

Tight Chang’s-lemma-type bounds for Boolean functions

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Abstract

Chang’s lemma (Duke Mathematical Journal, 2002) is a classical result in mathematics, with applications spanning across additive combinatorics, combinatorial number theory, analysis of Boolean functions, communication complexity and algorithm design. For a Boolean function f that takes values in $\{-1, 1\}$ let $r(f)$ denote its Fourier rank (i.e., the dimension of the span of its Fourier support). For each positive threshold t , Chang’s lemma provides a lower bound on $\delta(f) := \Pr[f(x) = -1]$ in terms of the dimension of the span of its characters with Fourier coefficients of magnitude at least $1/t$. In this work we examine the tightness of Chang’s lemma with respect to the following three natural settings of the threshold:

- the Fourier sparsity of f , denoted $k(f)$,
- the Fourier max-supp-entropy of f , denoted $k'(f)$, defined to be the maximum value of the reciprocal of the absolute value of a non-zero Fourier coefficient,
- the Fourier max-rank-entropy of f , denoted $k''(f)$, defined to be the minimum t such that characters whose coefficients are at least $1/t$ in magnitude span a $r(f)$ -dimensional space.

In this work we prove new lower bounds on $\delta(f)$ in terms of the above measures. One of our lower bounds, $\delta(f) = \Omega(r(f)^2/(k(f) \log^2 k(f)))$, subsumes and refines the previously best known upper bound $r(f) = O(\sqrt{k(f) \log k(f)})$ on $r(f)$ in terms of $k(f)$ by Sanyal (Theory of Computing, 2019). We improve upon this bound and show $r(f) = O(\sqrt{k(f) \delta(f) \log k(f)})$. Another lower bound, $\delta(f) = \Omega(r(f)/(k''(f) \log k(f)))$, is based on our improvement of a bound by Chattopadhyay, Hatami, Lovett and Tal (ITCS, 2019) on the sum of absolute values of level-1 Fourier coefficients in terms of \mathbb{F}_2 -degree. We further show that Chang’s lemma for the above-mentioned choices of the threshold is asymptotically outperformed by our bounds for most settings of the parameters involved.

Next, we show that our bounds are tight for a wide range of the parameters involved, by constructing functions witnessing their tightness. All the functions we construct are modifications of the Addressing function, where we replace certain input variables by suitable functions. Our final contribution is to construct Boolean functions f for which our lower bounds asymptotically match $\delta(f)$, and for any choice of the threshold t , the lower bound obtained from Chang’s lemma is asymptotically smaller than $\delta(f)$.

Our results imply more refined deterministic one-way communication complexity upper bounds for XOR functions. Given the wide-ranging application of Chang’s lemma to areas like additive

¹ Work mostly done while the author was a postdoc at Georgetown University.



46 combinatorics, learning theory and communication complexity, we strongly feel that our refinements
47 of Chang’s lemma will find many more applications.

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58 1 Introduction

59 Chang’s lemma [8, 14] is a classical result in additive combinatorics. Informally, the lemma
60 states that all the large Fourier coefficients of the indicator function of a large subset of
61 an Abelian group reside in a low dimensional subspace. The discovery of this lemma was
62 motivated by an application to improve Frieman’s theorem on set additions [8]. The lemma
63 has subsequently found many applications in additive combinatorics and combinatorial
64 number theory. Chang’s lemma and the ideas developed in Chang’s paper [8] have been used
65 to prove theorems about arithmetic progressions in sumsets [12, 23], structure of Boolean
66 functions with small spectral norm [16], and improved bounds for Roth’s theorem on three-
67 term arithmetic progressions in the integers [24, 4, 5]. Green and Ruzsa [15] used the ideas of
68 Chang’s lemma to prove a generalization of Frieman’s theorem for arbitrary Abelian groups.
69 The lemma is known to be sharp for various settings of parameters for the group \mathbb{Z}_N [13].

70 In this paper, our focus is a specialization of Chang’s lemma for the Boolean hypercube.
71 Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a Boolean function. For any positive real number t (which
72 we refer to as the *threshold*) define $\mathcal{S}_t := \{S \subseteq [n] : |\hat{f}(S)| \geq \frac{1}{t}\}$.^{2,3} Viewing elements of
73 \mathcal{S}_t as vectors in \mathbb{F}_2^n , Chang’s lemma gives a lower bound on $\delta(f) := \Pr[f(x) = -1]$ (called
74 the weight of f), in terms of t and the dimension of the span of \mathcal{S}_t (denoted by $\dim(\mathcal{S}_t)$).
75 Formally, we have the following lemma, referred to as Chang’s lemma in this paper. In the
76 literature Chang’s lemma is more commonly stated as an upper bound on d in terms of $\delta(f)$
77 and t . We refer the reader to the full version of our paper for a proof of the equivalence of
78 the two forms, and also for other missing proofs from this paper.

79 ► **Lemma 1.1** (Chang’s lemma [8]). *There exists a universal constant $c > 0$ such that the*
80 *following is true for every integer $n > 0$. Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function and t*
81 *be any positive real number. Let $\delta(f) := \Pr_x[f(x) = -1]$ and $d = \dim(\mathcal{S}_t) > 1$. If $\delta(f) < c$,*
82 *then $\delta(f) = \Omega\left(\frac{\sqrt{d}}{t\sqrt{\log(t^2/d)}}\right)$.*

83 This lemma has found numerous applications in complexity theory and algorithms [2, 7],

² The function f is implicit in the definition of \mathcal{S}_t and will be clear from context.

³ We refer the reader to Section A for preliminaries on Fourier analysis.

84 analysis of Boolean functions [16, 26], communication complexity [26, 19] and extremal
85 combinatorics [10]. See [20] for a proof of Lemma 1.1.

86 In this paper, we investigate the tightness of Lemma 1.1 for three natural choices of
87 the threshold t based on the Fourier spectrum of the function (see Section 1.1 for details
88 about these thresholds). We prove additional lower bounds on $\delta(f)$, and compare relative
89 performances of all the bounds under consideration. Our results imply that the bounds
90 given by Chang's lemma for the choices of the threshold that we consider are asymptotically
91 outperformed by one of the bounds we prove for a broad range of the parameters involved.
92 For most regimes of the parameters we are able to construct classes of functions that witness
93 the tightness of our bounds.

94 Interestingly, for each choice of threshold that we consider, $\dim(\mathcal{S}_t)$ equals the Fourier
95 rank of f (denoted by $r(f)$, see Definition A.12). In particular, setting t to be the Fourier
96 sparsity of f (denoted by $k(f)$) leads to a very natural question about the relationship among
97 $r(f)$, $k(f)$ and $\delta(f)$ for a Boolean function f . The best known upper bound on $r(f)$ in terms
98 of $k(f)$ is $r(f) = O(\sqrt{k(f)} \log k(f))$ [25]. We improve upon this bound by incorporating $\delta(f)$
99 into it, and show $r(f) = O(\sqrt{k(f)} \delta(f) \log k(f))$. Moreover, we also show that this bound is
100 tight; see Section 1.2 for a detailed discussion.

101 Throughout this paper, we assume that f is not a constant function or a parity or a
102 negative parity (unless mentioned otherwise). In other words, $k(f), r(f) \geq 2$.

103 1.1 Thresholds considered for Chang's lemma

104 For a Boolean function f , let $\text{supp}(f)$ denote the Fourier support of f . In this section, we
105 discuss and motivate the choices of the threshold t considered in this work.

