Bulk behaviour of some patterned block matrices

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Abstract

We investigate the bulk behaviour of singular values and/or eigenvalues of two types of block random matrices. In the first one, we allow unrestricted structure of order $m \times p$ with $n \times n$ blocks and in the second one we allow $m \times m$ Wigner structure with symmetric $n \times n$ blocks. Different rows of blocks are assumed to be independent while the blocks within any row satisfy a weak dependence assumption that allows for some repetition of random variables among nearby blocks. We prove that when the input random variables are i.i.d. with mean 0 and variance 1 with finite moments of all orders, the Marchenko-Pastur and the semicircular type results still hold when $m \to \infty$ and $\frac{m}{p} \to c \in (0, \infty)$ and $m \to \infty$ in the first and second models respectively. These in particular generalize the bulk behaviour results of Loubaton [10] (2015).

1 Introduction

Let A_n be any $n \times n$ real symmetric or hermitian matrix with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. The *empirical spectral measure* μ_n of A_n is the measure on $\mathbb{R}$ given by

$$\mu_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}, \tag{1.1}$$

where δ_x is the Dirac delta measure at x. The corresponding probability distribution function F^{A_n} on $\mathbb{R}$ is known as the *empirical spectral distribution* (ESD) of A_n. For us the entries of

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A_n shall be random, and hence F^{A_n} shall be a *random distribution function*. If F^{A_n} converges weakly almost surely to a non-random distribution function F, then it is called the *limiting spectral distribution* (LSD) of A_n (almost surely).

Two basic results on LSD are the following. In particular they can be found in Bose and Sen [8] (2008). Suppose W_n is a sequence of symmetric matrices with real entries that are i.i.d. with mean 0 and variance 1. Then the LSD of $\frac{1}{\sqrt{n}} W_n$ (they are called Wigner matrices) is Wigner's semicircular law which is supported on the interval $[-2, 2]$ and has density

$$f_W(x) = \frac{1}{2\pi} \sqrt{4 - x^2}, \ \mathbb{I}(|x| \le 2).$$

On the other hand suppose $X_{n,p}$ is an $n \times p$ random matrix whose all entries are i.i.d. with mean 0 and variance 1. If $n \to \infty$ and $\frac{n}{p} \to c \in (0, \infty)$ then the LSD of $\frac{1}{p} X_{n,p} X_{n,p}^*$ (called the sample covariance matrix or the Wishart matrix) is the Marchenko-Pastur law ($MP(c)$ with parameter c which is defined as follows: it has a mass $1 - \frac{1}{c}$ at the origin if $c > 1$ and has a density

$$f_c(x) = \begin{cases} \frac{1}{2\pi xc} \sqrt{(b-x)(x-a)} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

where $a = (1 - \sqrt{c})^2$ and $b = (1 + \sqrt{c})^2$.

This article is concerned with the following two kinds of block matrices.

$$\mathbb{B}_n(m, p) := \begin{pmatrix} A_{n,(1,1)} & \dots & A_{n,(1,p)} \\ A_{n,(2,1)} & \dots & A_{n,(2,p)} \\ & \vdots & \\ A_{n,(m,1)} & \dots & A_{n,(m,p)} \end{pmatrix} \tag{1.2}$$

and

$$\mathbb{B}_n(m) := \begin{pmatrix} A_{n,(1,1)} & A_{n,(1,2)} & \dots & A_{n,(1,m-1)} & A_{n,(1,m)} \\ A_{n,(2,1)} & A_{n,(2,2)} & \dots & A_{n,(2,m-1)} & A_{n,(2,m)} \\ & & \vdots & & \\ A_{n,(m,1)} & A_{n,(m,2)} & \dots & A_{n,(m,m-1)} & A_{n,(m,m)} \end{pmatrix}. \tag{1.3}$$

with the additional condition $A^*_{n,(i,j)} = A_{n,(j,i)}$ for (1.3). Here $A_{n,(i,j)}$'s are square matrices of order n. We use the terminology *inner dimension* and *outer dimension* for the block size and dimension of the block structure respectively. For example, the outer and inner dimension of $\mathbb{B}_n(m, p)$ are $m \times p$ and $n \times n$.

There is a growing literature on the LSD of block random matrices, under different assumptions on the structure of the blocks, on how the blocks are arranged in the matrix and on how the inner and outer dimensions grow.

Oraby [3] (2007) considered two types of block matrices. In one, the blocks are independent Wigner matrices with symmetric block structure. Assuming that the outer dimension remains fixed and the inner dimension tends to ∞, he proved the existence of the LSD. In particular, he found the exact form of LSD when the block structure is Symmetric Circulant. In the other, the blocks are Symmetric Circulant arranged in Wigner pattern. In this case an exact expression of the LSD was obtained when the outer dimension grows to infinity. These results were generalized by Banerjee and Bose [4] (2011) who replaced the Wigner blocks by a general class of

"Wigner type" blocks. They also proved that for certain symmetric block matrices with independent Wigner type blocks, the LSD is semicircular when both the outer and inner dimensions grow to infinity. Gazzah et al. [5] (2001) researched the asymptotic behaviour of eigenvalue distribution for deterministic block Toeplitz matrices. Rashidi Far et al. [6] (2008) considered block matrices where the entries are complex Gaussian and the blocks are arranged in certain patterns. They proved the existence of LSD and found functional equations for the Stieltjes transform for the LSD. The proofs use an operator-valued free probability approach and Wicks formula for moments of Gaussian variables. Li et al. [7] (2011) studied LSD of block Toeplitz and Hankel matrices. They established the LSD when the outer dimension and inner dimension both grow to infinity or when only the outer dimension does so. Basu et al. [9] (2012) studied the LSD of random block matrices where the blocks are arranged in a Toeplitz pattern. They considered two types of blocks, in the first one there is no further assumption on the structure of the blocks and in the second one the blocks are asymmetric Toeplitz; either the outer or the inner dimension is fixed or both of them grow to infinity. They showed that LSD exist in all the cases. Ding [12] (2014) considered hermitian matrices with independent rectangular blocks. He took the random variables inside a particular block to be i.i.d. with mean 0 and fixed variance and allowed the random variables in different blocks to have different variances. He proved the existence of LSD in this case and found the exact forms of the LSD for a few special cases.

