# INDIAN STATISTICAL INSTITUTE

Theoretical Statistics and Mathematics Unit, Kolkata

## Pre-Thesis Submission Seminar

**Date: May 15, 2024**
**Time: 03:00 PM**

**VENUE:**

**L-infinity**

**(5th Floor, A.N. Kolmogorov Bhavan), ISI Kolkata**

**TITLE:**

A^1-homotopy theory and Characterisation of A^2

**SPEAKER:**


**Biman Roy**

SRF, Stat-Math Unit, ISI Kolkata


**ABSTRACT:**

Attached below.

**ALL ARE CORDIALLY INVITED**

# ABSTRACT:

Characterisation of the affine $n$-space is one of the major problem in affine algebraic geometry. Miyanishi-Sugie showed that a smooth affine surface $X$ over an uncountable algebraically closed field $k$ of characteristic zero is isomorphic to $\mathbb{A}^2_k$ if $\mathcal{O}(X)$ is a U.F.D., $\mathcal{O}(X)^* = k^*$ and $X$ has a non-trivial $\mathbb{G}_a$-action. The last condition can be replaced by negativity of logarithmic Kodaira dimension. Since the orbits of a $\mathbb{G}_a$-action are the affine lines, existence of a non-trivial $\mathbb{G}_a$-action says that there is a non-constant $\mathbb{A}^1$ in $X$. Also by the results of Keel-Mckernan, Miyanishi negativity of the logarithmic Kodaira dimension of $X$ over such field $k$ is equivalent to the existence of a cylinder (i.e. $X$ contains an open subset isomorphic to $\mathbb{A}^1_k \times W$) in $X$. Ramanujam showed that a smooth complex surface is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ if it is topologically contractible and it is simply connected at infinity. Topological contractibility, in particular path connectedness says that there are non-constant intervals in $X$. On the other hand, $\mathbb{A}^1$-homotopy theory, developed by F.Morel and V.Voevodsky is a bridge between algebraic geometry and algebraic topology. An algebro-geometric analogue of topological connectedness is $\mathbb{A}^1$-connectedness. Asok-Morel, Balwe-Hogadi-Sawant showed that a smooth proper $k$-variety $X$ is $\mathbb{A}^1$-connected if and only if for every finitely generated separable field extension $F/k$, any two $F$-points of $X$ can be joined by a chain of $\mathbb{A}^1_F$'s (affine lines over $F$). Using ghost homotopy techniques, introduced by Balwe-Hogadi-Sawant, we have proved that if a smooth variety $X$ over an algebraically closed field $k$ is $\mathbb{A}^1$-connected, then $X$ is $\mathbb{A}^1$-uniruled i.e $X$ contains a family of affine lines. Therefore, any smooth $\mathbb{A}^1$-connected variety over an algebraically closed field $k$ of characteristic zero has negative logarithmic Kodaira dimension. As a consequence using the algebraic characterisation, we have proved that $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible smooth affine surface over a field $k$ of characteristic zero. We will also see some useful consequences of this result.

The quasi-affine varieties $\mathbb{A}^n_k \setminus \{(0,\ldots,0)\}$ are analouges of spheres in $\mathbb{A}^1$-homotopy theory. In the second part of the talk, we will discuss about the $\mathbb{A}^1$-homotopy type of $\mathbb{A}^n_k \setminus \{(0,\ldots,0)\}$. We have proved that in dimension 2, over a field $k$ of characteristic zero $\mathbb{A}^2_k \setminus \{(0,0)\}$ is the only open $k$-subvariety of a smooth affine $k$-surface, which is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^2_k \setminus \{(0,0)\}$. We have also shown that if $X$ is a Koras-Russell threefold of the first kind then, $X \setminus \{p\}$ ($p$ is a $k$-rational point of $X$) is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$ but $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$.