106 **a) The Fourier sparsity of f .** It was shown in [11, Theorem 3.3] that for all $S \in$
107 $\text{supp}(f)$, $|\hat{f}(S)| \geq \frac{1}{k(f)}$. It follows that $\mathcal{S}_{k(f)} = \text{supp}(f)$ and hence $\dim(\mathcal{S}_{k(f)}) = r(f)$.
108 Moreover, there exist functions (e.g. $f = \text{AND}_n$) for which $\dim(\mathcal{S}_t) = 0$ for $t = o(k(f))$,
109 justifying the choice of threshold $k(f)$.

110 This choice also leads us to a fundamental structural problem of bounding the weight
111 of a Boolean function f from below, in terms of its Fourier sparsity and Fourier rank. The
112 *uncertainty principle* (see, for example, [17] for a statement and a proof) asserts that $\delta(f) =$
113 $\Omega\left(\frac{1}{k(f)}\right)$. Chang's lemma with $t = k(f)$ and the fact that $\log(k(f)^2/r(f)) = \Theta(\log k(f))$
114 (Lemma A.16 (part 1)) implies that

$$115 \quad \delta(f) = \Omega\left(\frac{1}{k(f)} \sqrt{\frac{r(f)}{\log k(f)}}\right), \quad (1)$$

116 thereby subsuming the uncertainty principle (note that $r(f)/\log k(f) \geq 1$) and refining it by
117 incorporating $r(f)$ into the bound.

118 **b) The Fourier max-supp-entropy of f .** The next choice of the threshold that we
119 consider is the *Fourier max-supp-entropy* of f , denoted by $k'(f)$, which we define to be
120 $\max_{S \in \text{supp}(f)} \frac{1}{|\hat{f}(S)|}$ (Definition A.15). By its definition $k'(f)$ is the smallest value of t such
121 that $\mathcal{S}_t = \text{supp}(f)$. Since $k'(f) \leq k(f)$ (see the discussion in the last item), the knowledge
122 of $k'(f)$ can potentially offer us a more fine-grained lower bound on $\delta(f)$ than as in the
123 last item; Chang's lemma with $t = k'(f)$ and $\log(k'(f)^2/r(f)) = \Theta(\log k'(f))$ (Lemma A.16
124 (part 2)) implies

$$125 \quad \delta(f) = \Omega\left(\frac{1}{k'(f)} \sqrt{\frac{r(f)}{\log k'(f)}}\right). \quad (2)$$

126 Notice that Equation (2) subsumes the bound in Equation (1).

127 In [18] an equivalent statement of the well-known sensitivity conjecture was presented
128 in terms of $k'(f)$.⁴ Granularity is another widely-studied measure that is closely associated
129 with Fourier max-supp-entropy.

130 **c) The Fourier max-rank-entropy of f .** Our final choice of the threshold is the
131 *Fourier max-rank-entropy of f* , denoted by $k''(f)$, which we define to be the smallest positive
132 real number t such that $\dim(\mathcal{S}_t) = r(f)$ (Definition A.15). We have that $k''(f) \leq k'(f) \leq k(f)$
133 by their definitions. Amongst all settings of the threshold t for which $\dim(\mathcal{S}_t) = r(f)$, the
134 value $t = k''(f)$ yields the best lower bound from Chang's lemma. Chang's lemma with
135 $t = k''(f)$ implies

$$136 \quad \delta(f) = \Omega \left(\frac{1}{k''(f)} \sqrt{\frac{r(f)}{\log(k''(f)^2/r(f))}} \right), \quad (3)$$

137 which subsumes the bounds in Equations (2) and (1).

138 1.2 Our contributions

139 We prove the following results regarding the three natural instantiations of the threshold t
140 (mentioned in the preceding section) for Chang's lemma.

141 **a) The Fourier sparsity of f :** Recall that Chang's lemma with threshold $t = k(f)$
142 (Equation (1)) implies that $\delta(f) = \Omega \left(\frac{1}{k(f)} \sqrt{\frac{r(f)}{\log k(f)}} \right)$. It was shown in [1] that $\delta(f) =$
143 $\Omega \left(\frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)} \right) \right)$, improving upon this bound asymptotically (note that $r(f)/\log k(f) \geq 1$).
144 In this work we improve their bound further.

145 **► Theorem 1.2.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that $k(f) > 1$. Then*
146 $\delta(f) = \Omega \left(\frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)} \right)^2 \right)$.

147 Observe that the statement of Theorem 1.2 is equivalent to $r(f) = O(\sqrt{k(f)\delta(f)} \log k(f))$.
148 This bound subsumes the bound $r(f) = O(\sqrt{k(f)} \log k(f))$ shown by Sanyal [25]. We prove
149 Theorem 1.2 by incorporating $\delta(f)$ in Sanyal's arguments and thereby refining his proof. See
150 Section 2.1 for the proof of Theorem 1.2.

151 We also show that Theorem 1.2 is tight. For nearly all admissible values of ρ and κ
152 we construct many Boolean functions f such that $k(f) = O(\kappa)$, $r(f) = O(\rho)$ and $\delta(f) =$
153 $O \left(\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa} \right)^2 \right)$ (Theorem 1.4 and Claim B.2). For a comparison with Sanyal's bound see
154 Section 1.3.

155 **b) The Fourier max-supp-entropy of f :** Recall from Section 1.1 that the Fourier
156 max-supp-entropy of f , denoted $k'(f)$, is defined as $k'(f) = \max_{S \in \text{supp}(f)} \frac{1}{|\widehat{f}(S)|}$. It can be
157 shown that $\sqrt{k(f)} \leq k'(f) \leq k(f)/2$ (Lemma A.16 (part 2)). We prove the following lower
158 bound.

159 **► Theorem 1.3.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that $k(f) > 1$. Then,*
160 $\delta(f) = \Omega \left(\max \left\{ \frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)} \right)^2, \frac{k(f)}{k'(f)^2} \right\} \right)$.

⁴ In [18] $\log(k'(f)^2)$ is called the Fourier max-entropy while we refer to $k'(f)$ as the Fourier max-supp-entropy.

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161 As is evident from the statement, Theorem 1.3 presents two lower bounds, one of which is
 162 Theorem 1.2. The other lower bound $\delta(f) \geq \frac{k(f)}{k'(f)^2}$ is Claim A.17.

163 Chang's lemma with the threshold t set to $k'(f)$ (Equation (2)), together with the
 164 observation that $\log k(f) = \Theta(\log k'(f))$, implies $\delta(f) = \Omega\left(\frac{1}{k'(f)} \sqrt{\frac{r(f)}{\log k(f)}}\right)$. Theorem 1.3
 165 subsumes this bound since

$$166 \quad \delta(f) = \Omega\left(\frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)}\right)^2 \cdot \frac{k(f)}{k'(f)^2}\right)^{1/2} = \Omega\left(\frac{1}{k'(f)} \sqrt{\frac{r(f)}{\log k(f)}}\right),$$

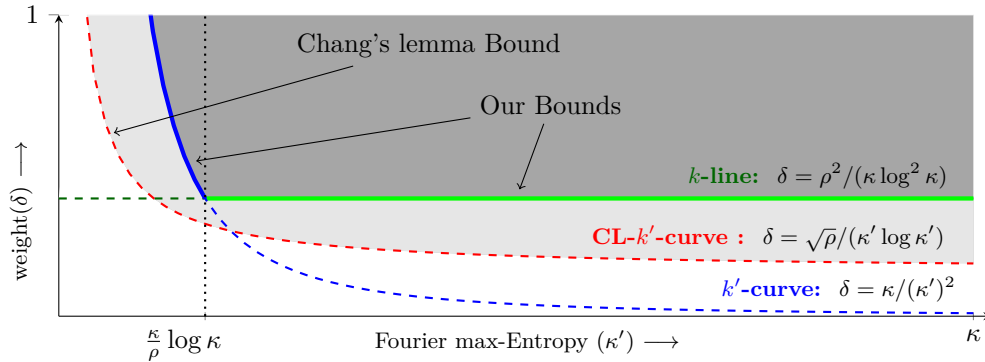
167 where the equality follows from $r(f)/\log k(f) \geq 1$.

