the tail behaviour when the random variables are exponentially distributed.

Then we consider $n \times n$ random k-circulant matrices with $n \to \infty$ and $k = k(n)$ whose input sequence $\{a_l\}_{l \geq 0}$ is i.i.d. with finite $(2 + \delta)$ moment. We study the asymptotic distribution of the spectral radius, when $n = k^g + 1$. We show that with appropriate scaling and centering, the limit distribution is Gumbel. We also identify the centering and scaling constants explicitly. The proof uses appropriate normal approximation techniques and the above tail behaviour.

Keywords Tail of product, Gumbel distribution, large dimensional random matrix, eigenvalues, k circulant matrix, spectral radius, normal approximation, linear process, spectral density.

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Contents

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1 Introduction

1.1 Tail of product

Several researchers have studied the distributional properties of the product of independent and identically distributed (i.i.d.) random variables. See for instance Springer and Thompson (1970)(33), Lomnicki (1967)(23) and Galambos and Simonelli (2004)(17). However, there does not seem to be in the literature any result quantifying the nature of the tail behaviour of product beyond two or three fold product of i.i.d. exponentials.

We first identify the tail behaviour of finite but arbitrary product of i.i.d. exponential random variables. As a consequence, it follows that this n fold product lies in the max domain of attraction of the Gumbel distribution for any n.

1.2 Application to random matrices

Next we turn to an interesting application of the above result to the study of the spectral radius of a class of random matrices. The k-circulant matrices are circulant matrices where, instead of shifting the elements in subsequent rows by one position, they are shifted by k positions. The value of k may change with the increasing dimension of the matrix. They and their block versions arise in many different areas of Mathematics and Statistics - from multi-level supersaturated design of experiment (Georgiou and Koukouvinos (2006)(18), Chen and Liu (2008)(11)) to spectra of De Bruijn graphs (Strok (1992)(34)) and (0, 1)-matrix solutions to $A^m = J_n$ (Wu, Jia and Li (2002)(38)) - just to name a few.

These matrices with i.i.d. input sequence have been studied by several authors. Some results for the spectral norm and radius for the reverse circulant and circulant were proved in Bose, Hazra and Saha (2009)(4), (5) and Bose, Mitra and Sen (2008)(7). The latter considered the k circulants with $n = k^2 + 1$ and established that the spectral norm, when appropriately scaled and centered, converges to the Gumbel distribution. Their result was proved by using, some combinatorics which helped in delineating the structure of the eigenvalues, the tail behaviour of the product of two i.i.d. exponentials and, appropriate normal approximation results.

We provide a significant extension of latter result to the case $sn = k^g + 1$ where g is any positive integer and suitable conditions on s are assumed. Our proof proceeds along lines similar to theirs. To achieve this, we establish some new auxiliary properties of the eigenvalues and use the Gumbel domain result mentioned above after bringing down everything to the exponential case via appropriate normal approximation results.

1.3 Outline of the article

In Section 2.1 we describe a few known methods for product of two exponentials. These methods do not seem to extend to the general case. In Theorem 1 of Section 2.2 we derive explicitly the tail behaviour of the n fold product of exponentials. In Section 2.3 we show that the $1/2g$-th root of product of g-many i.i.d. exponentials belongs to the max domain of attraction of the Gumbel distribution. In Section 2.4 we comment on some of the existing literature relating products and extreme values.

The rest of the sections are on the (random) k-circulant. In Section 3.2 we describe the general k circulant matrices and in Section 3.3 we give a brief description of the known results on spectral radius of circulant matrices. In Section 3.4 we present the known formula for the eigenvalues of the k circulant. A more detailed description of these eigenvalues for $n = k^g + 1$ is given in Section 3.5. Section 3.6 contains the properties of the eigenvalues when the input sequence is i.i.d. Gaussian. Section 3.7 has two preparatory Lemmas on truncation and normal approximation. Drawing on the developments of Section 2.3 and of Sections 3.4–3.7, in Section 3.8, we derive the limit behaviour of the spectral radius of k circulant matrices when $n = k^g + 1 \to \infty$, g being an integer held fixed. We show that the spectral radius, when scaled and centered appropriately, converges in distribution to the Gumbel distribution. In Section 3.9, we remark about the case $sn = k^g + 1$. In Section 3.10, we deal with the case where the input sequence is an infinite order linear process and derive the limit for the maxima of the eigenvalues when scaled by the spectral density.

2 Tail of product

Let $\{E_i\}$ be i.i.d. standard exponentials. Define

$$H_n(x) = \mathrm{P}[E_1 E_2 \cdots E_n > x]. \tag{1}$$

What is the behaviour of $H_n(x)$ as $x \to \infty$? It is easy to see that this tail becomes heavier as n increases but there does not appear to be any results in the literature quantifying the nature of the tail beyond the case $n = 2$.

2.1 Various methods for two fold product

There are several possible approaches that come to mind to solve this problem:

(a) **Mellin Transform:** The Mellin transform of any nonnegative function $f(x)$, $x \geq 0$, is defined as (see Springer and Thompson (1970)(33))

$$M(f(\cdot)|s) = \int_0^\infty x^{s-1} f(x)dx.$$

Under certain regularity conditions, this transform, considered as a function of the complex variable s, admits an inversion integral:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} M(f(\cdot)|s)ds,$$

where the path of integration is a line parallel to the imaginary axis and to the right of the origin. If X and Y are nonnegative independent random variables with p.d.f. $f(\cdot)$ and $g(\cdot)$ respectively and if $h(\cdot)$ is the p.d.f. of $Z = XY$, then

$$M\left(h(\cdot)|s\right) = M\left(f(\cdot)|s\right) M\left(g(\cdot)|s\right).$$

Thus the Mellin transform for the product plays a role similar to that played by the Fourier transform for sum. It can be easily seen that

$$M(\overline{F}(\cdot)|s) = s^{-1} M(f(\cdot)|s+1), \quad \text{where } \overline{F}(x) = \mathrm{P}[X > x]. \tag{2}$$

Using (2) and appropriate complex integration, Lomnicki (1967)(23) showed that,

$$M(H_n(\cdot)|s) = s^{-1}[\Gamma(s+1)]^n \ \text{ and } \ H_n(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} s^{-1}[\Gamma(s+1)]^n ds.$$

He also derived the following series representations of the above integral for $n = 2$ and 3.

$$H_2(x) = 1 - \sum_{j=1}^{\infty} \frac{x^j}{j\{j-1!\}^2}\{-\log x + 2\psi(j) + j^{-1}\}, \ \text{ and}$$

$$H_3(x) = 1 - \frac{1}{2}\sum_{j=1}^{\infty} \frac{x^j(-1)^{j-1}}{j\{(j-1)!\}^3}\left[\{-\log x + 3\psi(j) + j^{-1}\}^2 + 3\{\psi'(1) + \sum_{k=1}^{j-1} k^{-2}\} + j^{-2}\right],$$

where $\psi(\cdot)$ is the Euler psi function (digamma function) and $\psi'(\cdot)$ it's first derivative.

For the special case of $n = 2$, comparing the above series with the series expansion of the modified Bessel function of the second kind $K_0(x)$, it can be easily shown that

$$H_2(x) \sim \sqrt{\pi} x^{\frac{1}{4}} e^{-2x^{1/2}} \ \text{ as } \ x \to \infty.$$

However, this method does not seem to be easy to extend to other values of n.

(b) **Differential Equation:** The following differential equation can be easily derived for $H_2(\cdot)$:

$$x\frac{d^2}{dx^2}H_2(x) - H_2(x) = 0, \ \ H_2(0) = 0 \ \text{ and } \ H_2(\infty) = 1.$$

Standard theory of second order differential equations implies that the solution can be expressed in terms of the modified Bessel function of second kind and the tail behaviour follows from that. For $n \geq 3$ we obtain higher order differential equations and their solutions appear to be intractable.

(c) **Real analysis:** Tang (2008)(35) obtained a nice formula for $H_2(x)$ using simple integral substitutions. We reproduce the result and its proof since this will be useful to motivate our result and its proof for arbitrary n.

Lemma 1.

$$H_2(x) = e^{-2x^{1/2}} \int_0^{\infty} \frac{e^{-z}}{\sqrt{z}} \frac{z + 2x^{1/2}}{\sqrt{z^2 + 4zx^{1/2}}} dz \sim \sqrt{\pi} e^{-2x^{1/2}} x^{1/4} g_2(x), \ \ \textit{where} \ \ g_2(x) \to 1 \ \ \textit{as} \ \ x \to \infty.$$

Proof. First note that,

$$H_2(x) = \int_0^\infty e^{-y} e^{-x/y} dy = \int_0^{x^{1/2}} e^{-(y+\frac{x}{y})} dy + \int_{x^{1/2}}^\infty e^{-(y+\frac{x}{y})} dy.$$

Let $A(y) = y + \frac{x}{y}$. Then

$$\begin{aligned} A'(y) = 1 - \frac{x}{y^2} &> 0 \text{ if } y > x^{1/2} \\ &< 0 \text{ if } y < x^{1/2}. \end{aligned}$$

Hence in these two regions, consider separately, the monotone substitution $A(y) = t$. Let the corresponding unique solutions (inverses), be $y_i(t)$, $i = 1, 2$, so that

$$y_1(t) < x^{1/2} < y_2(t).$$

Observing that both the ranges transform to $(2x^{1/2}, \ \infty)$, we obtain,

$$H_2(x) = \int_{2x^{1/2}}^\infty e^{-t} \left[\left(1 - \frac{x}{y_2^2(t)}\right)^{-1} + \left(\frac{x}{y_1^2(t)} - 1\right)^{-1} \right] dt.$$

Since $y_i = t, \ i = 1, 2$ are the two solutions of the quadratic equation $A(y) = t$, it is easy to see that

$$\left[\left(1 - \frac{x}{y_2^2(t)}\right)^{-1} + \left(\frac{x}{y_1^2(t)} - 1\right)^{-1} \right] = \frac{t}{\sqrt{(t^2 - 4x)}}.$$

Now, making a further substitution $t = z + 2x^{1/2}$ we get,

$$\begin{aligned} H_2(x) &= e^{-2x^{1/2}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \frac{z + 2x^{1/2}}{\sqrt{z^2 + 4zx^{1/2}}} dz \\ &= \sqrt{\pi} e^{-2x^{1/2}} x^{1/4} \underbrace{\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left(\frac{1 + z/2x^{1/2}}{\sqrt{1 + z/4x^{1/2}}} \right) dz.}_{g_2(x)} \end{aligned}$$

Now a straightforward application of the Dominated Convergence Theorem (DCT) implies

$$\lim_{x\to\infty} g_2(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-z} z^{-1/2} dz = 1.$$