Loubaton [10] (2015) considered matrices $W_N = (W_N^{(1)T}, \ldots, W_N^{(M)T})^T$ where $(W_N^{(m)})_{m=1}^M$ are independent $L \times N$ block Hankel matrices with i.i.d. complex Gaussian entries. Under the assumption $\frac{LM}{N} \to c \in (0, \infty)$ and $M \to \infty$, he showed that the LSD of $W_N W_N^*$ is almost surely $MP(c)$. The proof is based on analysing the Stieltjes transform (the resolvent) of the ESD.

Observe that if $n = 1$, $\mathbb{B}_n(m, p)\mathbb{B}_n(m, p)^*$ and $\mathbb{B}_n(m)$ reduce to the Wishart and Wigner matrices respectively. Thus it is natural to ask what happens for general n. We let $m \to \infty$ and $\frac{m}{p} \to c \in (0, \infty)$ and in Theorems 2.1 and 2.2, we provide sufficient conditions so that the LSD of $\frac{1}{pn}\mathbb{B}_n(m, p)\mathbb{B}_n(m, p)^*$ and $\frac{1}{\sqrt{mn}}\mathbb{B}_n(m)$ are $MP(c)$ and the semicircular law respectively. Our major assumption is that different rows of blocks are independent whereas within each row the blocks are K-dependent. Loubaton's [10] (2015) model satisfies all the assumptions of Theorem 2.1. In Corollary 2.1, we derive his LSD result. However, it must be noted that we deal with only the LSD whereas he also dealt with the almost sure location of eigenvalues.

2 Main results

Instead of using the Stieltjes transform, we take a combinatorial approach to our results. It is convenient to bring the different patterns such as the Toeplitz, Hankel, Symmetric Circulant under a common umbrella. A patterned matrix is defined through a link function L_n. For each n, $L_n : \{0, 1, \ldots, n\} \to \mathbb{Z}^d$ is a function ($d = 1$ or 2). A patterned matrix A_n of order $n \times n$ with link function L_n is defined as $A_n = ((x_{L_n(i,j)}))_{1 \le i,j \le n}$. Here $\{x_{i,j}\}$ or $\{x_i\}$ is defined to be the input sequence of random variables. For notational convenience we write L for L_n. Some common (symmetric) link functions are given by

$$
\begin{aligned}
L_W(i,j) &= (\min(i,j), \max(i,j)), \quad \text{(Wigner)} \\
L_T(i,j) &= |i-j|, \quad \text{(Toeplitz)} \\
L_H(i,j) &= i+j, \quad \text{(Hankel)} \\
L_{RC}(i,j) &= (i+j) \mod n, \quad \text{(Reverse Circulant)} \\
L_{SC}(i,j) &= n/2 - |n/2 - |i-j|| \quad \text{(Symmetric Circulant).}
\end{aligned} \tag{2.1}
$$

Note that the block matrices also have a corresponding link function with a corresponding input sequence of matrices. We now introduce an important property for link functions.

Definition 2.1. (Property C) The link function L satisfies Property C if

$$\#\{l \mid 1 \leq l \leq n, L(k,l) = t\} \vee \#\{l \mid 1 \leq l \leq n, L(l,k) = t\} \leq 1 \;\; \forall \;\; t \in \mathbb{Z}^d \;\; \forall \;\; 1 \leq k \leq n \in \mathbb{N}. \tag{2.2}$$

This essentially says that along any row or column there is at most one occurrence of a particular input random variable. Note that the Wigner, Hankel and Reverse Circulant link functions satisfy Property C whereas the Symmetric Circulant and Toeplitz link functions do not.

Assumption 2.1. The input random variables are i.i.d. (possibly) complex valued with common law X such that $\mathrm{E}[X] = 0$, $\mathrm{E}[|X|^2] = 1$ and $\mathrm{E}[|X|^h] < \infty \;\; \forall h \in \mathbb{N}$.

Theorem 2.1. *Suppose $\mathbb{B}_n(m,p)$ satisfies Property C, $m \to \infty$ and $\frac{m}{p} \to c, 0 < c < \infty$. Suppose the input random variables satisfy Assumption 2.1 and the matrices $\{A_{n,i}\}_{i \in \mathbb{N}^2}$ satisfy the following: the entries of the matrices $A_{n,(i_1,i_2)}$ and $A_{n,(i_3,i_4)}$ are independent if $i_1 \neq i_3$ for any i_2 and i_4. Also, the entries of $A_{n,(i,j_1)}$ and $A_{n,(i,j_2)}$ are independent if $|j_1 - j_2| \geq K$ for some fixed $K \in \mathbb{N}$. Then almost surely the LSD of $\frac{1}{pn}\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*$ is $MP(c)$.*

The LSD result of Loubaton [10] (2015) follows as a simple Corollary to Theorem 2.1.

