LIMITING SPECTRAL DISTRIBUTION OF REVERSE CIRCULANT MATRIX WITH DEPENDENT ENTRIES

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Abstract. In this article, we derive the limiting spectral distribution of the *reverse circulant matrix* when the input sequence is a stationary infinite order two sided moving average process.

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1. Introduction and Main result

Suppose $\lambda_1, \lambda_2, ..., \lambda_n$ are all the real eigenvalues of a real symmetric square matrix A_n of order n. Then the *empirical spectral distribution function (ESDF)* of A_n is defined as

$$F_n(x) = n^{-1} \sum_{i=1}^{n} I\{\lambda_i \leq x\}.$$

Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of square matrices with the corresponding ESDF $\{F_n\}_{n=1}^{\infty}$. The Limiting Spectral Distribution (or measure) (LSD) of the sequence is defined as the weak limit of the sequence $\{F_n\}_{n=1}^{\infty}$, if it exists.

If $\{A_n\}$ are random, the limit is understood to be in some probabilistic sense, such as "almost surely" or "in probability". Suppose elements of $\{A_n\}$ are defined on some probability space $(\Omega, \mathcal{F}, P)$, that is $\{A_n\}$ are random. Let F be a nonrandom distribution function. We say the ESD of A_n converges to the *limiting spectral distribution* (LSD) F in L_2 if

$$\int_{\omega} \big(F_n(x) - F(x)\big)^2 dP(\omega) \to 0 \text{ as } n \to \infty$$

and converges in probability to F if for every $\epsilon > 0$

$$P\big(|F_n(x) - F(x)| > \epsilon\big) \to 0 \text{ as } n \to \infty.$$

For detailed information on limiting spectral distributions of large dimensional random matrices see [Bai(1999)] and also [Bose and Sen (2008)].

In this article we focus on obtaining the LSD of the *reverse circulant matrix* (RC_n) is given by

$$RC_n \;= \tfrac{1}{\sqrt{n}} \begin{bmatrix} x_0 & x_1 & x_2 & \ldots & x_{n-2} & x_{n-1} \\ x_1 & x_2 & x_3 & \ldots & x_{n-1} & x_0 \\ x_2 & x_3 & x_4 & \ldots & x_0 & x_1 \\ & & & \vdots & & \\ x_{n-1} & x_0 & x_1 & \ldots & x_{n-3} & x_{n-2} \end{bmatrix}.$$

So, the $(i,j)th$ element of the matrix is $x_{(i+j-2) mod\ n}$. It is not hard to calculate the eigenvalues of RC_n, for example see [Bose and Mitra(2002)] . The eigenvalues are given by :

$$\begin{cases} \lambda_0 & = n^{-1/2}\sum_{t=0}^{n-1} x_t \\ \lambda_{n/2} & = n^{-1/2}\sum_{t=0}^{n-1}(-1)^t x_t, \quad \text{if} \quad n \quad \text{is even} \\ \lambda_k = -\lambda_{n-k} & = \sqrt{I_n(\omega_k)}, \ 1 \le k \le [\frac{n-1}{2}]. \end{cases}$$

where,

$$\omega_k = \frac{2\pi k}{n}, \ I_n(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2.$$

Note that $[x]$ is the largest integer less than or equal to x. The existence of the LSD of RC_n is given by the following theorem of [Bose and Mitra(2002)].

Theorem 1.1. *Let $\{x_i\}$ be a sequence of independent random variables with mean 0 and variance 1 and $\sup_i E \mid x_i \mid^3 < \infty$. Then the ESD of RC_n converges in L_2 to F with density f given by*

$$f(x) = |x| \exp(-x^2), \ -\infty < x < \infty.$$

We investigate the existence of LSD of this matrix under a dependent situation. Let $\{x_n; n \ge 0\}$ be a two sided moving average process,