This proves the Lemma. □

2.2 Tail behaviour for n fold product

Theorem 1. *There exists constants* $\{C_k, N_k, M_k, \alpha_k, \beta_k\}$ *such that*

$$H_k(x) = C_k x^{\alpha_k} e^{-kx^{\frac{1}{k}}} g_k(x), \quad k \geq 2, \tag{3}$$

where for $k \geq 2$,

$$C_k = \frac{2}{\sqrt{k}} \left(\frac{\pi}{2}\right)^{\frac{k-1}{2}}, \quad \alpha_k = \frac{k-1}{2k}, \quad g_k(x) \to 1 \ \text{as} \ x \to \infty, \ \text{and}$$

$$g_k(x) \leq M_k + \frac{N_k}{x^{\beta_k}}, \ \text{with} \ \beta_k \leq \alpha_k. \tag{4}$$

Proof. The case $k = 2$ is Lemma 1 except the bound on $g_2(x)$. From that lemma it follows that,

$$
\begin{aligned}
g_2(x) &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} \left(\frac{1 + z/2x^{1/2}}{\sqrt{1 + z/4x^{1/2}}} \right) dz \\
&\leq \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-z}}{\sqrt{z}} 2 \left(1 + \frac{\sqrt{z}}{4x^{1/4}} \right) dz \\
&= 2 + \frac{1}{2\sqrt{\pi} x^{1/4}} = M_2 + \frac{N_2}{x^{\beta_2}}
\end{aligned}
$$

so that $M_2 = 2$, $N_2 = \frac{1}{2\sqrt{\pi}}$ and $\beta_2 = \frac{1}{4} = \alpha_2$. Hence the result holds for $k = 2$.
Now suppose it holds for some $k > 2$. We then show that it holds also for $k + 1$. Observe that,

$$
\begin{aligned}
H_{k+1}(x) &= \int_0^\infty e^{-y} H_k(\frac{x}{y}) dy \\
&= C_k \int_0^\infty e^{-y} (\frac{x}{y})^{\alpha_k} e^{-k(\frac{x}{y})^{1/k}} g_k(\frac{x}{y}) dy \\
&= xkC_k \int_0^\infty e^{-(kt + \frac{x}{t^k})} t^{k\alpha_k - k - 1} g_k(t^k) dt, \quad \text{(substituting } x/y = t^k \text{ in the integral)} \\
&= xkC_k \int_0^{x^{1/k+1}} e^{-(kt + \frac{x}{t^k})} t^{k\alpha_k - k - 1} g_k(t^k) dt + xkC_k \int_{x^{1/k+1}}^\infty e^{-(kt + \frac{x}{t^k})} t^{k\alpha_k - k - 1} g_k(t^k) dt.
\end{aligned}
$$

Now let $A(t) = \frac{x}{t^k} + tk$. Then

$$
\begin{aligned}
A'(t) = \frac{-kx}{t^{k+1}} + k &> 0 \;\text{ if }\; t > x^{1/k+1} \\
&< 0 \;\text{ if }\; t < x^{1/k+1}.
\end{aligned}
$$

So, again as in Lemma 1 consider the substitution $A(t) = s$, monotone in the above two regions. Let $t = f_1(s)$ and $t = f_2(s)$ be solution to $A(t) = s$ in the two regions so that

$$
f_1(s) < x^{1/k+1} < f_2(s).
$$

Then

$$
\begin{aligned}
H_{k+1}(x) &= kC_k x \left[\int_{(k+1)x^{1/k+1}}^\infty e^{-s} \frac{f_1(s)^{k\alpha_k - k - 1}}{k[\frac{x}{f_1(s)^{k+1}} - 1]} g_k(f_2(s)^k) ds + \int_{(k+1)x^{1/k+1}}^\infty e^{-s} \frac{f_2(s)^{k\alpha_k - k - 1}}{k[1 - \frac{x}{f_2(s)^{k+1}}]} g_k(f_2(s)^k) ds \right] \\
&= H_{k+1,1}(x) + H_{k+1,2}(x).
\end{aligned}
$$

Now we use the substitution $z = s - (k + 1)x^{1/k+1}$ in both integrals. For simplicity write $f_i(z + (k + 1)x^{1/k+1}) = f_i,\ i = 1, 2$. Then,

$$
H_{k+1,1}(x) = C_k x e^{-(k+1)x^{1/k+1}} \int_0^\infty e^{-z} \frac{f_1^{k\alpha_k - k - 1}}{[\frac{x}{f_1^{k+1}} - 1]} g_k(f_1^k) dz.
$$

$$
H_{k+1,2}(x) = C_k x e^{-(k+1)x^{1/k+1}} \int_0^\infty e^{-z} \frac{f_2^{k\alpha_k - k - 1}}{[1 - \frac{x}{f_2^{k+1}}]} g_k(f_2^k) dz.
$$

Combining all transformations we have

$$z + (k+1)x^{1/k+1} = \frac{x}{t^k} + kt \;\text{ or, }\; x + kt^{k+1} - t^k[z + (k+1)x^{1/k+1}] = 0. \tag{5}$$

We now deal with the two integrals.

The first integral $H_{k+1,1}$. Let

$$\frac{f_1}{x^{1/k+1}} = 1 - \eta(z,x) = 1 - \eta,\; 0 < \eta < 1. \tag{6}$$

Now write

$$H_{k+1,1}(x) = C_k x e^{-(k+1)x^{1/k+1}} \frac{1}{k+1} x^{\frac{k\alpha_k - k - 1}{k+1}} x^{\frac{1}{2(k+1)}} \left(\frac{k(k+1)}{2}\right)^{1/2} \sqrt{\pi}\, g_{k+1,1}(x)$$

where

$$\begin{aligned} g_{k+1,1}(x) &= x^{-\frac{1}{2(k+1)}} \int_0^\infty \frac{k+1}{(k(k+1)/2)^{\frac{1}{2}}} \frac{e^{-z}}{\sqrt{\pi}} \frac{(1-\eta)^{k\alpha_k - k - 1}}{(1-(1-\eta)^{k+1})} (1-\eta)^{k+1} g_k(f_1^k) dz \quad (7)\\ &= \int_0^\infty \frac{e^{-z}}{\sqrt{\pi}} I(x,z) dz \;\; (say). \quad (8)\end{aligned}$$

Our aim is to show that as $x \to \infty$,

(i) $I(x,z) \to z^{-1/2}$.

(ii) The bound in (4) holds, and hence DCT applies.

This will imply that

$$\lim_{x\to\infty} g_{k+1,1}(x) = \int_0^\infty \frac{e^{-z}}{\sqrt{\pi}} z^{-1/2} dz = 1.$$

To prove (i), note that,

$$x + kf_1^{k+1} - f_1^k[z + (k+1)x^{1/k+1}] = 0 \Rightarrow 1 + k\left(\frac{f_1}{x^{1/k+1}}\right)^{k+1} - \left(\frac{f_1}{x^{1/k+1}}\right)^k \left[\frac{z}{x^{1/k+1}} + k + 1\right] = 0. \tag{9}$$

Therefore, using (9)

$$1 + k(1-\eta)^{k+1} - (1+k)(1-\eta)^k = (1-\eta)^k \frac{z}{x^{1/k+1}}. \tag{10}$$

Fix z and let $x \to \infty$. Then right side of (10) goes to 0 and hence the left side also goes to 0. Then it is easy to conclude that $\eta \to 0$. Using this and elementary expansions, it then easily follows that

$$\frac{\eta}{(\frac{z}{x^{1/k+1}})^{1/2}} \to \left(\frac{2}{k(k+1)}\right)^{1/2} \;\text{ as }\; x \to \infty. \tag{11}$$

Now since z is fixed and $\eta \to 0$, it easily follows that,

$$x^{1/2(k+1)}\left(\frac{1}{(1-\eta)^{k+1}} - 1\right) = (k+1)x^{1/2(k+1)}\eta + o(\eta^2). \tag{12}$$

Using (11) and (12) we have

$$x^{1/2(k+1)}(\frac{1}{(1-\eta)^{k+1}}-1) \to (k+1)z^{1/2}\left(\frac{2}{k(k+1)}\right)^{1/2}$$

and since $g_k(x) \to 1$, (i) follows. To verify that the conditions for applying DCT holds, we first derive a lower bound for the denominator in the integrand. Now,

$$\begin{aligned}
\text{Equation (10)} \quad &\Rightarrow \quad 1-(1-\eta)^k[1+k\eta+\frac{z}{x^{1/k+1}}] = 0 \\
&\Rightarrow \quad (1-\eta)^{k+1}[1+k\eta+\frac{z}{x^{1/k+1}}] = 1-\eta \\
&\Rightarrow \quad 1-(1-\eta)^{k+1} = (1-\eta)^{k+1}[k\eta+\frac{z}{x^{1/k+1}}]+\eta \\
&\Rightarrow \quad \frac{1}{1-(1-\eta)^{k+1}} \leq \frac{1}{\eta}.
\end{aligned}$$

Now we consider two cases:

Case 1. $\eta < 1-\epsilon_0$ for some $\epsilon_0 > 0$. Note that for some large L

$$1+k(1-\eta)^{k+1}-(k+1)(1-\eta)^k < L\eta^2.$$

Since the first two terms vanish after expansion,

$$\begin{aligned}
&(1-\eta)^k\frac{z}{x^{1/k+1}} \leq L\eta^2 \\
\Rightarrow \quad &\eta^2 \geq \frac{1}{L}(1-\eta)^k\frac{z}{x^{1/k+1}} \geq \frac{1}{L}\epsilon_0^k\frac{z}{x^{1/k+1}} \\
\Rightarrow \quad &\eta \geq \left(\frac{\epsilon_0}{L}\right)^{1/2}\frac{z^{1/2}}{x^{1/2(k+1)}}.
\end{aligned}$$

Hence,

$$\frac{1}{1-(1-\eta)^{k+1}} \leq \left(\frac{L}{\epsilon_0}\right)^{1/2} x^{1/2(k+1)}z^{-1/2}. \tag{13}$$

Case 2. $\eta > 1-\epsilon_0$. Then $\frac{1}{\eta} < \frac{1}{(1-\epsilon_0)}$. So for large x,

$$g_k(f_1^k) \leq \epsilon_1, \text{ say.}$$

Hence we have by (13), lower bound in Case 2 and induction hypothesis,

$$\begin{aligned}
e^{-z}I(x,z) \quad &\leq \quad e^{-z}\frac{(1-\eta)^{k\alpha_k}}{\eta}g_k((1-\eta)^k x^{k/k+1})x^{-\frac{1}{2(k+1)}} \\
&\leq \quad e^{-z}\frac{(1-\eta)^{k\alpha_k}}{\eta}x^{-\frac{1}{2(k+1)}}\Big[M_k+\frac{N_k}{(1-\eta)^{k\beta_k}x^{\frac{k\beta_k}{k+1}}}\Big] \\
&= \quad M_k e^{-z}\frac{(1-\eta)^{k\alpha_k}}{\eta}x^{-\frac{1}{2(k+1)}} + N_k 2e^{-z}\frac{(1-\eta)^{k(\alpha_k-\beta_k)}}{\eta}x^{-\left(\frac{1}{2(k+1)}+\frac{k\beta_k}{k+1}\right)} \\
&\leq \quad \tilde{M}_k e^{-z}z^{-1/2}+\tilde{N}_k e^{-z}\frac{z^{-1/2}}{x^{\frac{k\beta_k}{k+1}}}, \text{ say.}
\end{aligned}$$