Corollary 2.1. *(Loubaton(2014)) Let $W_N = (W_N^{(1)T}, \ldots, W_N^{(M)T})^T$ be an $ML \times N$ block matrix where $(W_N^{(i)})_{i=1}^M$ are independent $L \times N$ Hankel matrices. Let $W_N^{(m)}(i,j) := w_{m,i+j-1}$. Assume that $(w_{m,n})_{1 \leq m \leq M, 1 \leq n \leq N+L-1}$ are i.i.d. complex Gaussian with $\mathrm{E}[|w_{m,n}|^2] = \frac{\sigma^2}{N}$ and $\mathrm{E}[w_{m,n}^2] = 0$. If $M \to \infty$ and $\frac{ML}{N} = c_N \to c \in (0,\infty)$ then the LSD of $\frac{1}{\sigma} W_N W_N^*$ is almost surely $MP(c)$.*

Theorem 2.2. *Suppose $\mathbb{B}_n(m)$ satisfies Property C. Suppose $m \to \infty$, the input random variables satisfy Assumption 2.1 and the following condition is satisfied.*

1. *The entries of the matrices $A_{n,(i_1,i_2)}$ and $A_{n,(i_3,i_4)}$ are independent if $i_1 \neq i_3$, $i_2 \geq i_1$ and $i_4 \geq i_3$. Also, the entries of $A_{n,(i,j_1)}$ and $A_{n,(i,j_2)}$ are independent if $i \leq \min\{j_1, j_2\}$ and $|j_1 - j_2| \geq K$ for some fixed $K \in \mathbb{N}$.*

Then almost surely the LSD of $\frac{1}{\sqrt{mn}}\mathbb{B}_n(m)$ is the semicircular law.

Remark 2.1. *We emphasize that Theorems 2.1 and 2.2 hold for any value of n. In particular, n can be fixed or grow to infinity. However, if n is fixed then the results may not hold in the absence of Property C. This is clear from Corollary 1 of Oraby [3] (2007) where he considers large Wigner block structure with independent finite dimensional Symmetric Circulant blocks and shows that the LSD is not the semicircle law. As discussed earlier, Symmetric Circulant matrices do not satisfy Property C. Nevertheless, from Theorem 3.1 it is clear that if $n \to \infty$ then the LSD of Wigner matrices with independent Symmetric Circulant blocks is indeed semicircular.*

3 Proofs

Proofs of Theorems 2.1, 2.2 and 3.1 (given later) are similar. We only prove Theorem 2.1. We shall need a few preliminaries; for more details see Bose and Sen [8] (2008).

Definition 3.1. (Property B) The link function L is said to satisfy Property B if

$$\sup_n \sup_{t\in\mathbb{Z}^d} \sup_{1\le k\le n} \#\{l \mid 1 \le l \le n, L(k,l) = t\} \vee \#\{l \mid 1 \le l \le n, L(l,k) = t\} = \Delta < \infty. \tag{3.1}$$

Note that Property C implies Property B.

Suppose A_n is hermitian. The h-th moments of the ESD is given by

$$h\text{-th moment of} \quad F^{A_n} = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^h = \frac{1}{n}\mathrm{Tr}(A_n^h) = \beta_h(A_n) \text{ (say)}. \tag{3.2}$$

where Tr denotes the trace of a matrix. To show the almost sure existence of the LSD, it is enough to show:

1. (M1) For every $h \ge 1$, $\mathrm{E}[\beta_h(A_n)] \to \beta_h$
2. (M4) $\sum_{n=1}^{\infty} \mathrm{E}[\beta_h(A_n) - \mathrm{E}(\beta_h(A_n))]^4 < \infty$ for every $h \ge 1$.
3. (C) The sequence $\{\beta_h\}$ satisfies Carleman's condition $\sum_{h=1}^{\infty} \beta_{2h}^{-1/2h} = \infty$.

The R.S. of (3.2) is often represented in terms of circuits and words. A *circuit* of length $l(\pi) := h$ is any function $\pi : \{0, 1, 2, \ldots, h\} \to \{1, 2, \ldots, n\}$ with $\pi(0) = \pi(h)$. For a patterned matrix A_n with link function L the R.S. of (3.2) equals

$$\frac{1}{n}\sum_{\pi:\ l(\pi)=h} x_{(L(\pi(0),\pi(1)))} x_{L(\pi(1),\pi(2))} \cdots x_{L(\pi(h-1),\pi(h))} := \frac{1}{n}\sum_{\pi:\ l(\pi)=h} x_\pi \quad \text{(say)}.$$

k circuits $\pi_1, \ldots, \pi_k$ are *jointly matched* if each L-value occurs at least twice across all circuits. They are *cross matched* if each circuit has at least one L-value which occurs in at least one of the other circuits.

Circuits π_1 and π_2 are said to be *equivalent* if and only if their L values match at the same locations. That is, for all i, j, $\{L(\pi_1(i-1), \pi_1(i)) = L(\pi_1(j-1), \pi_1(j))\} \Leftrightarrow \{L(\pi_2(i-1), \pi_2(i)) = L(\pi_2(j-1), \pi_2(j))\}$.

This defines an equivalence relation. Any equivalence class of circuits can be indexed by a partition of $\{1, 2, \ldots, h\}$. Each block of a given partition identifies the positions where the L-matches take place. We can label these partitions by *words* w of length h of letters where the first occurrence of each letter is in alphabetic order. For example if $h = 5$ then the partition $\{\{1, 3, 5\}\{2, 4\}\}$ is represented by the word *ababa*. This identifies the circuits π such that $L(\pi(0), \pi(1)) = L(\pi(2), \pi(3)) = L(\pi(4), \pi(5))$ and $L(\pi(1), \pi(2)) = L(\pi(3), \pi(4))$.

The set of words of length $2k$ such that each letter is repeated at least twice is denoted by $\mathcal{W}_{2k}$. A word $w \in \mathcal{W}_{2k}$ is called pair matched if each letter is repeated exactly twice. The set of such words is denoted by $\mathcal{P}_{2k}$. A word $w \in \mathcal{P}_{2k}$ is called *Catalan* if there are no four positions $i_1 < i_2 < i_3 < i_4$ such that $w[i_1] = w[i_3]$ and $w[i_2] = w[i_4]$. The set of all Catalan words of length $2k$ is denoted by C_{2k}. For example, the words *aabb* and *abba* are "Catalan words". However the word *abab* is not.