$$x_n = \sum_{i=-\infty}^{\infty} a_i \epsilon_{n-i}$$

where $\{a_n; n \in \mathbb{Z}\} \in l_1$, that is $\sum_n |a_n| < \infty$, are nonrandom and $\{\epsilon_i; i \in \mathbb{Z}\}$ are iid random variables with mean zero and variance one. We show that the LSD of RC_n continues to exist in this dependent situation. Define $\gamma_h = Cov\,(x_{t+h}, x_t)$. Then it is easy to see that $\sum_{j\in\mathbb{Z}} |\gamma_j| < \infty$ and the *spectral density function* of $\{x_n\}$ is given by

$$f(\omega) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} \gamma_k \exp(ik\omega) = \frac{1}{2\pi}\big[\gamma_0 + 2\sum_{k\ge 1}\gamma_k \cos(k\omega)\big] \text{ for } \ \omega \in [0, 2\pi].$$

Let $f^* = \inf_{\omega\in[0,2\pi]} f(\omega)$. For $k = 1, 2, \cdots, [\frac{n-1}{2}]$, define

$$\omega_k = \frac{2\pi k}{n}, \quad \xi_{2k-1} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} \epsilon_t \cos(\omega_k t), \quad \xi_{2k} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1} \epsilon_t \sin(\omega_k t),$$

$$I_n(\omega_k) = \frac{1}{n}|\sum_{t=0}^{n-1} x_t e^{-it\omega_k}|^2 = \frac{1}{n}\big[\big(\sum_{t=0}^{n-1} x_t\cos(\omega_k t)\big)^2 + \big(\sum_{t=0}^{n-1} x_t\sin(\omega_k t)\big)^2\big].$$

Theorem 1.2. *Suppose $\{\epsilon_i\}$ are iid with $E|\epsilon_i|^{(2+\delta)} < \infty$. Then the ESD of RC_n converges in L_2 to the LSD*

$$H(x) = \begin{cases} 1 - \int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x > 0 \\ \int_0^\pi \frac{1}{2\pi} e^{-\frac{x^2}{2\pi f(\omega)}} d\omega & \text{if } x \le 0. \end{cases}$$

It may be noted that integrand is zero whenever $f(\omega) = 0$.

Proof of the theorem mainly depends on following two lemmas. The proof of Lemma 1.3 is given in [Fan and Yao(2003)] (Theorem 2.14(ii), page 63). The proof of Lemma 1.4 follows easily from [Bhattacharya and Ranga Rao(1976)] (Corollary 18.3, page 184).

Lemma 1.3. *Let $x_t = \sum_{j=-\infty}^{\infty} a_t \epsilon_{t-j}$ for $t \ge 0$, where $\{\epsilon_t\} \sim IID(0,1)$ and $\sum_{j=-\infty}^{\infty} |a_j| < \infty$. Then for $k = 1, 2, \cdots, [\frac{n-1}{2}]$,*

$$I_n(\omega_k) = 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) + R_n(\omega_k)$$

and $\max_{1\le k\le[\frac{n-1}{2}]} E|R_n(\omega_k)| \to 0$ as $n \to \infty$.

Lemma 1.4. *Let $X_1, \ldots, X_k$ be independent random vectors with values in $\mathbb{R}^d$, having zero means and an average positive-definite covariance matrix $V_k = k^{-1}\sum_{j=1}^k CovX_j$. Let G_k denote the distribution of $k^{-1/2}T_k(X_1 + \ldots + X_k)$, where T_k is the symmetric, positive-definite matrix satisfying $T_k^2 = V_k^{-1}$, $n \ge 1$. If for some $\delta > 0$, $E \parallel X_j \parallel^{(2+\delta)} < \infty$, then*

$$\begin{aligned} \sup_{C\in\mathcal{C}} |G_k(C) - \Phi_{0,I}(C)| &\le ck^{-\delta/2}\big[k^{-1}\sum_{j=1}^k E \parallel T_k X_j \parallel^{(2+\delta)}\big] \\ &\le ck^{-\delta/2}(\lambda_{\min}(V_k))^{-(2+\delta)}\big[k^{-1}\sum_{j=1}^k E \parallel X_j \parallel^{(2+\delta)}\big] \end{aligned}$$

where $\Phi_{0,I}$ is the normal probability function with mean zero and identity covariance matrix, $\mathcal{C}$, the class of all Borel-measurable convex subsets of $\mathbb{R}^d$ and c is a constant, depending only on d.