It now easily follows that as $x \to \infty$, DCT applies. Further,

$$g_{k+1,1}(x) \le \tilde{M}_k + \frac{\tilde{N}_k}{x^{\frac{k\beta_k}{k+1}}}. \tag{14}$$

The second integral: The analysis for this integral is similar. Let

$$\frac{f_2}{x^{1/k+1}} = 1 + \delta(z,x) = 1 + \delta, \text{ say.} \tag{15}$$

Write

$$\begin{aligned} H_{k+1,2}(x) &= xC_k e^{-(k+1)x^{1/k+1}} \int_0^\infty e^{-z} \left(\frac{f_2}{x^{1/k+1}}\right)^{k\alpha_k - k - 1} \frac{g_k(f_2^k)}{(1 - \frac{x}{f_2^{k+1}})} x^{\frac{k\alpha_k - k - 1}{k+1}} dz \\ &= \left[C_k e^{-(k+1)x^{1/k+1}} \frac{x^{\frac{k\alpha_k}{k+1}}}{k+1} \left(\frac{k(k+1)}{2}\right)^{1/2} \sqrt{\pi} x^{\frac{1}{2k+1}}\right] g_{k+1,2}(x) \end{aligned}$$

where

$$g_{k+1,2}(x) = \int_0^\infty \frac{x^{-\frac{1}{2k+1}}(k+1)}{\left(\frac{k(k+1)}{2}\right)^{1/2} \sqrt{\pi}} e^{-z} \frac{(1+\delta)^{k\alpha_k - k - 1}}{(1-(1+\delta)^{-(k+1)}} g_k(f_2^k) dz.$$

Then equations (5) and (15) imply that,

$$1 + k(1+\delta)^{k+1} - (1+\delta)^k (k+1) = (1+\delta)^k \frac{z}{x^{k+1}}.$$

Expanding the terms in left hand side and simplifying,

$$\delta^2 \frac{k(k+1)}{2} + \sum_{j=3}^{k+1} \frac{(k+1)!(j-1)}{j!(k-j+1)!} \delta^j = (1+\delta)^k \frac{z}{x^{k+1}}.$$

When z is fixed, first observe that, δ is bounded as a function of x. Hence as $x \to \infty$, $\delta \to 0$. Therefore it is easy to see that,

$$\left(\frac{2}{k(k+1)}\right)^{1/2} \left(\frac{z}{x^{1/k+1}}\right)^{1/2} = \delta(1 + O(\delta)).$$

So for fixed z, as $x \to \infty$,

$$x^{1/2(k+1)}[1 - (1+\delta)^{-(k+1)}] \longrightarrow (k+1)\left(\frac{2}{k(k+1)}\right) z^{1/2}.$$

Now using calculations similar to those for $g_{k+1,1}$, the integrand in $g_{k+1,2}$ is bounded above by

$$\epsilon \frac{e^{-z} (\frac{z+k+1}{k})^{k\alpha_k}}{\sqrt{m}(k+1) z^{1/2}}, \text{for some } \epsilon > 0 \text{ and } m > 0$$

and which is integrable. Now using DCT, it follows that $g_{k+1,2}(x) \to 1$ as $x \to \infty$. So we have

$$H_{k+1,2}(x) \approx \frac{C_k}{k+1} \left(\frac{k(k+1)}{2}\right)^{1/2} \sqrt{\pi} x^{(\frac{k\alpha_k}{k+1} + \frac{1}{2(k+1)})} e^{-(k+1)x^{1/k+1}}$$

$$= \frac{C_{k+1}}{2} x^{\alpha_{k+1}} e^{-(k+1)x^{1/k+1}} g_{k+1,2}(x)$$

where

$$C_{k+1} = 2\left[\sqrt{\pi} C_k \left(\frac{k}{2(k+1)}\right)^{1/2}\right] \text{ and } \alpha_{k+1} = \frac{k\alpha_k}{k+1} + \frac{1}{2(k+1)}. \tag{16}$$

The above recursion can be solved easily.

If we take $\beta_{k+1} = \frac{k\beta_k}{k+1}$, then $\beta_{k+1} \leq \alpha_{k+1}$. Further, by applying arguments similar to those given for $g_{k+1,1}$, $g_{k+1,2}$ also obeys the same bound as obtained for $g_{k+1,1}$, with possibly constants different from $\tilde{M}_k$ and $\tilde{N}_k$. So we have

$$H_{k+1}(x) = C_{k+1} x^{\alpha_{k+1}} e^{-(k+1)x^{1/k+1}} g_{k+1}(x)$$

where

$$g_{k+1}(x) = \frac{1}{2}[g_{k+1,1}(x) + g_{k+1,2}(x)] \to 1, \text{ as } x \to \infty,$$

satisfying the required upper bound. Hence the result is true for $k+1$, proving the theorem. □

2.3 Tail of product and extreme values

Definition 1. *A probability distribution is said to be Gumbel with parameter $\theta > 0$ if its cumulative distribution function is given by*

$$\Lambda_\theta(x) = \exp\{-\theta \exp(-x)\}, \quad x \in \mathbb{R}.$$

Λ_1 is known as the (standard) Gumbel distribution.

Theorem 2. *Let $\{X_n\}$ be a sequence of i.i.d. nonnegative random variables with distribution F and let $F^{(n)} = \max_{1\leq i\leq n} X_i$. If $1 - F(x) \sim Cx^b e^{-ax^2}$ as $x \to \infty$, then*

$$\frac{F^{(n)} - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda_1,$$

where

$$c_n = \frac{1}{2a^{1/2}(\log n)^{1/2}} \text{ and } d_n = \frac{\log c - \frac{b}{2}\log a}{2a^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{a}\right)^{1/2}\left[1 + \frac{b \log\log n}{4\log n}\right].$$

Proof. Let $\overline{F} = 1 - F$. Then

$$\overline{F}(x) = \theta(x)\overline{F}_\#(x) \text{ where } \theta(x) \to \theta = Ce^{-a} \text{ and } \overline{F}_\#(x) = x^b \exp(-a(x^2-1)). \tag{17}$$

By invoking Proposition 1.1 given in Resnick (1987)(29) , it is now enough to show that, there exists some x_0 and a function f such that $f(y) > 0$ for $y > x_0$ and such that f has an absolute continuous density with $f'(x) \to 0$ as $x \to \infty$ so that

$$1 - F_\#(x) = \exp\Big(-\int_{x_0}^{x}(1/f(y))dy\Big), \quad x > x_0. \tag{18}$$

Further, a choice for the normalizing constants c_n and d_n is then given by

$$d_n^* = \left(1/(\overline{F}_\#)\right)^{-1}(n), \quad c_n^* = f(d_n^*). \tag{19}$$

Comparing the two representations of $\overline{F}_{\#}$ given in (17) and (18) implies that we may choose

$$f(x) = \frac{x}{2ax^2 - b} \sim \frac{1}{2ax} \text{ as } x \to \infty.$$

Hence we have (noting that $d_n^* \to \infty$),

$$c_n^* = f(d_n{}^*) \sim \frac{1}{2ad_n{}^*}.$$

On the other hand, since $\overline{F}_{\#}(d_n^*) = \frac{1}{n}$, we have

$$(d_n{}^*)^b \exp(-a((d_n^*)^2 - 1)) = \frac{1}{n}.$$

Taking logarithms on both sides we have

$$ad_n^{*2} - b\log d_n^* - a = \log n. \tag{20}$$

Since $d_n^* \to \infty$, $d_n^* \sim \left(\frac{\log n}{a}\right)^{1/2}$. Let $d_n^* = \left(\frac{\log n}{a}\right)^{1/2}(1+\delta_n)$. Using this in (20) we get

$$\delta_n = \frac{\frac{b}{2}\log\log n + \epsilon_n}{2\log n} + \mathrm{O}\left(\frac{(\log\log n)^2}{(\log n)^2}\right),$$

where $\epsilon_n = -b\log(1+\delta_n) - \frac{b}{2}\log a + a$. So we get

$$\begin{aligned} d_n^* &= \left(\frac{\log n}{a}\right)^{1/2}(1+\delta_n) \\ &= \left(\frac{\log n}{a}\right)^{1/2}\left[1 + \frac{b}{4}\frac{\log\log n}{\log n} + \frac{a - \frac{b}{2}\log a - b\log(1+\delta_n)}{2\log n}\right] + \mathrm{O}\left(\frac{(\log\log n)^2}{(\log n)^{3/2}}\right). \end{aligned}$$

Neglecting the lower order terms and denoting

$$\hat{d}_n = \left(\frac{\log n}{a}\right)^{1/2}\left[1 + \frac{b}{4}\frac{\log\log n}{\log n} + \frac{a - \frac{b}{2}\log a}{2\log n}\right] \text{ and } \hat{c}_n = \frac{1}{2a^{1/2}(\log n)^{1/2}}$$

we have

$$\frac{F^{(n)} - \hat{d}_n}{\hat{c}_n} \xrightarrow{\mathcal{D}} \Lambda_{Ce^{-a}}.$$

Now letting $c_n = \hat{c}_n$ and $d_n = c_n\log(Ce^{-a}) + \hat{d}_n$ and using convergence of types result, we have

$$\frac{F^{(n)} - d_n}{c_n} \xrightarrow{\mathcal{D}} \Lambda_1.$$

□

The following corollary and lemma follow immediately using Theorem 1.

Corollary 1. *Let $\{X_n\}$ be a sequence of i.i.d. random variables where $X_i \stackrel{\mathcal{D}}{=} (E_1E_2\ldots E_k)^{1/2k}$ and $\{E_i\}_{1\le i\le k}$ are i.i.d. $Exp(1)$ random variables. Then*

$$\frac{\max_{1\le i\le n} X_i - d_n}{c_n} \stackrel{\mathcal{D}}{\to} \Lambda_1,$$

where

$$c_n = \frac{1}{2k^{1/2}(\log n)^{1/2}},\ d_n = \frac{\log C_k - \frac{k-1}{2}\log k}{2k^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{2k}\right)^{1/2}\left[1 + \frac{(k-1)\log\log n}{4\log n}\right],\ C_k = \frac{2}{\sqrt{k}}\left(\frac{\pi}{2}\right)^{\frac{k-1}{2}}.$$

Lemma 2. *Let $\{E_i\}$, c_n and d_n be as in Corollary 1. Let $\sigma_n^2 = n^{-c}$, $c > 0$. Then there exists some positive constant $K = K(x)$, such that for all large n we have*

$$\mathrm{P}\Big((E_1E_2\ldots E_k)^{1/2k} > (1+\sigma_n^2)^{-1/2}(c_nx + d_n)\Big) \le \frac{K}{n},\quad x \in \mathbb{R}.$$

2.4 Tail of product and some related problems

If maximum of $\{X_i\}$ i.i.d. with distribution function F converges to G after appropriate scaling and centering, then F and X with distribution function F, are said to be belong to the max domain of attraction of G. It is very well known that there are three types of G namely, Frechet, Weibull and Gumbel (see Resnick (1985)(29) for details).