Define the following classes of circuits:

$$\begin{aligned}\Pi(w) &= \{\pi : w(x) = w(y) \Leftrightarrow L(\pi(x-1), \pi(x)) = L(\pi(y-1), \pi(y))\},\\ \Pi^*(w) &= \{\pi : w(x) = w(y) \Rightarrow L(\pi(x-1), \pi(x)) = L(\pi(y-1), \pi(y))\}.\end{aligned}$$

Any i (or $\pi(i)$ by abuse of notation) will be called a *vertex*. It is *generating* if either $i = 0$ or $w[i]$ is the first occurrence of a letter. For example, if $w = abbcab$ then $\pi(0), \pi(1), \pi(2), \pi(4)$ are generating. For a word w, $l(w)$ and $d(w)$ stand respectively for the length of the word and the number of distinct letters in it.

If the link function satisfies Property B then a circuit is completely determined up to a finitely many choices by its generating vertices. Hence $\#\Pi^*(w) = O(n^{d(w)+1})$. In fact it is easy to see that $\#\Pi^*(w) \leq \Delta^{l(w)-d(w)-1} n^{d(w)+1}$. Moreover, for any word w, $\#\left(\Pi^*(w) \backslash \Pi(w)\right) = O(n^{d(w)})$. As a consequence, $\lim_{n\to\infty} \frac{\#(\Pi^*(w)\backslash\Pi(w))}{n^{d(w)+1}} \to 0$. Let

$$p(w) := \lim_{n\to\infty} \frac{\#\Pi^*(w)}{n^{d(w)+1}} \text{ (provided the limit exists).} \tag{3.3}$$

One can express often express the final form of an LSD in terms of these $p(w)$'s. See Bose and Sen [8] (2008). For example, for the Wigner matrix, $p(w) = 1$ if $w \in C_{2k}$ and is 0 otherwise. In fact, for all the link functions in (2.1), $p(w) = 1$ for all $w \in C_{2k}$.

Even though $X_{n,p}$ is not symmetric, there is a suitable extension of the concept of link function and all related concepts to the Wishart matrix $\frac{1}{p} X_{n,p} X_{n,p}^*$. We give a description of the LSD of the Wishart matrix in terms of $p(w)$. Its proof is included in the proof of Theorem 5 of Bose and Sen [8] (2008).

Lemma 3.1. *The k th moment of* $MP(c)$ *is given by* $\beta_k = \sum_{w \in C_{2k}} p(w)$ *where*

$$p(w) = \begin{cases} c^t & \text{if } w \in C_{2k} \text{ and has } (t+1) \text{ and } (k-t) \text{ generating verices at even and odd positions} \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

So one way of identifying the Marchenko-Pastur law as LSD is to show that $p(w)$ defined in (3.3) satisfies (3.4).

3.1 Proof of Theorem 2.1

It is enough to verify (M1) (with β_k as in Lemma 3.1), (M4) and (C). Observe that

$$\begin{aligned}\operatorname{Tr}\left[(\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*)^k\right] &= \sum_{i_0, i_1, \ldots, i_{2k-1}} \operatorname{Tr}\left[A_{n,(i_0,i_1)} A^*_{n,(i_2,i_1)} \ldots A_{n,(i_{2k-2},i_{2k-1})} A^*_{n,(i_0,i_{2k-1})}\right] \\ &= \sum_{\pi:\ l(\pi)=2k} \operatorname{Tr}\left[A_{n,(\pi(0),\pi(1))} \ldots A^*_{n,(\pi(2k),\pi(2k-1))}\right] \\ &= \sum_{\pi:\ l(\pi)=2k} \operatorname{Tr}(A_\pi) \ (say).\end{aligned} \tag{3.5}$$

Observe that in the monomial $(\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*)^k$ the odd and the even positions are occupied by $\mathbb{B}_n(m,p)$ and $\mathbb{B}_n(m,p)^*$ respectively. So we need to modify the concepts of words, circuits etc. For convenience we shall continue to use the earlier terminology. The (i, j) th of block $\mathbb{B}_n(m, p)^*$ is $(A_{n,(j,i)})^*$. So now the earlier role of $(\pi(2i-1), \pi(2i))$ for symmetric matrices is now taken by $(\pi(2i), \pi(2i-1))$ and in the definition of word any letter at the even positions will denote the transpose of the corresponding matrix. For example if $w = aabb$ then it will represent all circuits π of the form $(\pi(0), \pi(1)) = (\pi(2), \pi(1))$, $(\pi(2), \pi(3)) = (\pi(4), \pi(3))$ and $(\pi(0), \pi(1)) \neq (\pi(2), \pi(3))$. Also for any such π, A_π will be MM^*NN^* where $M = A_{n,(\pi(0),\pi(1))}$ and $N = A_{n,(\pi(2),\pi(3))}$. To keep track of these transformations, we define

$$f(\pi, i) := \begin{cases} (\pi(i), \pi(i+1)) & \text{if } i \text{ is even} \\ (\pi(i+1), \pi(i)) & \text{otherwise} \end{cases}$$

and

$$G(\pi, i) := \begin{cases} A_{n,(\pi(i),\pi(i+1))} & \text{if } i \text{ even} \\ (A_{n,(\pi(i+1),\pi(i))})^* & \text{otherwise.} \end{cases}$$

Hence the equivalence class $\Pi(w)$ induced by a word w is now defined as

$$\Pi(w) := \{\pi \mid w[i] = w[j] \Leftrightarrow f(\pi, i-1) = f(\pi, j-1)\}.$$

Also let

$$A_\pi = \prod_{i=1}^{2k} G(\pi, i-1).$$

We now complete the proof in the following four steps.

Step 1: Accounting for the K-dependence of the blocks in a row, we construct an apporpriate equivalence relation on the circuits that will be vital in our proof. Call two matrices $A_{n,(i_1,i_2)}$ and $A_{n,(i_3,i_4)}$ as "strongly related" if $i_1 = i_3$ and $|i_2 - i_4| < K$. We denote this by $A_{n,(i_1,i_2)} \sim_s A_{n,(i_3,i_4)}$.