Proof of Theorem 1.2: To prove the theorem it suffices to show that for each x,

$$E(F_n(x)) \to H(x) \text{ and } V(F_n(x)) \to 0.$$

From the structure of the eigenvalues, the LSD, if it exists, is going to that of a symmetric distribution. So, it is enough to concentrate on the case $x > 0$. Also note that we may ignore the two eigenvalues λ_0 and $\lambda_{n/2}$ since they contribute $2/n$ to the ESD F_n.

Hence for $x > 0$,

$$E(F_n(x)) \quad \sim \quad 1/2 + n^{-1} \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} P(I_n(\omega_k) \le x^2).$$

From Lemma 1.3, it is intutively clear that for large n, $I_n(\omega_k) \sim 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2)$. So first we show that for large n

$$\frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(I_n(\omega_k) \le x^2) \sim \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2).$$

Let $L_n(\omega_k) = 2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2)$ for $1 \le k \le [\frac{n-1}{2}]$. Then

$$\begin{aligned}
& |\frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(I_n(\omega_k) \le x^2) - \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2)| \\
\le \ & \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |P(L_n(\omega_k) + R_n(\omega_k) \le x^2) - P(L_n(\omega_k) \le x^2)| \\
\le \ & \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(|R_n(\omega_k)| \ge \epsilon) + \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |P(L_n(\omega_k) \le x^2) - P(L_n(\omega_k) \le x^2 \pm \epsilon)| \\
\le \ & T_1 + T_2 + T_3 + T_4, \text{ say,}
\end{aligned}$$

where

$$T_1 = \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} P(|R_n(\omega_k)| \ge \epsilon), \quad T_2 = \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |P(L_n(\omega_k) \le x^2) - \Phi_{0,I}(A_{kn})|,$$

$$T_3 = \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |P(L_n(\omega_k) \le x^2 \pm \epsilon) - \Phi_{0,I}(A_{kn}^\epsilon)|, \quad T_4 = \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} |\Phi_{0,I}(A_{kn}) - \Phi_{0,I}(A_{kn}^\epsilon)|,$$

$$A_{kn} = \{(r_1, r_2) : \frac{r_1^2}{2} + \frac{r_2^2}{2} \le \frac{x^2}{2\pi f(\omega_k)}\}, \ A_{kn}^\epsilon = \{(r_1, r_2) : \frac{r_1^2}{2} + \frac{r_2^2}{2} \le \frac{x^2 \pm \epsilon}{2\pi f(\omega_k)}\}.$$

For convenience we assume that $f^* > 0$. If there exists ω such that $f(\omega) = 0$, then the proof given below can be easily modified.

Now as $n \to \infty$,

$$T_1 \le \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} \epsilon^{-1} E|R_n(\omega_k)| \le \frac{1}{\epsilon} \max_{1 \le k \le [\frac{n-1}{2}]} E|R_n(\omega_k)| \to 0.$$

$$T_4 \le \frac{1}{n} \sum_{k=1}^{[\frac{n-1}{2}]} \frac{\epsilon}{2\pi f(\omega_k)} e^{-\frac{(x^2-\epsilon)}{2\pi f(\omega_k)}} \le C\epsilon,$$

where C is a constant and right side can be make arbitrarily small by choosing ϵ small enough. To show $T_2, T_3 \to 0$ define for $k = 1, 2, \cdots, [\frac{n-1}{2}]$ and $l = 0, 1, 2, \cdots, n-1$,

$$X_{l,k} = (\sqrt{2}\epsilon_l \cos(l\omega_k), \quad \sqrt{2}\epsilon_l \sin(l\omega_k))'.$$

Note that

$$E(X_{l,k}) = 0 \ \ \forall \ \ l, \ k, \ n. \tag{1.1}$$

$$n^{-1}\sum_{l=0}^{n-1} Cov(X_{l,k}) = I \ \ \forall \ \ k, \ n. \tag{1.2}$$

Note that

$$\{2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2\} = \{n^{-1/2}\sum_{l=0}^{n-1} X_{l,k} \in A_{kn}\}.$$