It is also well known that X is in Frechet domain of attraction if and only if it has regularly varying tail. From Breiman (1965)(9) and Embrechets and Goldie (1980)(15), it is known that product of independent regularly varying random variables is again regularly varying and hence again in the Frechet domain of attraction. The results on Weibull domain can be sorted out from the results on Frechet domain.

A similar question arises when one considers the Gumbel distribution. As pointed out earlier, the tail behaviour of products does not seem to be known in any generality. Hence it is not known when a product belongs to the max domain of attraction of the Gumbel distribution. From the results proved so far, the product of finitely many exponentials belongs to the max domain of the Gumbel. The more general questions shall be addressed in a separate article.

3 Spectral radius of random k circulant matrices

3.1 Spectral norm and spectral radius in random matrices

For any matrix B, its *spectral radius* $\mathtt{sp}(B)$ is defined as

$$\mathtt{sp}(B) := \max\Big\{|\mu| : \mu \text{ is an eigenvalue of } B\Big\},$$

where $|z|$ denotes the modulus of $z \in \mathbb{C}$.

A related quantity is the *spectral norm*. For any matrix A with complex entries, its spectral norm, $||A||$ is the square root of the largest eigenvalue of the positive semidefinite matrix A^*A:

$$||A|| = \sqrt{\lambda_{max}(A^*A)}$$

where A^* denotes the conjugate transpose of A. Therefore if A is an $n\times n$ real symmetric matrix or A is a normal matrix, with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$||A|| = \mathtt{sp}(A).$$

The spectral radius and spectral norm have been important objects of study in random matrix theory. Bai and Yin (1988)(3) and Silverstein (1996)(30) study the spectral norm of the Wigner matrices. The same was done for sample covariance matrices by Bai, Yin and Krishnaiah (1988)(39). For work on spectral norm of (symmetric) Toeplitz matrices, see Meckes(2007)(25), Adamczak (2008)(1) and Bose and Sen (2007)(8). Results on spectral norm and spectral radius for circulant matrices are described in Section 3.3.

3.2 The random k circulant matrix

Suppose $\underline{a} = \{a_l\}_{l\geq 0}$ is a sequence of real numbers. For positive integers k and n, the k-circulant matrix with *input* sequence $\{a_l\}$ is defined as

$$A_{k,n}(\underline{a}) = \begin{bmatrix} a_0 & a_1 & \dots & a_{n-1} \\ a_{n-k} & a_{n-k+1} & \dots & a_{n-k-1} \\ a_{n-2k} & a_{n-2k+1} & \dots & a_{n-2k-1} \\ & & \vdots & \end{bmatrix}_{n\times n}.$$

We write $A_{k,n}(\underline{a}) = A_{k,n}$. The subscripts appearing in the matrix entries above are calculated modulo n and the convention is to start the row and column indices from zero. Thus, the 0-th row of $A_{k,n}(\underline{a})$ is $(a_0, a_1, a_2, \ldots, a_{n-1})$. For $0 \leq j < n-1$, the $(j+1)$-th row of $A_{k,n}$ is a right-circular shift of the j-th row by k positions (equivalently, $k \bmod n$ positions). Note that $A_{1,n}$ is the well-known circulant matrix (C_n) and $A_{n-1,n}$ is the *reverse circulant matrix* (RC_n). Without loss of generality, k may always be reduced modulo n.

3.3 Spectral norm and spectral radius for random circulant type matrices

The study of spectral radius of k circulant matrices was initiated in Bose, Mitra and Sen (2008)(7) for the case $n = k^2 + 1$. The spectral norm of different circulant type matrices was considered in Bose, Hazra and Saha (2009)(5), Meckes (2008)(26) and Bryc and Sethuraman (2009)(10). The spectral norm with heavy tailed entries for the circulant and Toeplitz matrices was studied in Bose, Hazra and Saha (2009)(4).

To put our main result (Theorem 5) in perspective, we now briefly describe the known results on various circulant type matrices when the entries do not have heavy tails. The following result can be derived following the argument given for symmetric Toeplitz matrix in Bose and Sen (2007)(8).

Theorem 3. *Consider RC_n with the input sequence $\{x_i\}$ which is i.i.d. with $E(x_0) = \mu$ and $Var(x_0) = 1$. Let $RC_n^0 = RC_n - \mu n u_n u_n^T$ with $u_n = (1, \cdots, 1)^T$. If $\mu > 0$, then*

$$\frac{\|RC_n\|}{n} \to \mu \text{ almost surely and } \|\frac{RC_n^0}{\|RC_n\|}\| \to 0 \text{ almost surely.}$$

Similar results hold for C_n also.

Theorem 4. *(Bose, Hazra and Saha (2009)(5)) Consider (RC_n) and (C_n) with the input sequence $\{x_i\}$ which is i.i.d. with mean μ and $E|x_i|^{2+\delta} < \infty$ for some $\delta > 0$.*

(i) If $\mu \neq 0$ then,

$$\frac{\|RC_n\| - |\mu| n}{\sqrt{n}} \xrightarrow{\mathcal{D}} N(0,1).$$

(ii) If $\mu = 0$ then,

$$\frac{\|\frac{1}{\sqrt{n}}RC_n\| - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1$$

where

$$q = q(n) = \lfloor \frac{n-1}{2} \rfloor, \;\; d_q = \sqrt{\ln q} \;\; and \;\; c_q = \frac{1}{2\sqrt{\ln q}} \, .$$

The above conclusions continue to hold for C_n also.

We now state the following significant generalisation of a result of Bose, Mitra and Sen (2008)(7) who proved the result for the case $g = 2$. See Sections 3.9 and 3.10 for extensions to the case $sn = k^g + 1$ and to the case of dependent entries.

Theorem 5. *Suppose $\{a_l\}_{l\geq 0}$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathrm{E}\,|a_l|^\gamma < \infty$ for some $\gamma > 2$. If $n = k^g + 1$ for some fixed positive integer g, then as $n \to \infty$,*

$$\frac{sp(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1$$

where $q = q(n) = \frac{n}{2g}$ and the normalizing constants c_n and d_n can be taken as follows

$$c_n = \frac{1}{2g^{1/2}(\log n)^{1/2}} \;\; and \;\; d_n = \frac{\log C_g - \frac{g-1}{2}\log g}{2g^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{2g}\right)^{1/2}\left[1 + \frac{(g-1)\log\log n}{4\log n}\right], \; C_g = \frac{2}{\sqrt{g}}\left(\frac{\pi}{2}\right)^{\frac{g-1}{2}} .$$

The proof of the above theorem is developed in the next sections. It involves some intricate study of the structure of the eigenvalues, the behaviour of $H_n(\cdot)$ that we have developed in Section 2.2, and some normal approximation methods.

3.4 Description of eigenvalues of k circulant

The formula solution by Zhou (1996)(40), given below in Theorem 6, for the eigenvalues of a k-circulant is our starting point. A proof is also provided in Bose, Mitra and Sen (2008)(7). Let

$$\omega = \omega_n := \cos(2\pi/n) + i\sin(2\pi/n), \;\; i^2 = -1 \;\; \text{and} \;\; \lambda_t = \sum_{l=0}^{n-1} a_l \omega^{tl}, \;\; 0 \leq t < n. \tag{21}$$

Note that $\{\lambda_t, 0 \leq t < n\}$ are eigenvalues of the usual circulant matrix $A_{1,n}$. Let $p_1 < p_2 < \cdots < p_c$ be all the common prime factors of n and k. Then we may write,

$$n = n' \prod_{q=1}^{c} p_q^{\beta_q} \;\; \text{and} \;\; k = k' \prod_{q=1}^{c} p_q^{\alpha_q}. \tag{22}$$

Here α_q, $\beta_q \geq 1$ and n', k', p_q are pairwise relatively prime. For any positive integer m, let

$$\mathbb{Z}_m = \{0, 1, 2, \ldots, m-1\}.$$

We introduce the following family of sets

$$S(x) := \{xk^b \mod n' : b \geq 0\}, \quad x \in \mathbb{Z}_{n'}. \tag{23}$$

Note that $x \in S(x)$ for every x. Suppose $S(x) \cap S(y) \neq \emptyset$. Then, $xk^{b_1} = yk^{b_2} \mod n'$ for some integers $b_1, b_2 \geq 1$. Multiplying both sides by $k^{g_x - b_1}$ we see that, $x \in S(y)$ so that, $S(x) \subseteq S(y)$. Hence, reversing the roles, $S(x) = S(y)$. Thus, the distinct sets in $\{S(x)\}_{x \in \mathbb{Z}_{n'}}$ forms a partition, called the *eigenvalue partition*, of $\mathbb{Z}_{n'}$. Denote the partitioning sets and their sizes by

$$\mathcal{P}_0 = \{0\}, \mathcal{P}_1, \ldots, \mathcal{P}_{\ell-1} \text{ and } k_j = \#\mathcal{P}_j,\ 0 \leq j < \ell. \tag{24}$$

Define

$$\Pi_j := \prod_{t \in \mathcal{P}_j} \lambda_{tn/n'}, \quad j = 0, 1, \ldots, \ell - 1. \tag{25}$$

Theorem 6 (Zhou (1996)(40))**.** *The characteristic polynomial of $A_{k,n}$ is given by*

$$\chi(A_{k,n})(\lambda) = \lambda^{n-n'} \prod_{j=0}^{\ell-1} \left(\lambda^{k_j} - \Pi_j\right). \tag{26}$$

3.5 Additional description of eigenvalues when $n = k^g + 1$

Let $g_x = \#S(x)$. We call g_x the *order* of x. Note that $g_0 = 1$. It is easy to see that

$$g_x = \min\{b > 0 : b \text{ is an integer and } xk^b = x \mod n'\} \tag{27}$$

and

$$S(x) = \{xk^b \mod n' : 0 \leq b < g_x\}.$$

Define

$$J_k := \{\mathcal{P}_i : \ \#\mathcal{P}_i = k\},\ n_k := \#J_k,\ \ X(k) := \{x : \ x \in \mathbb{Z}_n \text{ and } x \text{ has order } k\},$$

$$v_{k,n} = \frac{1}{n}\#\{x : \ x \in \mathbb{Z}_n \text{ and } g_x < g_1\}. \tag{28}$$

Lemma 3. *The eigenvalues $\{\eta_i\}$ of the k-circulant with $n = k^g + 1$, $g \geq 2$, satisfy the following:*

(a) $\eta_0 = \sum_{t=0}^{n-1} a_t$, is always an eigenvalue and if n is even, then $\eta_{\frac{n}{2}} = \sum_{t=0}^{n-1}(-1)^t a_t$, is also an eigenvalue and both have multiplicity one.

(b) For $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = g_1$ or $\frac{g_1}{b}$ for some $b \geq 2$ and $\frac{g_1}{2b}$ is an integer.