168 In addition, observe from the last equality above that the bound of Theorem 1.3 is asymp-
 169 totically larger than the bound obtained from Chang's lemma for $t = k'(f)$ (Equation (2))
 170 except when $r(f)/\log k(f) = \Theta(1)$. Theorem 1.4 complements Theorem 1.3 by showing that
 171 for nearly all admissible values of $r(f), k(f)$ and $k'(f)$, there exists a function for which the
 172 larger of the two bounds presented in Theorem 1.3 is tight.

► **Theorem 1.4.** *For all $\rho, \kappa, \kappa' \in \mathbb{N}$ such that κ is sufficiently large, for all constants $\epsilon > 0$ such that $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ and $\kappa^{\frac{1}{2}} \leq \kappa' \leq \kappa$, there exists a Boolean function $f_{\rho, \kappa, \kappa'}$ such that $r(f_{\rho, \kappa, \kappa'}) = \Theta(\rho)$, $k(f_{\rho, \kappa, \kappa'}) = \Theta(\kappa)$, $k'(f_{\rho, \kappa, \kappa'}) = \Theta(\kappa')$ and*

$$\delta(f_{\rho, \kappa, \kappa'}) = \Theta\left(\max\left\{\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa}\right)^2, \frac{\kappa}{\kappa'^2}\right\}\right).$$

173 The range of parameters considered in Theorem 1.4 is justified by Lemma A.16. We prove
 174 Theorem 1.4 in two parts. Fix any ρ, κ such that $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ for some constant $\epsilon > 0$.
 175 First, for each value of $\kappa' \in [\frac{\kappa \log \kappa}{\rho}, \kappa]$ we construct a function f for which the first lower
 176 bound on $\delta(f)$ from Theorem 1.3 is tight (Claim B.2). Next, for each value of $\kappa' \in [\kappa^{\frac{1}{2}}, \frac{\kappa \log \kappa}{\rho}]$
 177 we construct a function f for which the second lower bound on $\delta(f)$ from Theorem 1.3 is
 178 tight (Claim B.3). See Figure 1 for a graphical visualization of the bounds in Theorem 1.3
 179 for any fixed values of ρ and κ .



■ **Figure 1** This plot is constructed for any fixed values of ρ, κ for which $\log \kappa \leq \rho \leq \sqrt{\kappa}$, and depicts the relationship between $\delta(f)$ and $k'(f)$ for functions f with $r(f) = \Theta(\rho)$ and $k(f) = \Theta(\kappa)$. For any fixed values of ρ, κ , we will refer to this plot as the (ρ, κ) - k' -plot. Chang's lemma implies that Boolean functions lie above both the k -line and the k' -curve, highlighted by the dark grey region in the figure. Roughly speaking, Theorem 1.4 exhibits functions that lie on the boundary of the dark grey region described by the k -line and the k' -curve.

180 **c) The Fourier max-rank-entropy of f :** Recall from Section 1.1 that the Fourier
 181 max-rank-entropy of f , denoted $k''(f)$, is the smallest positive real number t such that

182 $\dim(\mathcal{S}_t) = r(f)$. It can be shown that $\max\left\{\sqrt{r(f)}, \frac{r(f)}{\log k(f)}\right\} \leq k''(f) \leq k(f)$ (Lemma A.16
183 (part 2)). We prove the following lower bound.

184 ► **Theorem 1.5.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that $k(f) > 1$. Then,*

$$185 \quad \delta(f) = \Omega\left(\max\left\{\frac{1}{k(f)}\left(\frac{r(f)}{\log k(f)}\right)^2, \frac{r(f)}{k''(f)\log k(f)}\right\}\right).$$

186 Theorem 1.5 yields a better lower bound than Chang's lemma with the threshold $t = k''(f)$
187 (Equation (3)), except when $r(f) < (\log k(f))^2$ (see the caption of Figure 2). Theorem 1.5
188 presents two lower bounds: the first one is Theorem 1.2, and the second one is Lemma 2.5.
189 Lemma 2.5 is proven by strengthening a bound due to [9] on the sum of absolute values
190 of level-1 Fourier coefficients of a Boolean function in terms of its \mathbb{F}_2 -degree. A proof of
191 Theorem 1.5 can be found in Section 2.2.

192 We also show that for nearly all admissible values of $r(f)$, $k(f)$ and $k''(f)$, there exist
193 functions for which the larger of the two bounds presented in Theorem 1.5 is nearly tight.

► **Theorem 1.6.** *For all $\rho, \kappa, \kappa'' \in \mathbb{N}$ such that κ is sufficiently large, for all $\epsilon > 0$ such
that $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ and $\rho \leq \kappa'' \leq \kappa$ there exists a Boolean function $f_{\rho, \kappa, \kappa''}$ such that
 $r(f_{\rho, \kappa, \kappa''}) = \Theta(\rho)$, $k(f_{\rho, \kappa, \kappa''}) = \Theta(\kappa)$, $k''(f_{\rho, \kappa, \kappa''}) = \Theta(\kappa'')$ and*

$$\delta(f_{\rho, \kappa, \kappa''}) = \Theta\left(\max\left\{\frac{1}{\kappa}\left(\frac{\rho}{\log \kappa}\right)^2, \frac{\rho}{\kappa'' \log(\kappa''/\rho)}\right\}\right).$$

194 The range of parameters considered in Theorem 1.6 is justified by Lemma A.16. The
195 theorem 1.6 is proved in two parts. Fix any ρ, κ such that $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ for some constant
196 $\epsilon > 0$. First, for each value of $\kappa'' \in [\frac{\kappa \log \kappa}{\rho}, \kappa]$ we construct a function f for which the first
197 lower bound on $\delta(f)$ from Theorem 1.5 is tight (Claim B.5). In fact these are the same
198 functions that are used to prove the first bound in Theorem 1.4. Next, for each value of
199 $\kappa'' \in [e\rho, \frac{\kappa \log \kappa}{\rho}]$ we construct a function f for which $\delta(f) = \Theta(\frac{\rho}{\kappa'' \log(\kappa''/\rho)})$ (Claim B.4).
200 From the above discussion one may verify that for every ρ, κ that we consider and for every
201 $\kappa'' \geq \rho \cdot \kappa^{\Omega(1)}$, the function that we construct witnesses tightness of the lower bound in
202 Theorem 1.5.

203 In general, for all settings of ρ, κ and κ'' that we consider, the upper bound on $\delta(f)$
204 from Theorem 1.6 is off by a factor of at most $O(\log \kappa)$ from the lower bound in Theorem 1.5.
205 See Figure 2 for a graphical visualization of the bounds in Theorem 1.5 for any fixed values
206 of ρ and κ .