For any circuit π, consider all the matrices in the monomial A_π. Two matrices $A_{n,f(\pi,i)}$ and $A_{n,f(\pi,j)}$ are called "weakly related", if there exists a sequence of integers $i = i_1 < i_2 \ldots < i_r = j$ such that

$$A_{n,f(\pi,i_1)} \sim_s A_{n,f(\pi,i_2)} \sim_s \ldots \sim_s A_{n,f(\pi,i_{r-1})} \sim_s A_{n,f(\pi,i_r)}.$$

We denote this equivalence relation by $A_{n,f(\pi,i)} \sim_{w,\pi} A_{n,f(\pi,j)}$.

We call two circuits π_1 and π_2 "R equivalent" if for all i, j

$$A_{n,f(\pi_1,i)} \sim_{w,\pi_1} A_{n,f(\pi_1,j)} \Leftrightarrow A_{n,f(\pi_2,i)} \sim_{w,\pi_2} A_{n,f(\pi_2,j)}.$$

The "R equivalence" is clearly an equivalence relation on the class of circuits of a fixed length $2k$. Every equivalence class is a partition of $\{1, 2, \ldots, 2k\}$ where the "weakly related" matrices are in the same blocks. Given a word w we denote the class of all "R equivalent" circuits corresponding to w by $\Pi_R(w)$. For example, if $w = aaa$, then $\Pi_R(w)$ will contain the following four kind of circuits:

(a) $A_{n,f(\pi,0)} \sim_s A_{n,f(\pi,1)}$ and $A_{n,f(\pi,1)} \sim_s A_{n,f(\pi,2)}$ but $A_{n,f(\pi,0)} \nsim_s A_{n,f(\pi,2)}$.

(b) $A_{n,f(\pi,0)} \sim_s A_{n,f(\pi,1)}$ and $A_{n,f(\pi,0)} \sim_s A_{n,f(\pi,2)}$ but $A_{n,f(\pi,1)} \nsim_s A_{n,f(\pi,2)}$.

(c) $A_{n,f(\pi,0)} \sim_s A_{n,f(\pi,2)}$ and $A_{n,f(\pi,1)} \sim_s A_{n,f(\pi,2)}$ but $A_{n,f(\pi,0)} \nsim_s A_{n,f(\pi,1)}$.

(d) $A_{n,f(\pi,0)} \sim_s A_{n,f(\pi,1)}$ and $A_{n,f(\pi,1)} \sim_s A_{n,f(\pi,2)}$ and $A_{n,f(\pi,1)} \sim_s A_{n,f(\pi,2)}$.

Now observe that if there exists a circuit π of length $2k$ such that $A_{n,(i_1,i_2)} \sim_{w,\pi} A_{n,(j_1,j_2)}$ then $i_1 = j_1$ and $|i_2 - j_2| < 2kK$. As a consequence,

$$\#\Pi_R(w) \leq (4kK)^{2k-d(w)} m^{t(w)+1} p^{d(w)-t(w)} \tag{3.6}$$

where $t(w) + 1$ is the number of generating vertices at the even positons of w.

Step 2: We now show that it is enough to consider only $\pi \in \Pi_R(w)$ where $w \in C_{2k}$. From (3.5)

$$\mathrm{E}\left[\frac{1}{mn}\,\mathrm{Tr}\left(\frac{1}{pn}\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*\right)^k\right] = \frac{1}{mp^k n^{k+1}} \sum_{\{w:\ l(w)=2k\}} \sum_{\pi\in\Pi_R(w)} \mathrm{E}\,\mathrm{Tr}(A_\pi). \tag{3.7}$$

We can break $\mathrm{E}\,\mathrm{Tr}(A_\pi)$ in the following way

$$\begin{aligned}\frac{\mathrm{E}\,\mathrm{Tr}(A_\pi)}{n^{k+1}} &= \frac{1}{n^{k+1}} \sum_{\pi'} \left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right) \\ &= \frac{1}{n^{k+1}} \sum_{w'} \sum_{\pi'\in\Pi(w')} \mathrm{E}\left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right).\end{aligned} \tag{3.8}$$

where π' is a circuit of length $2k$. Let X be the common distribution of the input random variables from Assumption 2.1. Now $\mathrm{E}[X] = 0$, and hence if $w' \notin \mathcal{W}_{2k}$, then

$$\mathrm{E}\left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right) = 0.$$

However, if $w' \notin \mathcal{W}_{2K}$, then $\#\Pi(w') \leq n^{k+1}$ and Holder's inequality implies that

$$\mathrm{E}\left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right) \leq \mathrm{E}[|X|^{2k}].$$

As a consequence, the R.S. of (3.8) (in absolute value) is bounded by

$$\sum_{w'\in\mathcal{W}_{2k}} \frac{\#\Pi(w')}{n^{k+1}}\,\mathrm{E}[|X|^{2k}] \leq B(2k)\,\mathrm{E}[|X|^{2k}]$$

where $B(2k)$ is the number of all possible partitions of the set $\{1, 2, \ldots, 2k\}$.

Now fix a particular w. If w has a letter occurring exactly once at position j, then for any $\pi \in \Pi_R(w)$ the matrix $G(\pi, j-1)$ has all elements independent of the matrix $G(\pi, z-1)$ for all $1 \leq z \neq i \leq 2k$. As a consequence, $\mathrm{E}\,\mathrm{Tr}\,[A_\pi] = \mathrm{E}\,\mathrm{Tr}\left[\prod_{i=1}^{2k} G(\pi, i)\right] = 0$. Hence to get a non-trivial value of $\mathrm{E}\,\mathrm{Tr}[A_\pi]$ the corresponding w must have all letters repeated at least twice. That in turn shows that the L.S. of (3.7) is bounded by

$$\frac{1}{mp^k} \sum_{w:\ l(w)=2k} \#\Pi_R(w)\,\mathrm{E}[|X|^{2k}]B(2k).$$

Now we make the following claim.

Claim: If $w \in \mathcal{W}_{2k} \backslash C_{2k}$ then $\frac{\#\Pi_R(w)}{mp^k} \to 0$ as $m \to \infty$.
Proof: The proof is similar to the proof of part(i) of Theorem 1 in Banerjee and Bose [4]. However, due to difference in the context and notation we give a proof here.