(c) For all large n, $g_1 = 2g$. Hence from (b), for $x \in \mathbb{Z}_n \setminus \{0, \frac{n}{2}\}$, $g_x = 2g$ or $\frac{2g}{b}$. The total number of eigenvalues corresponding to J_{2g} is

$$2g \times \#J_{2g} = \#X(2g) \sim n.$$

(d) $X(\frac{2g}{b}) = \emptyset$ for $2 \leq b < g$, b even. If g is even then $X(\frac{2g}{g}) = X(2)$ is either empty or contains exactly two elements with eigenvalues

$$\eta_l = |\lambda_l|,\ \eta_{n-l} = -|\lambda_l|,\ \textit{for some } 1 \leq l \leq \frac{n}{2}.$$

(e) Suppose b is odd, $3 \leq b \leq g$ and $\frac{g}{b}$ is an integer. For each $\mathcal{P}_j \in J_{\frac{2g}{b}}$ there are $\frac{2g}{b}$ eigenvalues given by the $\frac{2g}{b}$-th roots of Π_j. Total number of eigenvalues corresponding to the set $J_{\frac{2g}{b}}$ is

$$\frac{2g}{b} \times \#J_{\frac{2g}{b}} = \#X(\frac{2g}{b}) \sim (k^{g/b} + 1)(1 + n^{-a}) \ \textit{for some } \ a > 0.$$

There are no other eigenvalues.

Proof. Since $n = k^g + 1$, n and k are relatively prime and hence $n' = n$.
(a) $\mathcal{P}_0 = S(0) = \{0\}$ and the corresponding eigenvalue is $\eta_0 = \sum_{t=0}^{n-1} a_t$ with multiplicity one. Similarly if n is even then k is odd and hence $S(n/2) = \{\frac{n}{2}\}$, and the corresponding eigenvalue is $\eta_{\frac{n}{2}} = \sum_{t=0}^{n-1}(-1)^t a_t$ of multiplicity one.

(b) From (27) it is easy to see that g_x divides g_1 and hence $g_x = g_1$ or $g_x = \frac{g_1}{b}$ for some $b \geq 2$. Also for every integer $t \geq 0$, $tk^g = (-1+n)t = -t \mod n$. Hence λ_t and λ_{n-t} belong to same partition block $S(t) = S(n-t)$. Thus each $S(t)$ contains even number of elements, except for $t = 0, \frac{n}{2}$. Hence $\frac{g_1}{b}$ must be even, that is, $\frac{g_1}{2b}$ must be an integer.

(c) From Lemma 5(i) of Bose, Mitra and Sen (2008)(7), $g_1 = 2g$ for all but finitely many n and $v_{k,n} \to 0$ as $n \to \infty$. For each $\mathcal{P}_j \in J_{2g}$ we have $2g$ many eigenvalues and which are $2g$-th roots of Π_j. Now the result follows from the fact that

$$n = 2g\#J_{2g} + n\upsilon_{k,n}.$$

(d) Suppos $b = 2$ and $x \in X(\frac{g_1}{2}) = X(\frac{2g}{2})$. Then $xk^{\frac{g_1}{2}} = xk^g = x \mod n$. But $k^g = -1 \mod n$ and so, $xk^g = -x \mod n$. Therefore $2x = 0 \mod n$ and x can be either 0 or $n/2$. But we have already seen in part (a) that $g_0 = g_{n/2} = 1$. Hence $X(\frac{2g}{2}) = \emptyset$.

Now suppose $b > 2$, even. From Lemma 3(ii) Bose, Mitra and Sen (2008)(7), $\#X(\frac{2g}{b}) \leq \gcd(k^{2g/b} - 1, k^g + 1)$ for $b \geq 3$. Now observe that for b even,

$$\gcd(k^{2g/b} - 1, k^g + 1) = \begin{cases} 1 & \text{if } k \text{ even,} \\ 2 & \text{if } k \text{ odd.} \end{cases}$$

So we have $\#X(\frac{2g}{b}) \leq 2$ for $b > 2$ and b even.

Suppose if possible, there exist $x \in \mathbb{Z}_n$ such that $g_x = \frac{2g}{b}$. Then $\#S(x) = \frac{2g}{b}$ and for all $y \in S(x)$, $g_y = \frac{2g}{b}$. Hence

$$\#\left\{y: \ g_y = \frac{2g}{b}\right\} \geq \frac{2g}{b} > 2 \ \text{ for } \ g > b > 2, \ b \textit{ even.}$$

This contradicts the fact that $\#X(\frac{2g}{b}) \leq 2$ for $g > b > 2$, b even. Hence $X(\frac{2g}{b}) = \emptyset$ for b even and $g > b > 2$.

If $b = g$ and it is even, then from previous discussion $\#X(\frac{2g}{g}) = 0$ or 2. In the later case there are exactly two elements in $\mathbb{Z}_n$ whose order is 2 and there will be only one partitioning set containing them. So corresponding eigenvalues will be

$$\eta_l = |\lambda_l|, \ \eta_{n-l} = -|\lambda_l|, \ \text{ for some } 1 \leq l \leq \frac{n}{2}.$$

(e) We first show that for b odd,

$$(k^{g/b} + 1) - \sum_{\substack{b_i > b, \ b_i \ odd, \\ \frac{g}{b_i} \ integer}} (k^{g/b_i} + 1) \leq \#X(\frac{2g}{b}) \leq k^{g/b} + 1.$$

Note that (e) is a simple consequence of this. Let

$$Z_{n,b} = \left\{x: \ x \in \mathbb{Z}_n \text{ and } xk^{2g/b} = x \mod (k^g + 1)\right\}.$$

Then it is easy to see that

$$X(\frac{2g}{b}) \subseteq Z_{n,b}. \tag{29}$$

Let $x \in Z_{n,b}$ and $\frac{g}{b} = m$. Then

$$\begin{aligned} & k^g + 1 \mid x(k^{2g/b} - 1) \\ \Rightarrow \;& k^{bm} + 1 \mid x(k^{2m} - 1) \\ \Rightarrow \;& k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1 \mid x(k^m - 1). \end{aligned}$$

But $\gcd(k^m - 1, k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1) = 1$, and therefore x is a multiple of $(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)$. Hence

$$\begin{aligned} \#Z_{n,b} &= \left\lfloor \frac{k^{bm} + 1}{(k^{(b-1)m} - k^{(b-2)m} + k^{(b-3)m} - \cdots - k + 1)} \right\rfloor \\ &= k^m + 1 = k^{g/b} + 1 \end{aligned}$$

and combining with (29),

$$\#X(\frac{2g}{b}) \leq \#Z_{n,b} = k^{g/b} + 1.$$

On the other hand, if $x \in Z_{n,b}$ then either $g_x = \frac{2g}{b}$ or $g_x < \frac{2g}{b}$. For the second case $g_x = \frac{2g}{b_i}$ for some $b_i > b$, b_i odd and therefore $x \in Z_{n,b_i}$. Hence

$$\begin{aligned} \#X(\frac{2g}{b}) &\geq \#Z_{n,b} - \sum_{\substack{b_i > b,\ b_i\ odd, \\ \frac{g}{b_i}\ integer}} \#Z_{n,b_i} \\ &\geq (k^{g/b} + 1) - \sum_{\substack{b_i > b,\ b_i\ odd, \\ \frac{g}{b_i}\ integer}} (k^{g/b_i} + 1). \end{aligned}$$

□

3.6 Properties of eigenvalues of Gaussian circulant matrices

Suppose $\{a_l\}_{l \geq 0}$ are independent, mean zero and variance one random variables. Fix n. For $1 \leq t < n$, let us split λ_t into real and complex parts as $\lambda_t = a_{t,n} + ib_{t,n}$, that is,

$$a_{t,n} = \sum_{l=0}^{n-1} a_l \cos\left(\frac{2\pi tl}{n}\right), \quad b_{t,n} = \sum_{l=0}^{n-1} a_l \sin\left(\frac{2\pi tl}{n}\right). \tag{30}$$

For $z \in \mathbb{C}$, $\bar{z}$ denotes its complex conjugate. For all $0 < t, t' < n$, the following identities can easily be verified using the orthogonality relations of sine and cosine functions.

$$\mathrm{E}(a_{t,n} b_{t,n}) = 0, \quad \text{and} \quad \mathrm{E}(a_{t,n}^2) = \mathrm{E}(b_{t,n}^2) = n/2,$$

$$\bar{\lambda}_t = \lambda_{n-t}, \quad \mathrm{E}(\lambda_t \lambda_{t'}) = n\mathbb{I}(t + t' = n), \quad \mathrm{E}(|\lambda_t|^2) = n.$$

The following Lemma is due to Bose, Mitra and Sen (2008)(7).

Lemma 4. *Fix k and n. Suppose that* $\{a_l\}_{0\le l<n}$ *are i.i.d. standard normal random variables.*

(a) For every n, $n^{-1/2}a_{t,n}, n^{-1/2}b_{t,n}$*,* $0 \le t \le n/2$ *are i.i.d. normal with mean zero and variance* $1/2$*. Consequently, any subcollection* $\{\Pi_{j_1}, \Pi_{j_2}, \ldots\}$ *of* $\{\Pi_j\}_{0\le j<\ell}$*, so that no member of the corresponding partition blocks* $\{\mathcal{P}_{j_1}, \mathcal{P}_{j_2}, \ldots\}$ *is a conjugate of any other, are mutually independent.*

(b) Suppose $1 \le j < \ell$ *and* $\mathcal{P}_j = n - \mathcal{P}_j$ *and* $n/2 \notin \mathcal{P}_j$*. Then* $n^{-n_j/2}\Pi_j$ *are distributed as* $(n_j/2)$*-fold product of i.i.d. exponential random variables with mean one.*

3.7 Preparatory lemmas: truncation and normal approximation

3.7.1 Truncation

From Section 3.5, $n = n'$ and $S(t) = S(n-t)$ except for $t = 0,\ n/2$. So for $\mathcal{P}_j \ne S(0),\ S(n/2)$, we can define $\mathcal{A}_j$ such that

$$\mathcal{P}_j = \{x : x \in \mathcal{A}_j \text{ or } n - x \in \mathcal{A}_j\} \ \text{ and } \ \#\mathcal{A}_j = \frac{1}{2}\#\mathcal{P}_j. \tag{31}$$

For any sequence of random variables $b = \{b_l\}_{l\ge 0}$, define

$$\beta_{b,g}(j) = \prod_{t\in\mathcal{A}_j} \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} b_l \omega^{tl} \right|^2, \text{ where } \omega = \exp\left(\frac{2\pi i}{n}\right), \quad 1 \le j \le q. \tag{32}$$

For each $n \ge 1$, define a triangular array of centered random variables $\{\bar{a}_l^{(n)}\}_{0\le l<n}$ by

$$\bar{a}_l = \bar{a}_l^{(n)} = a_l I_{|a_l|\le n^{1/\gamma}} - \mathrm{E}\, a_l I_{|a_l|\le n^{1/\gamma}}.$$