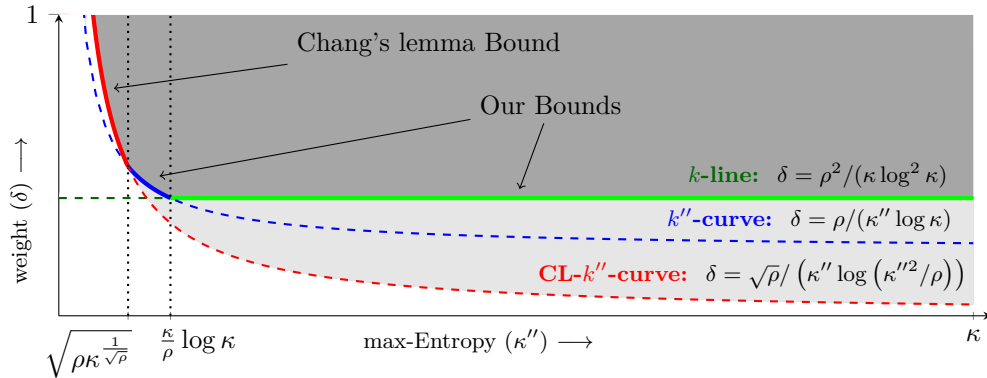
207 **Dominating Chang's lemma for all thresholds.** Our final contribution is to show
208 that there exists a function for which: our lower bounds (Theorem 1.3 and 1.5) asymptotically
209 match the weight, but for any choice of the threshold the lower bound obtained from Chang's
210 lemma (Lemma 1.1) is asymptotically smaller than the weight. See [6, Section 7] in the full
211 version of our paper for a proof of the below claim.

212 ► **Claim 1.7 (Beating Chang's lemma for all thresholds).** For any integer $t > 4$ there exists a
213 function $f : \{-1, 1\}^{\log t} \times \{-1, 1\}^{t \log t} \rightarrow \{-1, 1\}$ such that

214 ■ $\delta(f) = \frac{1}{t}$.

215 ■ For all real $x > 0$ for which $\dim(\mathcal{S}_x) > 1$, we have $\frac{\sqrt{\dim(\mathcal{S}_x)}}{x\sqrt{\log(x^2/\dim(\mathcal{S}_x))}} = O\left(\frac{1}{t^{3/2}}\right)$.

216 ■ $\frac{1}{k(f)}\left(\frac{r(f)}{\log k(f)}\right)^2 = \Omega\left(\frac{1}{t}\right)$, $\frac{k(f)}{k'(f)^2} = \Omega\left(\frac{1}{t}\right)$ and $\frac{r(f)}{k''(f)\log k(f)} = \Omega\left(\frac{1}{t}\right)$.



■ **Figure 2** This plot is constructed for any fixed values of ρ, κ for which $\log \kappa \leq \rho \leq \sqrt{\kappa}$, and depicts the relationship between $\delta(f)$ and $k''(f)$ for functions f with $r(f) = \Theta(\rho)$ and $k(f) = \Theta(\kappa)$. For any fixed values of ρ, κ , we will refer to this plot as (ρ, κ) - k'' -plot. Chang’s lemma implies that Boolean functions lie above the CL- k'' -curve. Theorem 1.5 improves upon Chang’s lemma and shows that Boolean functions lie above both the k -line and the k'' -curve, highlighted by the dark grey region in the figure. Although the picture indicates that the CL- k'' -curve is better than the k'' -curve for certain ranges of κ'' , this is actually only possible for certain values of ρ and κ . This is because the CL- k'' -curve and the k'' -curve intersect at $\sqrt{\rho \kappa^{1/\sqrt{\rho}}}$, which is less than $\sqrt{\rho}$ if $\rho \geq (\log \kappa)^2$. By Lemma A.16 we know that for any function f on this plot, the range of $k''(f)$ is between $\max\{\sqrt{\rho}, \rho / \log \kappa\}$ and κ . Thus our bounds in Theorem 1.5 dominate those given by the CL- k'' -curve in all (ρ, κ) - k'' plots where $\rho \geq \log^2 \kappa$.

217 In particular, Claim 1.7 shows that our bounds can be strictly stronger than those given
 218 by Chang’s lemma, in the following sense.

- 219 ■ All the lower bounds on $\delta(f)$ from Theorems 1.3 and 1.5 are tight, as witnessed by f
 220 from Claim 1.7.
- 221 ■ For the function f from Claim 1.7, no matter what threshold x is chosen in Lemma 1.1,
 222 the best possible lower bound on $\delta(f)$ that we get can get from Lemma 1.1 is $\Omega\left(\frac{1}{t^{3/2}}\right)$.
 223 This is polynomially smaller than $1/t$, the actual weight of f .

224 1.3 Applications of our results

225 An application of our result is an enhanced understanding of the bound $r(f) = O(\sqrt{k(f)} \log k(f))$
 226 proven by Sanyal [25]. This bound is a special case of Theorem 1.2 for $\delta(f) = \Theta(1)$. It is
 227 not known whether the $\log k(f)$ term is required in Sanyal’s upper bound on $r(f)$ (when f
 228 equals the Addressing function, $r(f) = \Omega(\sqrt{k(f)})$, see Definition A.10 and Observation A.19).
 229 For all the functions we construct witnessing the tightness of the bound in Theorem 1.2,
 230 $\delta(f) = o(1)$. We prove Theorem 1.2 by generalizing Sanyal’s proof. As stated before,
 231 our bound is tight in this generality, i.e. the logarithmic factor is required in the upper
 232 bound on $r(f)$. This sheds light on the presence of the logarithmic term in the bound
 233 $r(f) = O(\sqrt{k(f)} \log k(f))$.

234 Also, Fourier sparsity and Fourier rank of f have intimate connections with the commu-
 235 nication complexity of functions of the form $F := f \circ \text{XOR}$. The Fourier sparsity of f equals
 236 the real rank ($\text{rank}(M_F)$) of the communication matrix M_F of F , and the Fourier rank of f
 237 equals the deterministic (and even exact quantum) one-way communication complexity of
 238 F [22]. Theorem 1.2 thus implies an improved upper bound of $O(\sqrt{k(f)} \delta(f) \log k(f))$ on
 239 the one-way communication complexity of F in these models, which asymptotically beats
 240 the best known upper bound of $O(\sqrt{\text{rank}(M_F)})$ even for two-way protocols [26, 21], for the
 241 special case of functions of this form (when $\delta(f) = o(1/\log k)$).

242 Given the wide-ranging application of Chang's lemma to areas like additive combinatorics,
 243 learning theory and communication complexity, we strongly feel that our refinements of
 244 Chang's lemma will find many more applications.

245 **2 Lower bound proofs**

246 For lower bounds on $\delta(f)$ of a Boolean function f , we need to prove two theorems: The-
 247 orems 1.3 and 1.5. The proof of Theorem 1.3 is given in Section 2.1 and the proof of
 248 Theorem 1.5 is given in Section 2.2.

249 **2.1 Proof of Theorem 1.3 (and Theorem 1.2)**

250 Remember that we defined the *Fourier max-supp-entropy* of a Boolean function f , denoted
 251 by $k'(f)$, to be $\max_{S \in \text{supp}(f)} \frac{1}{|\widehat{f}(S)|}$.

252 The main aim of this section is to give a lower bound on $\delta(f)$ with respect to $k'(f)$ for a
 253 Boolean function f (Theorem 1.3).