First observe that if any letter in w appears at least thrice then the number of generating vertices in w is at most k. As a consequence, $\#\Pi_R(w) = O(m^k) = o(mp^k)$. Hence we need to consider only $w \in \mathcal{P}_{2k}$. As $w \in \mathcal{P}_{2k} \backslash C_{2k}$, there exist $i < j < t < l$ such that $w[i] = w[t] = x$ (*say*) and $w[j] = w[l] = y$ (*say*). Without loss of generality, let i be the minimum such choice (for some j, t, l), and for this i, let j be the maximum such choice (for some t, l). Thus all letters of w in $\{w[j+1], \ldots, w[t-1]\}$ have both copies in $\{i+1, \ldots, t-1\}$. Now we observe the following for circuits $\pi \in \Pi_R(w)$.

(a) First fill up $\pi(0), \ldots, \pi(j-1)$. Let the number of generating vertices in $\{0, \ldots, j-1\}$ be p_1. Then there are at most $O(m^{p_1})$ choices for these vertices. In this procedure we have fixed the matrix $G(\pi, i-1)$ corresponding to the letter x upto finitely many choices.

(b) As $A_{n,f(\pi,i-1)} \sim_{w,\pi} A_{n,f(\pi,t-1)}$, there are at most $4kK$ choices for the vertices $\pi(t-1)$ and $\pi(t)$.

(c) Now we fill up $\pi(t), \ldots, \pi(j+1)$ in that order according to the following algorithm. We consider two sub cases.

(i) The position of the first occurrence of the letter at $w[t-1]$ is in $\{0, \ldots, j-1\}$.

(ii) The position of the first occurrence of the letter at $w[t-1]$ is in $\{j+1, \ldots, t-1\}$.

For sub case (i), clearly $\pi(t-2)$ has at most $4kK$ choices.

For sub case (ii), fill $\pi(t-2)$ arbitrarily. Observe that this specifies the matrix $A_{n,f(\pi,t-2)}$. As a consequence, if z is the first position of first occurrence of $w[t-1]$ then $\pi(z-1)$ and $\pi(z)$ have at most $4kK$ choices.

We fill the vertices $\pi(t-3)$ to $\pi(j+1)$ in similar fashion.

Now $w[j+1]$ is either the first occurrence or the second occurrence of a letter. Since we have specified $A_{n,f(\pi,j)}$ by backward traversal, if it is a first occurrence then the second letter is in $\{j+2, \ldots, t-1\}$. As a consequence, $(\pi(j), \pi(j+1))$ have at most $4kK$ choices. On the other hand if $w[j+1]$ is a second occurrence then also we have specified the matrix $A_{n,f(\pi,j)}$ by specifying the vertices $\{\pi(0), \ldots, \pi(j-1)\}$. As a consequence, the generating vertex $\pi(j)$ has only finitely many choices.

Let p_2 be the number of generating vertices in $\{j+1, \ldots, t-2\}$. Then the vertices $\{j, \ldots, t\}$ has at most $O(m^{p_2})$ choices. On the other hand if the number of generating vertices in between $\{t+1, \ldots, 2k\}$ is p_3 then rest of the vertices have at most $O(m^{p_3})$ choices.

So any $\pi \in \Pi_R(w)$ has at most $O(m^{p_1+p_2+p_3})$ many choices. However, the total number of generating vertices in w is $p_1 + p_2 + p_3 + 1 = k + 1$. Hence $\#\Pi_R(w) = O(m^k)$. This proves the claim and hence completes Step 2.

Step 3: Now fix $w \in C_{2k}$. We show that

$$\lim \frac{1}{mp^k n^{k+1}} \sum_{\pi \in \Pi_R(w)} \mathrm{E}\,\mathrm{Tr}(A_\pi) = c^{t(w)}. \tag{3.9}$$

Let $\pi \in \Pi_R(w)$ be any circuit. We first prove that

$$\frac{\mathrm{E}\,\mathrm{Tr}(A_\pi)}{n^{k+1}} = \begin{cases} 1 & \text{if } \pi \in \Pi(w) \cap \Pi_R(w) \\ 0 & \text{if } \pi \in \Pi_R(w) \backslash \Pi(w). \end{cases} \tag{3.10}$$

Consider $\pi \in \Pi_R(w)\backslash\Pi(w)$. As $w \in C_{2k}$, there exists a double letter at positions j and $j+1$. Without loss of generality we assume j to be even. The odd case will follow similarly. As j is even, $G(\pi, j-1) = A^*_{n,f(\pi,j-1)}$ and $G(\pi, j) = A_{n,f(\pi,j)}$. Recalling the definition of f we get that

$$A_{n,f(\pi,j-1)} = A_{n,(\pi(j),\pi(j-1))} \text{ and } A_{n,f(\pi,j)} = A_{n,(\pi(j),\pi(j+1))}.$$

By our assumption, $A_{n,(\pi(j),\pi(j-1))} \sim_{w,\pi} A_{n,(\pi(j),\pi(j+1))}$. Suppose they are not equal. By Property C of $\mathbb{B}_n(m,p)$, we get that

$$A_{n,(\pi(j),\pi(j-1))}(\pi'(j), \pi'(j-1)) \neq A_{n,(\pi(j),\pi(j+1))}(\pi'(j), \pi(j+1))$$

for any π'. As a consequence, $A^*_{n,(\pi(j),\pi(j-1))}(\pi'(j-1), \pi'(j))$ and $A_{n,(\pi(j),\pi(j+1))}(\pi'(j), \pi'(j+1))$ are independent.