Lemma 5. *Assume* $\mathrm{E}\,|a_l|^\gamma < \infty$ *for some* $\gamma > 2$*. Then, almost surely,*

$$\max_{1\le j\le q} (\beta_{a,g}(j))^{1/2g} - \max_{1\le j\le q} (\beta_{\bar{a},g}(j))^{1/2g} = o(1).$$

Proof. Since $\sum_{l=0}^{n-1} \omega^{tl} = 0$ for $0 < t < n$, it follows that $\beta_{\bar{a},n}(j) = \beta_{\tilde{a},n}(j)$ where

$$\tilde{a}_l = \tilde{a}_l^{(n)} = \bar{a}_l + \mathrm{E}\, a_l I_{|a_l|\le n^{1/\gamma}} = a_l I_{|a_l|\le n^{1/\gamma}}.$$

By Borel-Cantelli lemma, $\sum_{t=0}^{\infty} |a_l| I_{|a_l|>l^{1/\gamma}}$ is finite a.s. and has only finitely many non-zero terms. Thus there exists an integer $N \ge 0$, which may depend on the sample point, such that

$$\sum_{l=m}^{n-1} |\tilde{a}_l^{(n)} - a_l| = \sum_{l=m}^{n-1} |a_l| I_{|a_l|>n^{1/\gamma}} \le \sum_{t=m}^{\infty} |a_l| I_{|a_l|>t^{1/\gamma}} = \sum_{l=m}^{N} |a_l| I_{|a_l|>l^{1/\gamma}}. \tag{33}$$

Consequently, if $m > N$, the left side of (33) is zero. Therefore, the terms of the two sequences $\{a_l\}_{m\le l<n}$ and $\{\tilde{a}_l^{(n)}\}_{m\le l<n}$ are identical almost surely for all sufficiently large n and the assertion follows immediately. □

3.7.2 Normal approximation

For $d \geq 1$, and any distinct integers $i_1, i_2, \ldots, i_d$, from $\left\{1, 2, \ldots, \lceil \frac{n-1}{2} \rceil\right\}$, define

$$v_{2d}(l) = \left(\cos\left(\frac{2\pi i_j l}{n}\right), \sin\left(\frac{2\pi i_j l}{n}\right) : 1 \leq j \leq d\right)^T, \quad l \in \mathbb{Z}_n.$$

Let $\varphi_\Sigma(\cdot)$ denote the density of the $2d$-dimensional Gaussian vector having mean zero and covariance matrix Σ and let I_{2d} be the identity matrix of order $2d$.

Lemma 6 (Davis and Mikosch (1999)(27))**.** *Fix $d \geq 1$, $\gamma > 2$ and let $\tilde{p}_n$ be the density function of*

$$2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_{2d}(l),$$

where $\{N_l\}_{l\geq 0}$ is a sequence of i.i.d. $N(0,1)$ random variables, independent of $\{a_l\}_{l\geq 0}$ and $\sigma_n^2 = \mathrm{Var}(\bar{a}_0) s_n^2$. If $n^{-2c} \ln n \leq s_n^2 \leq 1$ with $c = 1/2 - (1-\delta)/\gamma$ for arbitrarily small $\delta > 0$, then

$$\tilde{p}_n(x) = \varphi_{(1+\sigma_n^2)I_{2d}}(x)(1 + \varepsilon_n) \quad \text{with } \varepsilon_n \to 0$$

holds uniformly for $\|x\|^3 = o_d(n^{1/2-1/\gamma})$, $x \in \mathbb{R}^{2d}$.

Corollary 2. *Let $\gamma > 2$ and $\sigma_n^2 = n^{-c}$ where c is as in Lemma 6. Then for an measurable $B \subseteq \mathbb{R}^{2d}$,*

$$\left| \int_B \tilde{p}_n(x)dx - \int_B \varphi_{(1+\sigma_n^2)I_{2d}}(x)dx \right| \leq \varepsilon_n \int_B \varphi_{(1+\sigma_n^2)I_{2d}}(x)dx + O_d(\exp(-n^\eta)),$$

for some $\eta > 0$ and uniformly over all the d-tuples of distinct integers $1 \leq i_1 < i_2 < \ldots < i_d \leq \lceil \frac{n-1}{2} \rceil$.

3.8 Proof of Theorem 5

Recall that $\{\beta_{a,g}(t)^{1/2g}\}$ are the eigenvalues corresponding to the set of partitions having cardinality $2g$. First we derive the behaviour of the maximum of these eigenvalues. Then we show that the maximum of the remaining eigenvalues is negligible compared to the above.

Lemma 7.

$$\frac{\max_{1\leq t\leq q} \beta_{a,g}(t)^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1$$

where d_q, c_q are as in Corollary 1, $q = q_n = \frac{n}{2g} - k_n$ and $\frac{k_n}{n} \to 0$ as $n \to \infty$. As a consequence,

$$\frac{\max_{1\leq t\leq q} \beta_{a,g}(t)^{1/2g} - d_{n/2g}}{c_{n/2g}} \xrightarrow{\mathcal{D}} \Lambda_1.$$

Proof. First assume that $\{a_l\}_{l\geq 0}$ are i.i.d. standard normal. Let $\{E_j\}_{j\geq 1}$ be i.i.d. standard exponentials. By Lemma 4, it easily follows that

$$\mathrm{P}\Big(\max_{1\leq t\leq q} (\beta_{a,g}(t))^{1/2g} > c_q x + d_q\Big) = \mathrm{P}\Big(\big(E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj}\big)^{1/2g} > c_q x + d_q \text{ for some } 1 \leq j \leq q\Big).$$

The Lemma then follows in this special case from Corollary 1.

For the general case we make use of the two results from Section 3.7. Fix $x \in \mathbb{R}$. For notational convenience, define

$$Q_1^{(n)} := \mathrm{P}\left(\max_{1\le j\le q}(\beta_{\bar{a}+\sigma_n N,g}(j))^{1/2g} > c_q x + d_q\right),$$
$$Q_2^{(n)} := \mathrm{P}\left(\max_{1\le j\le q}(1+\sigma_n^2)\left(E_{g(j-1)+1}E_{g(j-1)+2}\cdots E_{gj}\right)^{1/2g} > c_q x + d_q\right),$$

where $\{N_l\}_{l\ge 0}$ is a sequence of i.i.d. standard normal random variables. Our goal is to approximate $Q_1^{(n)}$ by the simpler quantity $Q_2^{(n)}$. By Bonferroni's inequality, for all $m \ge 1$,

$$\sum_{j=1}^{2m}(-1)^{j-1}S_{j,n} \le Q_1^{(n)} \le \sum_{j=1}^{2m-1}(-1)^{j-1}S_{j,n}, \tag{34}$$

where

$$S_{j,n} = \sum_{1\le t_1<t_2<\ldots<t_j\le q} \mathrm{P}\left((\beta_{\bar{a}+\sigma_n N,g}(t_i))^{1/2g} > c_q x + d_q,\ i=1,\ldots,j\right).$$

Similarly, we have

$$\sum_{j=1}^{2m}(-1)^{j-1}T_{j,n} \le Q_2^{(n)} \le \sum_{j=1}^{2m-1}(-1)^{j-1}T_{j,n}, \tag{35}$$

where

$$T_{j,n} = \sum_{1\le t_1<t_2<\ldots<t_j\le q} \mathrm{P}\left((1+\sigma_n^2)(E_{g(t_i-1)+1}E_{g(t_i-1)+2}\cdots E_{gt_i})^{1/2g} > c_q x + d_q, i=1,\ldots j\right).$$

Therefore, the difference between $Q_1^{(n)}$ and $Q_2^{(n)}$ can be bounded as follows:

$$\sum_{j=1}^{2m}(-1)^{j-1}(S_{j,n}-T_{j,n}) - T_{2m+1,n} \le Q_1^{(n)} - Q_2^{(n)} \le \sum_{j=1}^{2m-1}(-1)^{j-1}(S_{j,n}-T_{j,n}) + T_{2m,n}, \tag{36}$$

for each $m \ge 1$. By independence and Lemma 2, there exists $K = K(x)$ such that

$$T_{j,n} \le \binom{n}{j}\frac{K^j}{n^j} \le \frac{K^j}{j!} \quad \text{for all } n, j \ge 1. \tag{37}$$

Consequently, $\lim_{j\to\infty}\limsup_n T_{j,n} = 0$.

Now fix $j \ge 1$. Let us bound the difference between $S_{j,n}$ and $T_{j,n}$. Let $\mathcal{A}_t$ defined in (31) be represented as $\mathcal{A}_t = \{e_t^1, e_t^2, \ldots, e_t^g\}$. Also note $e_t^1, e_t^2, \ldots, e_t^g \in \{1, 2, \ldots, [\frac{n}{2}]\}$. For $1 \le t_1 < t_2 < \ldots < t_j \le q$, define

$$v_{2gj}(l) = \left(\cos\left(\frac{2\pi l e_{t_k}^1}{n}\right), \sin\left(\frac{2\pi l e_{t_k}^1}{n}\right), \cos\left(\frac{2\pi l e_{t_k}^2}{n}\right), \ldots, \cos\left(\frac{2\pi l e_{t_k}^g}{n}\right), \sin\left(\frac{2\pi l e_{t_k}^g}{n}\right);\ 1 \le k \le j\right).$$

Note that $\{e_{t_k}^1, \ldots, e_{t_k}^g :\ 1 \le k \le j\}$ is a set of distinct integers in $\{1, 2, \ldots, [\frac{n}{2}]\}$. Then,

$$\mathrm{P}\left((\beta_{\bar{a}+\sigma_n N,g}(t_i))^{1/2g} > c_q x + d_q, i=1,\ldots,j\right) = \mathrm{P}\left(2^{1/2}n^{-1/2}\sum_{l=0}^{n-1}(\bar{a}_l + \sigma_n N_l)v_{2gj}(l) \in B_n^{(j)}\right),$$

where

$$B_n^{(j)} := \left\{ y \in \mathbb{R}^{2gj} : \prod_{l=1}^{g} (y_{2gt+2l-1}^2 + y_{2gt+2l}^2)^{1/2g} > 2^{1/2}(c_q x + d_q); 0 \le t < j \right\}.$$

By Corollary 2 and the fact $N_1^2 + N_2^2 \stackrel{\mathcal{D}}{=} 2E_1$, we deduce that uniformly over all the d-tuples $1 \le t_1 < t_2 < \ldots < t_j \le q$,

$$\left| \mathrm{P}\left(2^{1/2} n^{-1/2} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) v_{2gj}(l) \in B_n^{(j)}\right) - \mathrm{P}\left((1+\sigma_n^2)^{1/2} (\prod_{i=1}^{g} E_{g(t_m-1)+i})^{1/2g} > c_q x + d_q, 1 \le m \le j\right) \right|$$
$$\le \varepsilon_n \mathrm{P}\left((1+\sigma_n^2)^{1/2} (E_{g(t_m-1)+1} E_{g(t_m-1)+2} \cdots E_{gt_m})^{1/2g} > c_q x + d_q, 1 \le m \le j\right) + O(\exp(-n^{\eta})).$$

Therefore, as $n \to \infty$,

$$|S_{j,n} - T_{j,n}| \le \varepsilon_n T_{j,n} + \binom{n}{j} O(\exp(-n^{\eta})) \le \varepsilon_n \frac{K^j}{j!} + o(1) \to 0, \tag{38}$$

where $O(\cdot)$ and $o(\cdot)$ are uniform over j. Hence using (34), (35), (37) and (38), we have

$$\limsup_n |Q_1^{(n)} - Q_2^{(n)}| \le \limsup_n T_{2m+1,n} + \limsup_n T_{2m,n} \quad \text{for each } m \ge 1.$$

Letting $m \to \infty$, we conclude $\lim_n (Q_1^{(n)} - Q_2^{(n)}) = 0$.