254 We first prove Theorem 1.2 which implies Theorem 1.3 (together with Claim A.17).

255 Theorem 1.2 can be viewed as an upper bound of $O(\sqrt{k(f)\delta(f)} \log k(f))$ on the Fourier
 256 rank of f . In order to prove Theorem 1.2, we give an algorithm (Algorithm 1) which takes a
 257 Boolean function f as input and outputs a set of $O(\sqrt{\delta(f)k(f)} \log k(f))$ parities such that
 258 any assignment of these parities makes the function constant. From Observation A.14, this
 259 implies an upper bound of $O(\sqrt{\delta(f)k(f)} \log k(f))$ on Fourier rank of the function. We start
 260 by formally describing this algorithm. The central ingredient in the algorithm is a lemma
 261 in [26, Lemma 28].

262 **► Lemma 2.1 ([26]).** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ a function. There is an affine subspace*
 263 *$V \subseteq \{-1, 1\}^n$ of co-dimension at most $3\sqrt{\delta(f)k(f)}$ such that f is constant on V .*

264 Recall that for a function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, a set of parities Γ and an assignment
 265 $b \in \{-1, 1\}^\Gamma$, we define the restriction $f|_{(\Gamma, b)} := f|_{\{x \in \{-1, 1\}^n : \chi_\gamma(x) = b_\gamma \text{ for all } \gamma \in \Gamma\}}$. Also let
 266 $\mathcal{B}_\Gamma := \{b \in \{-1, 1\}^\Gamma : f|_{(\Gamma, b)} \text{ is not constant}\}$.

267 **Algorithm 1**

Input: A function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$.

Output: A set Γ of parities whose evaluation determines f .

Initialization: $f_{\min} \leftarrow f$, $\Gamma \leftarrow \emptyset$.

while \mathcal{B}_Γ *is non-empty* **do**

(a) **Update** Γ : Let Γ' be the smallest set of parities, such that, there exists $b \in \{-1, 1\}^{\Gamma'}$
 for which $f_{\min}|_{(\Gamma', b)}$ is constant,

$\Gamma \leftarrow \Gamma \cup \Gamma'$.

(b) **Update** f_{\min} : Define $b^* := \operatorname{argmin}_{b \in \mathcal{B}_\Gamma} \left\{ \frac{\delta(f|_{(\Gamma, b)})}{k(f|_{(\Gamma, b)})} \right\}$, and update

$f_{\min} \leftarrow f|_{(\Gamma, b^*)}$.

end

Return Γ .

268 Since number of parities are finite and we fix at least one parity at each iteration of
 269 Step (a) of the **while** loop, the algorithm terminates. The termination condition implies

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270 that the algorithm outputs a set of parities Γ such that for any assignment $b \in \{-1, 1\}^\Gamma$ of
 271 Γ , the restricted function $f_{(\Gamma, b)}$ becomes constant.

272 The only remaining step is to show is that the number of parities fixed in Algorithm 1
 273 is $O(\sqrt{\delta(f)}k(f) \log k(f))$. For this we define an equivalence relation and observe a few
 274 properties of restricted functions (restricted according to an assignment of a set of parities).

275 Equivalence relation for a set of parities

276 Let f be the input to Algorithm 1, first we define an equivalence relation given a set of
 277 parities over the variables of f . Given a set of parities Γ , define the following equivalence
 278 relation among parities in $\text{supp}(f)$.

$$279 \quad \forall \gamma_1, \gamma_2 \in \text{supp}(f), \gamma_1 \equiv \gamma_2 \text{ iff } \gamma_1 + \gamma_2 \in \text{span}(\Gamma). \quad (4)$$

280 Let ℓ be the number of equivalence classes according to the equivalence relation for Γ .
 281 For $j \in [\ell]$, let k_j be the size of the j -th equivalence class. Since the equivalence classes form
 282 a partition of $\text{supp}(f)$, we have

283 ► **Observation 2.2.** *Following the notation of the paragraph above, $\sum_{j=1}^{\ell} k_j = k(f)$.*

284 Let $\beta_1, \dots, \beta_\ell \in \text{supp}(f)$ be some representatives of the equivalence classes. For $j \in [\ell]$,
 285 let $\beta_j + \alpha_{j,1}, \dots, \beta_j + \alpha_{j,k_j}$ be the elements of the j -th equivalence class. This notation gives
 286 a compact representation of f in terms of these equivalence classes. For all $x \in \{-1, 1\}^n$,

$$287 \quad f(x) = \sum_{j=1}^{\ell} P_j(x) \chi_{\beta_j}(x), \quad (5)$$

288 where

$$290 \quad P_j(x) = \sum_{r=1}^{k_j} \widehat{f}(\beta_j + \alpha_{j,r}) \cdot \chi_{\alpha_{j,r}}(x). \quad (6)$$

291 Note that P_j are non-zero multilinear polynomials and depend only on the parities in Γ . So,
 292 fixing parities in Γ collapses all the parities in an equivalence class to their representative,
 293 thereby making P_j 's constant.

294 We will denote Γ after the i -th iteration of the **while** loop by $\Gamma^{(i)}$ (so $\Gamma^{(0)} = \emptyset$). Let $f_{\min}^{(i)}$
 295 be the selected function f_{\min} after the i -th iteration (thus $f_{\min}^{(0)} = f$).

296 With the above properties of restricted functions we are ready to prove the main technical
 297 lemma needed to show Theorem 1.2.

298 ► **Lemma 2.3.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ a function. Suppose Γ be a set of parities and
 299 ℓ be the number of equivalence classes of $\text{supp}(f)$ under the equivalence relation defined by
 300 in Equation (4). Then, there exists a $b \in \{-1, 1\}^\Gamma$ such that $f|_{(\Gamma, b)}$ is non-constant and
 301 $\frac{\delta(f|_{(\Gamma, b)})}{k(f|_{(\Gamma, b)})} \leq \frac{4k(f)\delta(f)}{\ell^2}$.*

302 **Proof.** For the sake of succinctness, when Γ is clear from the context, let $V_b = \{x \in \{-1, 1\}^n :
 303 \forall \gamma \in \Gamma, x_\gamma = b_\gamma\}$, for all $b \in \{-1, 1\}^\Gamma$, and $f|_b = f|_{\{x: x \in V_b\}}$.

304 Since we are interested in a non-constant $f|_b$, define $k_{\{\emptyset\}^c}(f)$ to be the number of non-zero
 305 non-empty monomials in Fourier representation of f . We first need to prove the following
 306 two bounds on the expected values of $\delta(f|_b)$ and $k_{\{\emptyset\}^c}(f|_b)$.

- 307 — $\mathbb{E}_b [\delta(f|_b)] = \delta(f)$,
- 308 — $\mathbb{E}_b [k_{\{\emptyset\}^c}(f|_b)] \geq \frac{\ell^2}{4k(f)}$.

