# INDIAN STATISTICAL INSTITUTE

## Theoretical Statistics and Mathematics Unit, Kolkata

## Thesis Defence Seminar

**Date: December 10, 2024**
**Time: 11:00 AM**

**VENUE/MODE: Online**

**Link :** https://meet.google.com/dfn-obce-uau

## TITLE:

**$\mathbb{A}^1$-homotopy types of $\mathbb{A}^2$ and $\mathbb{A}^2 \setminus \{(0,0)\}$**

## SPEAKER:


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## ABSTRACT:


Morel-Voevodsky developed $\mathbb{A}^1$-homotopy theory which is a bridge between algebraic geometry and algebraic topology. They defined homotopy theory on the category of simplicial presheaves on $Sm/k$ ($Sm/k$ is the category of smooth varieties over a field $k$), where the affine line $\mathbb{A}^1_k = Spec\ k[T]$ plays the role of unit interval $[0,1]$. An algebro-geometric analogue of topological connectedness is $\mathbb{A}^1$-connectedness. Asok-Morel, Balwe-Hogadi-Sawant showed that a smooth proper $k$-variety $X$ is $\mathbb{A}^1$-connected if and only if for every finitely generated separable field extension $F/k$, any two $F$-points of $X$ can be joined by a chain of $\mathbb{A}^1_F$'s (affine lines over $F$). In the first part of the talk, we will discuss that the $\mathbb{A}^1$-connected component of a variety $X$ contains the information about the existence of affine lines in $X$. Using ghost homotopy techniques, introduced by Balwe-Hogadi-Sawant, we have proved that if a smooth variety $X$ over an algebraically closed field $k$ is $\mathbb{A}^1$-connected, then $X$ is $\mathbb{A}^1$-uniruled i.e $X$ admits a dominant generically finite morphism $\mathbb{A}^1_k \times W \to X$, for some $k$-variety $W$. Therefore, a smooth $\mathbb{A}^1$-connected variety over an algebraically closed field $k$ of characteristic zero has negative logarithmic Kodaira dimension. As a consequence using Miyanishi-Sugie's algebraic characterisation of the affine plane, we have proved that $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible smooth affine surface over a field $k$ of characteristic zero. We will also discuss some useful consequences of this result.

The quasi-affine varieties $\mathbb{A}^n_k \setminus \{(0,\ldots,0)\}$ are analouges of spheres in $\mathbb{A}^1$-homotopy theory. In the second part of the talk, we will discuss on $\mathbb{A}^1$-homotopy type of $\mathbb{A}^n_k \setminus \{(0,\ldots,0)\}$. We have proved that in dimension 2, over a field $k$ of characteristic zero $\mathbb{A}^2_k \setminus \{(0,0)\}$ is the only open $k$-subvariety of a smooth affine $k$-surface, which is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^2_k \setminus \{(0,0)\}$. We have also shown that if $X$ is a Koras-Russell threefold of the first kind then, $X \setminus \{p\}$ ($p$ is a $k$-rational point of $X$) is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$ but $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$. These are joint works with Prof. Utsav Choudhury.


**ALL ARE CORDIALLY INVITED**