Since by Corollary 1,

$$\max_{1 \le j \le q} (E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} = O_p((\log n)^{1/2}) \text{ and } \sigma_n^2 = n^{-c},$$

it follows that

$$\frac{(1+\sigma_n^2)^{1/2} \max_{1 \le j \le q} (E_{g(j-1)+1} E_{g(j-1)+2} \cdots E_{gj})^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1$$

and consequently,

$$\frac{\max_{1 \le j \le q} (\beta_{\bar{a}+\sigma_n N, g}(j))^{1/2g} - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1.$$

In view of Lemma 5, it now suffices to show that

$$\max_{1 \le j \le q} (\beta_{\bar{a}+\sigma_n N, g}(j))^{1/2g} - \max_{1 \le j \le q} (\beta_{\bar{a}, g}(j))^{1/2g} = o_p(c_q).$$

Note that

$$\beta_{\bar{a}+\sigma_n N, g}(j) = \prod_{k=1}^{g} \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} (\bar{a}_l + \sigma_n N_l) \omega^{l e_j^k} \right|^2 = \prod_{k=1}^{g} \left| \alpha_{j,k} \right|^2, \text{ say,}$$

and

$$\beta_{\bar{a}, g}(j) = \prod_{k=1}^{g} \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \bar{a}_l \omega^{l e_j^k} \right|^2 = \prod_{k=1}^{g} \left| \gamma_{j,k} \right|^2, \text{ say.}$$

Now by the inequality

$$|\prod_{i=1}^{g} a_i - \prod_{i=1}^{g} b_i| \le \sum_{j=1}^{g} (\prod_{i=1}^{j-1} b_i) |a_j - b_j| (\prod_{i=j+1}^{g} a_i) \tag{39}$$

for nonnegative numbers $\{a_i\}$ and $\{b_i\}$, we have

$$\left|(\beta_{\bar{a}+\sigma_n N,g}(j)) - (\beta_{\bar{a},g}(j))\right| \leq \sum_{k=1}^{g} \left|\gamma_{j,1}\right|^2 \cdots \left|\gamma_{j,k-1}\right|^2 \left||\alpha_{j,k}|^2 - |\gamma_{j,k}|^2\right| \left|\alpha_{j,k+1}\right|^2 \cdots \left|\alpha_{j,g}\right|^2.$$

For any sequence of random variables $\{X_n\}_{n\geq 0}$, define

$$M_n(X) := \max_{1\leq t\leq n} \left| n^{-1/2} \sum_{l=0}^{n-1} X_l \omega^{tl} \right|.$$

As a trivial consequence of Theorem 2.1 of Davis and Mikosch (1999)(27), we have

$$M_n^2(\sigma_n N) = O_p(\sigma_n \log n) \text{ and } M_n^2(\bar{a} + \sigma_n N) = O_p(\log n).$$

Therefore $\left|\alpha_{j,k}\right| = O_p(\sqrt{\log n})$. Now,

$$\left|\gamma_{j,k}\right| \leq \left|\alpha_{j,k}\right| + \sigma_n \left| \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} N_l \omega^{l e_j^k} \right|$$

and therefore $\left|\gamma_{j,k}\right| = (1+\sigma_n) O_p(\sqrt{\log n}) = O_p(\sqrt{\log n})$. So we have

$$\begin{aligned}
\left| \max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j) - \max_{1\leq j\leq q} \beta_{\bar{a},g}(j) \right| &\leq \max_{1\leq j\leq q} \left|\beta_{\bar{a}+\sigma_n N,g}(j) - \beta_{\bar{a},g}(j)\right| \\
&\leq \max_{1\leq j\leq q} \sum_{k=1}^{g} \left(O_p(\log n)\right)^{g-1} \left|\alpha_{j,k} - \gamma_{j,k}\right| \left(|\alpha_{j,k}| + |\gamma_{j,k}|\right) \\
&\leq O_p(\log n)^{g-1} O_p(\sqrt{\log n}) \max_{1\leq j\leq q} \sum_{k=1}^{g} \left|\alpha_{j,k} - \gamma_{j,k}\right| \\
&\leq O_p(\log n)^{g-\frac{1}{2}} g \sigma_n M_n(N) \\
&\leq o_p\left(n^{-c/4} (\log n)^g\right).
\end{aligned}$$

$$\left| \max_{1\leq j\leq q} (\beta_{\bar{a}+\sigma_n N,g}(j))^{1/2g} - \max_{1\leq j\leq q} (\beta_{\bar{a},g}(j))^{1/2g} \right| \leq \left| \max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j) - \max_{1\leq j\leq q} \beta_{\bar{a},g}(j) \right| \frac{1}{\xi^{1/2g}}$$

where ξ lies between $\max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j)$ and $\max_{1\leq j\leq q} \beta_{\bar{a},g}(j)$. We know that

$$\frac{\max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j)}{(\log n)^g} \xrightarrow{\mathcal{P}} 1 \text{ and } \frac{\left|\max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j) - \max_{1\leq j\leq q} \beta_{\bar{a},g}(j)\right|}{(\log n)^g} \xrightarrow{\mathcal{P}} 0.$$

Therefore

$$\frac{\max_{1\leq j\leq q} \beta_{\bar{a},g}}{(\log n)^g} = \frac{\max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j)}{(\log n)^g} + \frac{\max_{1\leq j\leq q} \beta_{\bar{a},g}(j) - \max_{1\leq j\leq q} \beta_{\bar{a}+\sigma_n N,g}(j)}{(\log n)^g} \xrightarrow{\mathcal{P}} 1.$$

Hence

$$\frac{\xi}{(\log n)^g} \xrightarrow{\mathcal{P}} 1 \Rightarrow \frac{\xi^{1-1/2g}}{(\log n)^{g(1-1/2g)}} \xrightarrow{\mathcal{P}} 1 \Rightarrow \frac{1}{\xi^{1-1/2g}} = O_p((\log n)^{\frac{1}{2}-g}).$$

Combining all these we have

$$\left|\max_{1\le j\le q}\beta_{\bar{a}+\sigma_n N,g}(j)^{1/2g} - \max_{1\le j\le q}\beta_{\bar{a},g}(j)^{1/2g}\right| \le o_p(n^{-c/4}(\log n)^g) + O_p((\log n)^{\frac{1}{2}-g}) \le o_p(c_q).$$

This completes the proof of the first part. By convergence of type theorem, the second part follows since the following hold. We omit the tedious algebraic details.

$$\frac{c_q}{c_{n/2g}} \to 1 \text{ and } \frac{d_q - d_{n/2g}}{c_q} \to 0 \text{ as } n \to \infty. \tag{40}$$

□

The next Lemma is technical and is required in the proof of Theorem 5. Let

$$c_n(l) = \frac{1}{2l^{1/2}(\log n)^{1/2}},\ d_n(l) = \frac{\log C_l - \frac{l-1}{2}\log l}{2l^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{2l}\right)^{1/2}\left[1 + \frac{(l-1)\log\log n}{4\log n}\right],\ C_l = \frac{2}{\sqrt{l}}\left(\frac{\pi}{2}\right)^{\frac{l-1}{2}},$$

and

$$c_{n_{2j}} = c_{n_{2j}}(j),\ d_{n_{2j}} = d_{n_{2j}}(j),\ c_{n/2g} = c_{n/2g}(g) \text{ and } d_{n/2g} = d_{n/2g}(g).$$

Lemma 8. *Let $n = k^g + 1$ If $j < g$ and for some $a > 0$, $2jn_{2j} = (k^j + 1)(1 + n^{-a}) \sim n^{\frac{j}{g}}$ or is finite, then there exists a constant $K = K(j, g) \ge 0$ such that,*

$$\frac{c_{n/2g}}{c_{n_{2j}}} \to K \text{ and } \frac{d_{n/2g} - d_{n_{2j}}}{c_{n_{2j}}} \to \infty \text{ as } n \to \infty.$$

Proof. First observe that if n_j is finite then the result holds trivially. If $n_{2j} = \frac{(k^j+1)(1+n^{-a})}{2j}$ then

$$\log n_{2j} = j\log k + \left(\frac{1}{n^a} + \frac{1}{n^{j/g}}\right)(1 + \mathrm{o}(1)) - \log 2j$$

for some $a > 0$ and since $k = (n-1)^{\frac{1}{g}}$ we have

$$\frac{c_{n/2g}}{c_{n_{2j}}} \to \frac{j}{g} \text{ as } n \to \infty.$$

Similarly we get for some $a_0 > 0$,

$$\log\log n_{2j} = \log\log n^{\frac{j}{g}} + \left(\frac{1}{n^{a_0}\log n}\right)(1 + \mathrm{o}(1)) - \log 2j.$$

Now observe that $\frac{d_{n/2g} - d_{n_{2j}}}{c_{n_{2j}}}$ can be broken into the following three parts say J_i, $i = 1, 2$ or 3.

$$J_1 = 2j^{1/2}(\log n_{2j})^{1/2}\left[\frac{\log C_j - \frac{j-1}{2}\log j}{2j^{1/2}(\log n_{2j})^{1/2}} - \frac{\log C_g - \frac{g-1}{2}\log g}{2g^{1/2}(\log\frac{n}{2g})^{1/2}}\right] \to m_1 \text{ (some finite constant)}$$

$$J_2 = 2j^{1/2}(\log n_{2j})^{1/2}\left[\left(\frac{\log n/2g}{2g}\right)^{1/2} - \left(\frac{\log n_{2j}}{2j}\right)^{1/2}\right] \to m_2 \text{ (some finite constant).}$$

$$
\begin{aligned}
J_3 &= \sqrt{2} j^{1/2} (\log n_{2j})^{1/2} \left[\frac{(g-1)\log\log n/2g}{4(g\log n/2g)^{1/2}} - \frac{(j-1)\log\log n_{2j}}{4(j\log n_{2j})^{1/2}} \right] \\
&= \sqrt{2} j^{1/2} (\log n_{2j})^{1/2} \left[\frac{(g-1)\log\log n/2g}{4(g\log n/2g)^{1/2}} - \frac{(j-1)\sqrt{g}\log\log n_{2j}}{4j(\log n/2g)^{1/2}} + \mathrm{o}(1) \right] \\
&= \frac{\sqrt{2} j^{1/2} (\log n_{2j})^{1/2}}{4(g\log n/2g)^{1/2}} \left[(g-1)\log\log n/2g - \frac{(j-1)g}{j} \log\log n_{2j} + \mathrm{o}(1) \right] \\
&= \frac{\sqrt{2} j^{1/2} (\log n_{2j})^{1/2}}{4(g\log n/2g)^{1/2}} \left[\left((g-1) - \frac{g(j-1)}{j} \right) \log\log n/2g + \mathrm{o}(1) \right] \to \infty \text{ (since } g > j).
\end{aligned}
$$

□

Proof of Theorem 5. If $\#\mathcal{P}_i = j$, then the eigenvalues corresponding to $\mathcal{P}_i$'s are the j-th roots of Π_i and hence these eigenvalues have the same modulus. From Lemma 3, the possible values of $\#\mathcal{P}_i$ are $\{1, 2, 2g$ and $2g/b,\ 3 \le b < g, b$ odd, $\frac{g}{b} \in \mathbb{Z}\}$. Let $\beta_{a,j}(k)$ denote the modulus of the eigenvalue associated with the partition set $\mathcal{P}_k$, where $\#\mathcal{P}_k = 2j$.