310 **Expected value of $\delta(f|_b)$:**

311 Since $\{V_b : b \in \{-1, 1\}^{\Gamma(i)}\}$ form a partition on $\{-1, 1\}^n$ and all partitions are of the same
312 size, we get the expected value of $\delta(f|_b)$.

$$313 \quad \mathbb{E}_b [\delta(f|_b)] = \delta(f). \quad (7)$$

314 **Expected value of $k_{\{\emptyset\}^c}(f|_b)$:**

315 From Equation (5), for all $b \in \{-1, 1\}^\Gamma$ and for all $x \in \{-1, 1\}^n$,

$$316 \quad f|_b(x) = \sum_{j=1}^{\ell} P_j(b) \chi_{\beta_j}(x). \quad (8)$$

317

For each $j \in [\ell]$ and $b \in \{-1, 1\}^\Gamma$, let $I_j(b)$ be the indicator function for $P_j(b) \neq 0$,

$$I_j(b) = \begin{cases} 1 & \text{if } P_j(b) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

318 From Equation (6), each P_j is a polynomial having monomials $\{\chi_{\alpha_{j,r}} : r \in [k_j]\}$ with Fourier
319 sparsity of P_j being equal to k_j . Since each P_j is a non-zero polynomial, by Lemma A.2

$$320 \quad \mathbb{E}_b [I_j(b)] = \Pr_{b \sim \{-1, 1\}^\Gamma} [P_j(b) \neq 0] \geq \frac{1}{k_j}. \quad (9)$$

321

322 We calculate the expectation of $k_{\{\emptyset\}^c}(f|_b)$.

$$323 \quad \mathbb{E}_b [k_{\{\emptyset\}^c}(f|_b)] = \mathbb{E}_b \left[\sum_{j=1}^{\ell-1} I_j(b) \right] \quad \text{by Equation (8)}$$

$$324 \quad = \sum_{j=1}^{\ell-1} \mathbb{E}_b [I_j(b)] \quad \text{by linearity of expectation}$$

$$325 \quad \geq \sum_{j=1}^{\ell-1} \frac{1}{k_j} \quad \text{by Equation (9)}$$

$$326 \quad \geq \frac{(\ell-1)^2}{\sum_{j=1}^{\ell-1} k_j} \quad \text{by Cauchy-Schwarz inequality}$$

$$327 \quad \geq \frac{\ell^2}{4k(f)}. \quad \text{by Observation 2.2}$$

328

329 To finish the proof of the theorem, we use bounds on the two expected values,⁵

$$330 \quad \frac{\mathbb{E}_b [\delta(f|_b)]}{\mathbb{E}_b [k_{\{\emptyset\}^c}(f|_b)]} \leq \frac{4k(f)\delta(f)}{\ell^2}$$

$$331 \quad \iff \mathbb{E}_b \left[\delta(f|_{V_b}) - \frac{4k(f)\delta(f)}{\ell^2} k_{\{\emptyset\}^c}(f|_{V_b}) \right] \leq 0. \quad \text{by linearity of expectation}$$

⁵ this part of our proof is inspired by a proof of the Cheeger's inequality in spectral graph theory. See, for example, the proof of Fact 2 in <https://people.eecs.berkeley.edu/~luca/expanders2016/lecture04.pdf>.

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332
333

If $\delta(f|_{V_b}) - \frac{4k(f)\delta(f)}{\ell^2} k_{\{\emptyset\}^c}(f|_{V_b}) = 0$ for all b , then pick any non-constant $f|_b$. Otherwise, there exists a b_0 such that $\delta(f|_{V_{b_0}}) - \frac{4k(f)\delta(f)}{\ell^2} k_{\{\emptyset\}^c}(f|_{V_{b_0}}) < 0$. Since this equation can only be satisfied when $k_{\{\emptyset\}^c}(f|_{V_{b_0}}) > 0$, $f|_{V_{b_0}}$ is not constant. Dividing by $k_{\{\emptyset\}^c}(f|_{V_{b_0}})$,

$$\frac{\delta(f|_{b_0})}{k(f|_{b_0})} \leq \frac{\delta(f|_{b_0})}{k_{\{\emptyset\}^c}(f|_{b_0})} \leq \frac{4k(f)\delta(f)}{\ell^2},$$

334 and $f|_{b_0}$ is non-constant. ◀

335 Lemma 2.3 allows us to bound the number of parities fixed in the i -th iteration (in terms
336 of the decrease in number of equivalence classes).

► **Lemma 2.4.** *Suppose f is given as input to Algorithm 1. Consider the i -th iteration of Algorithm 1. Let q_i be the number of parities fixed in Step (a) of the i -th iteration of the while loop, and ℓ_i be the number of equivalence classes after Step (a) of the i -th iteration. Then*

$$\frac{q_i}{(\ell_{i-1} - \ell_i)} \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}}.$$

337 **Proof.** Recall that $\Gamma = \Gamma^{(i)}$ after the i -th of Step (a) of Algorithm 1. Again, for the sake
338 of succinctness, let $V_b = \{x \in \{-1, 1\}^n : \forall \gamma \in \Gamma^{(i)}, x_\gamma = b_\gamma\}$, for all $b \in \{-1, 1\}^{\Gamma^{(i)}}$, and
339 $f|_b = f|_{\{x: x \in V_b\}}$. Let f_{\min} be the function chosen after the i -th iteration of Step (b) of
340 Algorithm 1. Since Step (b) of Algorithm 1 chooses f_{\min} to be a non-constant function such
341 that weight-to-sparsity ratio is minimized, from Lemma 2.3 we have,

$$342 \frac{\delta(f_{\min})}{k(f_{\min})} \leq \frac{4k(f)\delta(f)}{\ell_{i-1}^2}. \quad (10)$$

344 Write every $f|_b$ as in Equation (5), and define $\mathcal{S}^{(i)} := \bigcup_{b \in \{-1, 1\}^{\Gamma^{(i)}}} \text{supp}(f|_b)$. We now
345 prove that $|\mathcal{S}^{(i)}| = \ell_i$.

- 346 ■ $|\mathcal{S}^{(i)}| \leq \ell_i$: Follows from the representation in Equation (5), since each $\text{supp}(f|_b)$ is a
347 subset of $\{\chi_{\beta_j^{(i)}} \mid j \in [\ell_i]\}$.
- 348 ■ $|\mathcal{S}^{(i)}| \geq \ell_i$: Since $P_j^{(i)}$ is a non-zero polynomial, there exists an assignment to parities in
349 $\Gamma^{(i)}$, such that, $P_j^{(i)}$ is non-zero. Thus, for all $j \in [\ell_i]$, we have $\chi_{\beta_j^{(i)}} \in \mathcal{S}^{(i)}$.

350 Since $|\mathcal{S}^{(i)}| = \ell_i$, Lemma 2.1 guarantees that $q_i \leq 3\sqrt{k(f_{\min})\delta(f_{\min})}$. Since f_{\min} becomes
351 constant after fixing these q_i parities, every parity in $\text{supp}(f_{\min})$ is paired with at least one
352 other parity in $\text{supp}(f_{\min})$ for the equivalence class with respect to $\Gamma^{(i)}$.⁶ This implies that
353 $\ell_{i-1} - \ell_i \geq \frac{k(f_{\min})}{2}$. Combining the two inequalities in the last paragraph we have,

$$354 \frac{q_i}{(\ell_{i-1} - \ell_i)} \leq 6\sqrt{\frac{\delta(f_{\min})}{k(f_{\min})}}.$$

356 From Equation (10),

$$357 \frac{q_i}{(\ell_{i-1} - \ell_i)} \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}}. \quad (11)$$

359 ◀

⁶ There is a boundary case ($k(f) = 1$) which can be dealt with separately, as in [25, Lemma 3.4]. For readability, we assume $k(f) \geq 2$.