As $w \in C_{2k}$, the double letter at positions j and $j+1$ have not appeared anywhere else in w. So all the elements of $G(\pi, j-1)$ and $G(\pi, j)$ are independent of the elements of the matrix $G(\pi, z)$ when $z \notin \{j-1, j\}$. As a consequence for any π', the random variable $A^*_{n,(\pi(j),\pi(j-1))}(\pi'(j-1), \pi'(j))$ occurs exactly once in the product $\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))$. Hence

$$\mathrm{E}\left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right) = 0$$

implying $\mathrm{E}\,\mathrm{Tr}(A_\pi) = 0$.

So we are left with the case $A_{n,(\pi(j),\pi(j-1))} = A_{n,(\pi(j),\pi(j+1))}$, which implies $\pi(j-1) = \pi(j+1)$. By applying Property C again we observe that one needs $\pi'(j-1) = \pi'(j+1)$ to get a non-trivial value of

$$\mathrm{E}\left(\prod_{i=1}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right).$$

As a consequence,

$$A^*_{n,(\pi(j),\pi(j-1))}(\pi'(j-1), \pi'(j)) = \overline{A_{n,(\pi(j),\pi(j+1))}(\pi'(j), \pi'(j+1))} := x \ (say).$$

In this case,

$$\sum_{\pi'(j-1)=\pi'(j+1)} \mathrm{E}\left(\prod_{i=1}^{j-1} G(\pi, i-1)(\pi'(i-1), \pi'(i)) \ \bar{x}x \prod_{i=j+2}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right) = \sum_{\pi'(j-1)=\pi'(j+1)} \mathrm{E}\left(\prod_{i=1}^{j-1} G(\pi, i-1)(\pi'(i-1), \pi'(i)) \prod_{i=j+2}^{2k} G(\pi, i-1)(\pi'(i-1), \pi'(i))\right). \tag{3.11}$$

Now we use a successive reduction argument.

Notice that $\pi(j-1) = \pi(j+1)$. So if we consider the circuit

$$\pi_{red}(i) = \begin{cases} \pi(i) & \text{if } i \leq j-1 \\ \pi(i+2) & \text{if } 2k-2 \geq i \geq j, \end{cases}$$

then

$$\prod_{i=1}^{j-1} G(\pi, i-1) \prod_{i=j+2}^{2k} G(\pi, i-1) = \prod_{i=1}^{2k-2} G(\pi_{red}, i-1).$$

Let w' be the reduced word removing letters $w[j]$ and $w[j+1]$. By definition $w' \in C_{2k-2}$ and also $\pi_{red} \in \Pi_R(w')$.

Define π'_{red} in a similar fashion. As $\pi'(j-1) = \pi'(j+1)$, the R.S. of (3.11) becomes

$$\sum_{\{\pi'_{red} \mid l(\pi'_{red})=2k-2\}} n \, \mathrm{E}\left(\prod_{i=1}^{2k-2} G(\pi_{red}, i-1)(\pi'_{red}(i-1), \pi'_{red}(i)) \right).$$

The factor n appears since $\pi'(j)$ can be chosen freely.

As $w \in C_{2k}$, we can repeat the above arguments till we get an empty word. Hence to get a non-zero value for $\mathrm{E}\,\mathrm{Tr}(A_\pi)$, the weakly related matrices in $\prod_{i=1}^{2k} G(\pi, i-1)$ are actually equal. Hence $\pi \in \Pi_R(w)$ and $\mathrm{E}\,\mathrm{Tr}(A_\pi) \neq 0$ implies $\pi \in \Pi(w)$. In that case $\mathrm{E}\,\mathrm{Tr}(A_\pi) = n^{k+1}$.

So (3.10) is established.

As equality of two matrices implies them to be strongly related, it is easy to observe that $\#(\Pi(w) \backslash \Pi_R)(w) = O(m^k) = o(mp^k)$. So $\lim \frac{\#\Pi(w)}{mp^k} = \lim \frac{\#(\Pi(w) \cap \Pi_R(w))}{mp^k}$.

As a consequence the L.S. of (3.9) becomes

$$\lim \frac{\#\Pi(w)}{mp^k}. \tag{3.12}$$

We can now recall the proof of part (a) of Theorem 5 from Bose and Sen [8] (2008) to conclude that the above expression converges $c^{t(w)}$. As a consequence, the L.S. of (3.7) converge to $\sum_{w \in C_{2k}} c^{t(w)}$ verifying the (M1) condition.

Step 4: We now verify (M4). Bose and Sen [8] (2008) showed the following: if A_n are hermitian matrices satisfying Property B with input sequence satisfying Assumption 2.1, then

$$\frac{1}{n^4} \mathrm{E}\left[\mathrm{Tr}\left(\frac{A_n}{\sqrt{n}} \right)^k - \mathrm{E}\left(\mathrm{Tr}\left(\frac{A_n}{\sqrt{n}} \right)^k \right) \right]^4 = O(n^{-2}). \tag{3.13}$$

Now, the matrix $\mathbb{B}_n(m,p)$ is not necessarily symmetric and hence we cannot apply directly the above result. However, we can still use Property B which is implied by Property C and modify the arguments of the proof of above result suitably to conclude that

$$\frac{1}{n^4 m^4} \mathrm{E}\left[\mathrm{Tr}\left(\frac{\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*}{pn} \right)^k - \mathrm{E}\left(\mathrm{Tr}\left(\frac{\mathbb{B}_n(m,p)\mathbb{B}_n(m,p)^*}{pn} \right)^k \right) \right]^4 = O(\frac{1}{n^2 m^2}). \tag{3.14}$$

This implies that the (M4) condition is satisfied.

Note that condition (C) is satisfied as β_k are the moments of the $MP(c)$ law which is bounded. This proves Theorem 2.1 completely. □

3.2 Proof of Corollary 2.1

First assume that N is divisible by L. Then we can write $W_N^{(i)}$ as $(W_N^{(i,1)} \ldots W_N^{(i,P)})$ where $P = \frac{N}{L}$ and $\{W_N^{(i,j)}\}_{j=1}^{P}$ have L columns. It is easy to check that the matrix W_N satisfies the conditions of Theorem 2.1 with $m = M$, $n = L$, $p = P$ and $N = pn$. Hence, the result follows in this case.