In case of Gaussian entries it easily follows that $\beta_{a,j}(k)$ is the product of j exponential random variables and they are independent as k takes $2j$ many distinct values. So from Corollary 1, if $n_{2j} \to \infty$ then the maximum of $\beta_{a,j}(k)^{1/2j}$ has a Gumbel limit. For more general entries the method as in the proof of Lemma 7 can be adopted to get the following limit:

$$
\max_{1 \le k \le n_{2j}} \frac{\beta_{a,j}(k)^{1/2j} - d_{n_{2j}}}{c_{n_{2j}}} \xrightarrow{\mathcal{D}} \Lambda_1, \text{ as } n_{2j} \to \infty, \tag{41}
$$

where $c_{n_{2j}}$ and $d_{n_{2j}}$ are as above.

Let $x_n = c_n x + d_n$, $q = q_n = \frac{n}{2g}$ and $\mathcal{B} = \{b : b \text{ odd},\ 3 \le b < g,\ \frac{g}{b} \in \mathbb{Z}\}$. Then

$$
\mathrm{P}\Big(\mathsf{sp}(n^{-1/2}A_{k,n}) > x_q\Big) \ge \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{a,g}(j)^{1/2g} > x_q \right)
$$

and

$$
\begin{aligned}
\mathrm{P}\Big(\mathsf{sp}(n^{-1/2}A_{k,n}) > x_q\Big) &= \mathrm{P}\left(\max_l \max_{j:\mathcal{P}_j \in J_l} \beta_{a,l}(j)^{1/2l} > x_q \right) \\
&\le \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{2g}} \beta_{a,g}(j)^{1/2g} > x_q \right) + \sum_{b \in \mathcal{B}} \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_{\frac{2g}{b}}} \beta_{a,\frac{g}{b}}(j)^{b/2g} > x_q \right) \\
&\quad + \mathrm{P}\Big(\beta_{a,1}(0) > x_q\Big) + \mathrm{P}\left(|n^{-1/2} \sum_{l=0}^{n-1} (-1)^l a_l| > x_q \right) + \mathrm{P}\left(\max_{j:\mathcal{P}_j \in J_2} \beta_{a,2}(j)^{1/2} > x_q) \right) \\
&=: A + B + C + D + E.
\end{aligned}
$$

From Lemma 3, the term D appears only when $\frac{n}{2} \in \mathbb{Z}$ and the term E appears only if g is even and in that case J_2 contains only one element. It is easy to see that C, D and E tend to zero since we are taking maximum of single element.

Note that B is sum of finitely many terms. Now suppose for $b \in \mathcal{B}$, we have some finite K_b such that

$$
\frac{c_{n/2g}}{c_{n_{2g/b}}} \to K_b \text{ and } \frac{d_{n/2g} - d_{n_{2g/b}}}{c_{n_{2g/b}}} \to \infty \text{ as } n \to \infty. \tag{42}
$$

Then from observation (41) and (42) we get that the probability in B goes to zero. So it remains to check that whether (42) holds for $b \in \mathcal{B}$. But (42) holds from Lemma 3(e) and Lemma 8.

Now the limit in A follows from Lemma 7, proving the result. □

3.9 Remark on k circulants with $sn = k^g + 1$

Bose, Mitra and Sen (2008)(7) show existence of the limiting spectral distribution of the k circulant matrix with $k^g = sn - 1$ assuming that $s = o(n^{p_1 - 1})$ where p_1 was the smallest prime factor of g. To derive the limit of the spectral radius, we need a slightly stronger assumption that $s = o(n^{p_1 - 1 - \epsilon})$ for some $0 < \epsilon < p_1$ and $s > 1$. This is essential since $s = o(n^{p_1 - 1})$ implies $v_{k,n} \to 0$ which is not enough to deal with the maximum. We need the stronger result $v_{k,n} = o(n^{-a_1})$ for some $a_1 > 0$, so that these terms are negligible in the log scale that we have. Note that with the above conditions $s = o(n^{p_1 - 1})$ and $\upsilon_{k,n} = O(n^{-\epsilon/p_1})$.

Since $s > 1$ it easy to see from Lemma 3 in Bose, Mitra and Sen (2008)(7) that

$$\#X(\frac{2g}{b}) \leq \gcd(k^{2g/b} - 1, \frac{k^g + 1}{s}) \leq \gcd(k^{2g/b} - 1, k^g + 1). \tag{43}$$

Also observe that,

$$\#\left\{x : \ x \in \mathbb{Z}_n \text{ and } xk^{2g/b} = x \mod (\frac{k^g + 1}{s})\right\} \geq \#Z_{n,b}. \tag{44}$$

From observations (43) and (44) it easily follows that Lemma 3(d) remains valid in this case. Further, for some $\alpha > 0$ we get that

$$1 \geq \frac{\#X(\frac{2g}{b})}{k^{g/b} + 1} \geq 1 - k^{-g\alpha}(1 + o(1)) = 1 - (sn)^{-\alpha}(1 + o(1)) \geq 1 - n^{-\alpha}(1 + o(1)).$$

Hence from the above discussions we have the following Theorem.

Theorem 7. *Suppose $\{a_l\}_{l \geq 0}$ is an i.i.d. sequence of random variables with mean zero and variance 1 and $\mathrm{E}\,|a_l|^\gamma < \infty$ for some $\gamma > 2$. If $s \geq 1$ and $sn = k^g + 1$ where $s = o(n^{p_1 - 1 - \epsilon})$, $0 < \epsilon < p_1$, and p_1 is the smallest prime factor of g, then as $n \to \infty$,*

$$\frac{sp(n^{-1/2}A_{k,n}) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1$$

where $q = q(n) = \frac{n}{2g}$ and c_n and d_n can be taken as follows

$$d_n = \frac{\log C_g - \frac{g-1}{2}\log g}{2g^{1/2}(\log n)^{1/2}} + \left(\frac{\log n}{2g}\right)^{1/2}\left[1 + \frac{(g-1)\log\log n}{4\log n}\right], \ \ C_g = \frac{2}{\sqrt{g}}\left(\frac{\pi}{2}\right)^{\frac{g-1}{2}}$$

and

$$c_n = \frac{1}{2g^{1/2}(\log n)^{1/2}}.$$

3.10 Result for dependent input

Now let $\{x_n; n \geq 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}, \tag{45}$$

where $\{a_n, n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are i.i.d. with $E(\epsilon_i) = 0$ and $V(\epsilon_i) = 1$. Let $f(\omega)$, $\omega \in [0, 2\pi]$ be the spectral density of $\{x_n\}$. Note that if $\{x_n\}$ is i.i.d. with mean 0 and variance σ^2, then $f \equiv \frac{\sigma^2}{2\pi}$.

In this case, the variance of each eigenvalue is actually of the order of the spectral density at the corresponding ordinate. Thus it is meaningful to rescale by the spectral density. This is, for example, the approach taken by Walker (1965)(37), Davis and Mikosch (1999)(13), Lin and Liu (2009)(21) while studying the periodogram. This rescaling by the spectral density makes them approximately same variance and that makes it relatively easy to handle their maxima. Define,

$$\widetilde{\beta}_{x,j}(t) := \frac{\beta_{x,j}(t)}{\prod_{l \in \mathcal{A}_t} 2\pi f(\omega_l)} \text{ and } \mathrm{M}(n^{-1/2} A_{k,n}, f) = \max_{l} \max_{j:\mathcal{P}_j \in J_l} (\widetilde{\beta}_{x,l}(j))^{1/2l}.$$

Theorem 8. *Let $\{x_n\}$ be the two sided moving average process (45) where $E(\epsilon_i) = 0$, $E(\epsilon_i^2) = 1$, $E|\epsilon_i|^{2+\delta} < \infty$ for some $\delta > 0$ and*

$$\sum_{j=-\infty}^{\infty} |a_j||j|^{1/2} < \infty \text{ and } f(\omega) > \alpha > 0 \text{ for all } \omega \in [0, 2\pi]. \tag{46}$$

Then

$$\frac{M(n^{-1/2} A_{k,n}, f) - d_q}{c_q} \xrightarrow{\mathcal{D}} \Lambda_1.$$

as $n \to \infty$ where $q = q(n) = \frac{n}{2g}$ and c_q, d_q are as defined in Theorem 5.

Proof. Since we shall be using the bounds given in Walker (1965)(37) we define a few relevant notation for convenience. Define

$$I_{x,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} x_l e^{i\omega_j l}\Big|^2, \quad I_{\epsilon,n}(\omega_j) = \frac{1}{n}\Big|\sum_{l=1}^{n} \epsilon_l e^{i\omega_j l}\Big|^2,$$

$$A(\omega_j) = \sum_{t=-\infty}^{\infty} a_t e^{i\omega_j t}, \quad T_n(\omega_j) = I_{x,n}(\omega_j) - |A(\omega_j)|^2 I_{\epsilon,n}(\omega_j).$$

To prove the result we use following facts:
(i) From Walker (1965)(37) (page 112),

$$\max_{1 \leq t \leq n} |T_n(\omega_t)| = O_p(n^{-\delta}(\log n)^{1/2}).$$

(ii) From Davis and Mikosch (1999)(13),

$$\max_{1 \leq t \leq n} |I_{\epsilon,n}(\omega_t)| = O_p(\log n) \text{ and } \max_{1 \leq t \leq n} |I_{x,n}(\omega_t)| = O_p(\log n).$$

Using these and inequality (39), it is easy to see that, for some $\delta_0 > 0$

$$\max_{l} \max_{j:\mathcal{P}_j \in J_l} \left|\widetilde{\beta}_{x,l}(t) - \beta_{\epsilon,l}(t)\right| = o_p(n^{-\delta_0}). \tag{47}$$

Now the results follows from Theorem 5 and (47). □

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