360 We are now ready to prove Theorem 1.2.

361 **Proof of Theorem 1.2.** We only need to show that the number of parities fixed in Algorithm 1
 362 is $O(\sqrt{\delta(f)k(f)} \log k(f))$ (Observation A.14). Suppose the while loop runs for t iterations.
 363 Let q_i be the number of queries made in Step (a) of Algorithm 1 in the i -th iteration. From
 364 Lemma 2.3, we have

$$365 \quad q_i \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}} (\ell_{i-1} - \ell_i)$$

367 Thus when Algorithm 1 is run of f , the total number of queries made by the algorithm is

$$368 \quad \sum_{i=1}^t q_i \leq 6\sqrt{\delta(f)k(f)} \sum_{i=1}^t \frac{(\ell_{i-1} - \ell_i)}{\ell_{i-1}}$$

$$369 \quad \leq 6\sqrt{\delta(f)k(f)} \sum_{i=1}^t \left(\frac{1}{\ell_{i-1}} + \frac{1}{\ell_{i-1}-1} \cdots + \frac{1}{\ell_i+1} \right)$$

$$370 \quad \leq 6\sqrt{\delta(f)k(f)} \sum_{i=1}^{\ell_0} \frac{1}{i}$$

$$371 \quad \leq 6\sqrt{\delta(f)k(f)} \log \ell_0 = 6\sqrt{\delta(f)k(f)} \log k(f).$$

373 Observation A.14 implies $r(f) = O(\sqrt{\delta(f)k(f)} \log k(f))$. ◀

374 Along with Theorem 1.2, this proves Theorem 1.3.

375 **Proof of Theorem 1.3.** The bound $\delta(f) = \Omega\left(\frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)}\right)^2\right)$ follows from Theorem 1.2
 376 and the bound $\delta(f) = \Omega\left(\frac{k(f)}{(k'(f))^2}\right)$ from Claim A.17. ◀

377 2.2 Proof of Theorem 1.5

378 Recall that we defined *max-rank-entropy* of a Boolean function f , denoted by $k''(f)$, to be
 379 $\arg\min_t \{\dim(\mathcal{S}_t)\} = r(f)$. The main aim of this section is to give a lower bound on $\delta(f)$ with
 380 respect to $k''(f)$ for a Boolean function f (Theorem 1.5). The second bound of Theorem 1.5
 381 is given by the following lemma.

382 ▶ **Lemma 2.5.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that $k(f) > 1$. Then,*
 383 $\delta(f) = \Omega\left(\frac{r(f)}{k''(f) \log k(f)}\right)$.

384 Together with Theorem 1.2 proved in Section 2.1, Lemma 2.5 implies Theorem 1.5. We
 385 now give the proof of Lemma 2.5.

386 Lemma 2.5 gives a lower bound of $\Omega\left(\frac{r(f)}{k''(f) \log k(f)}\right)$ on $\delta(f)$. The crucial ingredient for
 387 this lower bound is Lemma 2.7, which is a refinement of the following theorem.

388 ▶ **Theorem 2.6** ([9, Theorem 13]). *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that*
 389 $\deg_{\mathbb{F}_2}(f) = d$. *Then, $\sum_{i \in [n]} |\hat{f}(\{i\})| \leq 4d$.*

390 ▶ **Lemma 2.7.** *For any Boolean function f , $\sum_{i=1}^n |\hat{f}(i)| = O(\delta(f) \deg_{\mathbb{F}_2}(f))$.*

391 The proof of Lemma 2.7 for a Boolean function f essentially applies Theorem 2.6 on
 392 the xor of disjoint copies of f . The only difference in the statement of Lemma 2.7 and
 393 Theorem 2.6 is that the right hand side becomes $O(\delta(f) \cdot \deg_{\mathbb{F}_2}(f))$ instead of $4\deg_{\mathbb{F}_2}(f)$.

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394 **Proof of Lemma 2.7.** Assume $\delta(f) \leq 1/4$ (otherwise Theorem 2.6 implies $\sum_{i=1}^n |\widehat{f}(i)| =$
 395 $O(\delta(f)d)$). Define $F : \{-1, 1\}^{nt} \rightarrow \{-1, 1\}$ to be $F(x^{(1)}, \dots, x^{(t)}) := f(x^{(1)}) \times \dots \times f(x^{(t)})$,
 396 where t is a parameter to be fixed later, and $x^{(i)} \in \{-1, 1\}^n$ for all $i \in [t]$. Since $\deg_{\mathbb{F}_2}(F) =$
 397 $\deg_{\mathbb{F}_2}(f)$, Theorem 2.6 implies

$$398 \sum_{\substack{S \subseteq [nt] \\ |S|=1}} |\widehat{F}(S)| = O(d). \quad (12)$$

399 Since $(1-x)^{1/x}$ is a decreasing function in x for $x \in (0, 1/2]$, we have

$$400 (1-x)^{1/x} \geq 1/4 \quad \text{for all } x \in (0, 1/2]. \quad (13)$$

401 Expressing the Fourier coefficients of F in terms of the Fourier coefficients of f ,

$$402 \sum_{\substack{S \subseteq [nt] \\ |S|=1}} |\widehat{F}(S)| = t \cdot \widehat{f}(\emptyset)^{t-1} \sum_{i=1}^n |\widehat{f}(i)|$$

$$403 = \left(1 + \frac{1}{2\delta(f)}\right) \cdot (1 - 2\delta(f))^{\frac{1}{2\delta(f)}} \sum_{i=1}^n |\widehat{f}(i)|$$

Choosing $t = 1 + \frac{1}{2\delta(f)}$, and since $\widehat{f}(\emptyset) = 1 - 2\delta(f)$

$$404 \geq \left(1 + \frac{1}{2\delta(f)}\right) \cdot \left(\frac{1}{4}\right) \sum_{i=1}^n |\widehat{f}(i)| \quad \text{by Equation (13)}$$

$$405 \geq \frac{1}{8\delta(f)} \cdot \sum_{i=1}^n |\widehat{f}(i)|.$$

406

407 Now, Equation (12) implies the desired bound, $\sum_{i=1}^n |\widehat{f}(i)| = O(\delta(f)d)$. ◀

408 We would like to extend the upper bound of Lemma 2.7 to any basis of $\text{span}(\text{supp}(f))$
 409 instead of just the standard basis of the set of parities.

410 ► **Corollary 2.8.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function with $\deg_{\mathbb{F}_2}(f) = d$. Suppose*
 411 *$\mathcal{S} \subseteq \text{supp}(f)$ is a basis of $\text{span}(\text{supp}(f))$, then*

$$412 \sum_{S \in \mathcal{S}} |\widehat{f}(S)| = O(\delta(f)d) = O(\delta(f) \log k(f)).$$

413 **Proof.** The main idea of the proof is to do a basis change on parities and construct another
 414 function h , the corollary will follow by applying Lemma 2.7 on h .

415 Recall that we denote both a subset of $[n]$ and the corresponding indicator vector in \mathbb{F}_2^n ,
 416 by the same notation.

417 Let $\mathcal{S} = \{S_1, \dots, S_{r(f)}\}$, extend \mathcal{S} to $\mathcal{S}' = \{S_1, \dots, S_{r(f)}, S_{r(f)+1}, \dots, S_n\}$, a complete
 418 basis of \mathbb{F}_2^n . Observe that $\widehat{f}(S_i) = 0$, for $i \in \{r(f)+1, \dots, n\}$ (since \mathcal{S} spans $\text{supp}(f)$). Fix
 419 the change of basis matrix $B \in \mathbb{F}_2^{n \times n}$ with i -th column as S_i , $i \in [n]$.