Since A_{kn} is a convex set in R^2 and since $\{X_{l,k}, \ l = 0, 1, \ldots (n-1)\}$ satisfies (1.1) and (1.2), we can apply Lemma 1.4 to get

$$|P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2) - \Phi_{0,I}(A_{kn})| \le cn^{-\delta/2}[n^{-1}\sum_{l=0}^{n-1} E \parallel X_{lk} \parallel^{(2+\delta)}]$$

Note

$$\sup_{1 \le k \le [\frac{n-1}{2}]} [n^{-1}\sum_{l=0}^{n-1} E \parallel X_{lk} \parallel^{(2+\delta)}] \le M < \infty.$$

$$I_1 = \frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]} |P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2) - \Phi_{0,I}(A_{kn})| \le cMn^{-\delta/2} \to 0 \text{ as } n \to \infty.$$

Hence $T_2 \to 0$ and similarly $T_3 \to 0$. Therefore

$$E(F_n(x)) \sim 1/2 + n^{-1}\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2),$$

and also

$$|\frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]} P(2\pi f(\omega_k)(\xi_{2k-1}^2 + \xi_{2k}^2) \le x^2) - \Phi_{0,I}(A_{kn})| \to 0 \ \text{ as } \ n \to \infty.$$

Now

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]}\Phi_{0,I}(A_{kn}) &= \frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]}\left(1-e^{-\frac{x^2}{2\pi f(\omega_k)}}\right)\\ &= \frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]}\left(1-e^{-\frac{x^2}{\gamma_0+2\sum_{h=1}^{\infty}\gamma_h\cos\frac{2\pi hk}{n}}}\right)\\ &\to \frac{1}{2}-\int_0^{1/2} e^{-\frac{x^2}{\gamma_0+2\sum_{h=1}^{\infty}\gamma_h\cos 2\pi ht}}dt\\ &= \frac{1}{2}-\int_0^{\pi}\frac{1}{2\pi}e^{-\frac{x^2}{2\pi f(\omega)}}d\omega = G(x),\ \text{say.}\end{aligned}$$

Hence for $x \geq 0$,

$$E(F_n(x)) \to 1-\int_0^{\pi}\frac{1}{2\pi}e^{-\frac{x^2}{2\pi f(\omega)}}d\omega = H(x).$$

Now, to show $V(F_n(x)) \to 0$, it is enough to show that

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{\lfloor\frac{n-1}{2}\rfloor} Cov(J_k, J_{k'}) \to 0. \tag{1.3}$$

where for $1 \leq k \leq [\frac{n-1}{2}]$, J_k is the indicator that $\{I_n(\omega_k) \leq x^2\}$.

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{[\frac{n-1}{2}]} Cov(J_k, J_{k'}) = \frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{[\frac{n-1}{2}]}\left[E(J_k, J_{k'}) - E(J_k)E(J_{k'})\right].$$

Now as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{[\frac{n-1}{2}]} E(J_k)E(J_{k'}) = \big(\frac{1}{n}\sum_{k=1}^{[\frac{n-1}{2}]}E(J_k)\big)^2 - \frac{1}{n^2}\sum_{k=1}^{[\frac{n-1}{2}]}(E(J_k))^2 \to G(x)^2.$$

So to show (1.3), it is enough to show as $n \to \infty$,

$$\frac{1}{n^2}\sum_{k\neq k';k,k'=1}^{\lfloor\frac{n-1}{2}\rfloor} E(J_k, J_{k'}) \to G(x)^2.$$

Along the lines of the proof used to show $\frac{1}{n}\sum_{k=0}^{[\frac{n-1}{2}]}P(I_n(\omega_k) \leq x^2) \to G(x)$, one may now extend the vectors of two coordinates defined above to ones with four coordinates and proceed exactly as above to verify this. We omit the routine details. □

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