Now assume that N is not divisible by L and let $P = [\frac{N}{L}] + 1$. Hence

$$W_N = \left(\tilde{W}_N : \Delta_N\right) \text{ where } \Delta_N = \begin{pmatrix} W_N^{(1,P)} \\ W_N^{(2,P)} \\ \vdots \\ W_N^{(M,P)} \end{pmatrix}.$$

Now

$$W_N W_N^* = \tilde{W}_N \tilde{W}_N^* + \Delta_N \Delta_N^*.$$

Observe that one can directly use Theorem 2.1 for the matrix $\tilde{W}_N \tilde{W}_N^*$ to conclude that its LSD is $MP(c)$ as discussed earlier.

Let $\lambda_1 \geq \ldots \geq \lambda_{LM}$ and $\nu_1 \geq \ldots \geq \nu_{LM}$ be the eigenvalues of $W_N W_N^*$ and $\tilde{W}_N \tilde{W}_N^*$ respectively.

The LSD of $W_N W_N^*$ and $\tilde{W}_N \tilde{W}_N^*$ are identical almost surely. To verify this, it is enough to verify the following condition.

$$\frac{1}{LM} \sum_{i=1}^{LM} (\lambda_i - \nu_i)^2 \stackrel{a.s.}{\to} 0. \tag{3.15}$$

Now

$$\sum_{i=1}^{LM} (\lambda_i - \nu_i)^2 \leq \operatorname{Tr}\left[(W_N W_N^* - \tilde{W}_N \tilde{W}_N^*)(W_N W_N^* - \tilde{W}_N \tilde{W}_N^*)^*\right] = \operatorname{Tr}(\Delta_N \Delta_N^*)^2.$$

For a proof of the first inequality, see page 69 of Bhatia [11] (1997). Thus it is now enough to show $\frac{1}{ML} \operatorname{Tr}(\Delta_N \Delta_N^*)^2 \to 0$ almost surely.

Let $\tilde{\Delta}_N = \sqrt{N}\sigma\Delta_N$ so that the variance of each entry of $\tilde{\Delta}_N$ is 1. It is enough to show that

$$\frac{1}{MLN^2} \operatorname{Tr}(\tilde{\Delta}_N \tilde{\Delta}_N^*)^2 \stackrel{a.s.}{\to} 0. \tag{3.16}$$

Recall the definitions of f and $\Pi(w)$ from the proof of Theorem 2.1. Now

$$\begin{aligned} \frac{1}{MLN^2} \operatorname{E}\operatorname{Tr}(\tilde{\Delta}_N \tilde{\Delta}_N^*)^2 &= \frac{1}{MLN^2} \sum_{\pi:\ l(\pi)=4} \operatorname{E}\left[\tilde{\Delta}_N(f(\pi,0))\tilde{\Delta}_N^*(f(\pi,1))\tilde{\Delta}_N(f(\pi,2))\tilde{\Delta}_N^*(f(\pi,3))\right]. \\ &= \frac{1}{MLN^2} \sum_{\{w:\ l(w)=4\}} \sum_{\pi \in \Pi(w)} \operatorname{E}\left[\tilde{\Delta}_N(f(\pi,0)) \ldots \tilde{\Delta}_N^*(f(\pi,3))\right] \\ &= \frac{1}{MLN^2} \sum_{w \in \mathcal{P}_4} \sum_{\pi \in \Pi(w)} \operatorname{E}\left[\tilde{\Delta}_N(f(\pi,0)) \ldots \tilde{\Delta}_N^*(f(\pi,3))\right] + R \ \ (say). \end{aligned} \tag{3.17}$$

By applying the standard arguments used so far, the term R tends to 0. The first term is

$$O\left(\frac{1}{MLN^2} \sum_{w \in \mathcal{P}_4} \#\Pi(w)\right).$$

However, $\#\Pi(w) = O(L^2ML)$ or $O(L(ML)^2)$ depending on the word having 1 or 2 generating vertices. In each of the cases

$$\frac{\#\Pi(w)}{MLN^2} = O(\frac{L(ML)^2}{MLN}) = O(\frac{1}{M}\frac{(ML)^2}{N^2}).$$

As $M \to \infty$, the R.S. of (3.17) goes to 0. As a consequence, the expectation of L.S. of (3.16) goes to 0.

Now we verify (M4). As the number of columns in $\tilde{\Delta}_N$ is lesser than N, the number of jointly matched and cross matched circuits for $\tilde{\Delta}_N\tilde{\Delta}_N^*$ is lesser than that of the matrix $\tilde{W}_N\tilde{W}_N^*$. On the other hand the dimension of $\tilde{\Delta}_N\tilde{\Delta}_N^*$ is still $ML \times ML$. As a consequence, the arguments of verification of (M4) condition in Theorem 2.1 still hold here. Hence (3.16) is satisfied. As we have already proved the result for $\tilde{W}_N\tilde{W}_N^*$, the proof is complete. □

The following variant of Theorem 2.2 is now quite straightforward to prove.

Theorem 3.1. *Suppose and* $m, n \to \infty$, $\mathbb{B}_n(m)$ *satisfy and condition* 1 *of Theorem 2.2 holds. Suppose further that the blocks* $\{A_{n,i}\}_{i\in\mathbb{Z}^2}$ *are symmetric, of same pattern and*

1. *For any* $1 \le p_1, p_2, p_3 \le n$, $A_{n,(i_1,i_2)}(p_1, p_2) \ne A_{n,(i_1,i_3)}(p_1, p_3)$ *if* $i_2 \ne i_3$.
2. *For any* $w \in C_{2k}$, $p(w) = 1$ *for the patterned matrix* $\frac{1}{\sqrt{n}}A_{n,(1,1)}$ *as* $n \to \infty$.

Then the LSD of $\frac{1}{\sqrt{mn}}\mathbb{B}_n(m)$ *is semicircular.*

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