420 Consider the function $h : \{-1, 1\}^n \rightarrow \mathbb{R}$ satisfying $\widehat{h}(\alpha) = \widehat{f}(B\alpha)$, for all $\alpha \in \mathbb{F}_2^n$. By
 421 Claim A.4, h is Boolean and $\deg_{\mathbb{F}_2}(h) = \deg_{\mathbb{F}_2}(f)$. Using Lemma 2.7, $\sum_{i \in [n]} |\widehat{h}(\{i\})| =$
 422 $O(\delta(f)d)$. From the definition of h , $\widehat{h}(e_i) = \widehat{f}(S_i)$ for $i \in [r(f)]$ and $\widehat{h}(e_i) = 0$ for $i \in$
 423 $\{r(f)+1, \dots, n\}$, we have $\sum_{S \in \mathcal{S}} |\widehat{f}(S)| = O(\delta(f)d)$. The second equality in the statement
 424 of the lemma follows from Lemma A.3. ◀

425 **Proof of Lemma 2.5.** Observe that every term on the left hand side of Corollary 2.8 is
 426 bigger than $1/k''(f)$, giving the required lower bound on $\delta(f)$ and finishing the proof of
 427 Lemma 2.5. ◀

428 **Proof of Theorem 1.5.** From Lemma 2.5 we have $\delta(f) = \Omega\left(\frac{r(f)}{k''(f)\log k(f)}\right)$, and from The-
 429 orem 1.2 we have $\delta(f) = \Omega\left(\frac{r(f)^2}{k(f)\log^2 k(f)}\right)$. ◀

430 The following corollary combines the lower bounds on $\delta(f)$ from Theorem 1.5 and
 431 Lemma 1.1 by setting $k''(f)$ as the threshold.

432 ► **Corollary 2.9.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function such that $k(f) > 1$. Then,*

$$433 \quad \delta(f) = \Omega\left(\max\left\{\frac{r(f)^2}{k(f)\log^2 k(f)}, \frac{r(f)}{k''(f)\log k(f)}, \frac{\sqrt{r(f)}}{k''(f)\log(k''(f)^2/r(f))}\right\}\right).$$

434 **3 Proof techniques for upper bound results**

435 In this section we give the overview of our two upper bound results, Theorems 1.4 and
 436 1.6. For presenting the overview of the proofs of these theorems we will use (ρ, κ) - k' -plots
 437 (Figure 1) and (ρ, κ) - k'' -plots (Figure 2), respectively. In an (ρ, κ) - k' -plot ((ρ, κ) - k'' -plot,
 438 respectively) we will refer to the “intersection point” as the point of intersection between the
 439 k -line and k' -curve (the point of intersection between the k -line and k'' -curve, respectively).
 440 Which intersection point we are referring to should be clear from the context.

441 **3.1 Proof techniques for Theorem 1.4**

442 To prove Theorem 1.4, we split our goal into two natural parts: constructing functions on
 443 the k -line and constructing functions on the k' -curve. Both the classes of functions are
 444 modifications of the Addressing function (Definition A.10). In these modifications, all or
 445 some of the target variables of the Addressing function are replaced with an AND function or
 446 a Bent function or a combination of them. We first provide a description of some functions
 447 that lie on the intersection point. While we do not require this, we choose to describe these
 448 functions in order to provide more intuition.

449
 450 **Construction of functions at the intersection point in any (ρ, κ) - k' -plot:** Note that
 451 a function lies at the intersection point when

$$452 \quad k'(f) = \frac{k(f)\log(k(f))}{r(f)}. \quad (14)$$

453 Thus, we want to construct a function f with $k(f) = \Theta(\kappa)$, $r(f) = \Theta(\rho)$, $k'(f) = \Theta\left(\frac{\kappa\log\kappa}{\rho}\right)$
 454 and $\delta(f) = \rho^2/\kappa(\log^2\kappa)$. In particular, we want to construct functions for all ρ, κ satisfying
 455 $\log\kappa \leq \rho \leq \kappa^{\frac{1}{2}}$. Note that, the Addressing function $\text{AD}_t : \{-1, 1\}^{\log t + t} \rightarrow \{-1, 1\}$ has
 456 sparsity t^2 , rank $(t + \log t)$, max-supp-entropy t and weight $1/2$ (Observation A.19) and
 457 thus, AD_t satisfies Equation (14). This only gives functions on the intersection point on all
 458 (ρ, κ) - k' -plots where $\rho = \Theta(\sqrt{\kappa})$, while we have to exhibit such functions for all (ρ, κ) - k' -plots
 459 where $\log\kappa \leq \rho = O(\sqrt{\kappa})$.

460 Our next step is to tweak AD_t in such a way that the rank of the new function f does
 461 not change significantly while the sparsity and max-supp-entropy both increase by the same
 462 multiplicative factor. This would ensure that the resulting function satisfies Equation (14).

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463 If the resulting function's weight decreases to the required value, we would have a function
464 at the intersection point.

465 In order to tweak AD_t , we consider a special kind of composed function $f := \text{AD}_t \circ_{\text{target}} g$
466 (see Definition A.11 for a precise definition) obtained by replacing each target variable in the
467 addressing function with a function g where each copy of g acts on a set of new variables. We
468 prove a *composition lemma* (Lemma B.1) that gives the properties of such composed functions.
469 Due to the structure of the Fourier spectrum of the Addressing function, Lemma B.1 gives
470 us $r(f) \approx t \cdot r(g)$, $k(f) \approx t^2 \cdot k(g)$, $k'(f) = t \cdot k'(g)$ and $\delta(f) = \delta(g)$.

471 So, if g is a function on a small number of variables (say $\log t'$) with near-maximal sparsity
472 and max-supp-entropy ($\Theta(t')$), then the resulting function satisfies Equation (14). The AND
473 function is a natural choice for g . We denote the resulting function by $\text{AD}_{t,t'}$, and this is a
474 function at the intersection point for all plots by suitably varying t and t' .

475
476 **Constructing functions on the k -line:** We start with $\text{AD}_{t,t'}$, the function at the inter-
477 section point in (ρ, κ) - k' -plots. We modify $\text{AD}_{t,t'}$ in such a way that its sparsity, rank and
478 weight do not change much, while the max-supp-entropy increases. We replace a single
479 $\text{AND}_{\log t'}$ in $\text{AD}_{t,t'}$ by $\text{AND}_{\log a}$ for some suitable $a > t$, denote the new function by $\text{AD}_{t,t',a}$.
480 A suitable setting of the parameters t, t' and a yields functions on the k -line for all plots
481 (Claim B.2).

482
483 **Constructing functions on the k' -curve of the (ρ, κ) - k' -plot:** We start with $\text{AD}_{t,t'}$ at
484 the intersection point on $(\rho, \kappa/\ell)$ - k' -plot (for some parameter $\ell > 0$). We modify $\text{AD}_{t,t'}$ in
485 such a way that its rank and weight do not change, the sparsity increases by a multiplicative
486 factor of ℓ and the max-supp-entropy increases by a factor of $\sqrt{\ell}$. The new function f will
487 be on the k' -curve in the (ρ, κ) - k' -plot because $\frac{k(f)}{k'(f)^2} = \frac{k(\text{AD}_{t,t'})}{k'(\text{AD}_{t,t'})^2} = \delta(\text{AD}_{t,t'}) = \delta(f)$. Note
488 that $k'(f) \approx \frac{\kappa \log(\kappa)}{\rho \sqrt{\ell}}$, thus making ℓ suitably large yields functions on the k' -curve for all
489 $\rho \leq \kappa' \leq \frac{\kappa \log(\kappa)}{\rho}$ for all plots.

490 We now change $\text{AD}_{t,t'}$ to have the properties mentioned above. We modify each $\text{AND}_{\log t'}$
491 in $\text{AD}_{t,t'}$ as follows: replace a single variable x by $x \cdot B$, where B is a bent function on
492 $\log \ell$ new variables. We denote this new inner function by $\text{AB}_{t',\ell}$, and $\text{AD}_t \circ_{\text{target}} \text{AB}_{t',\ell}$ by
493 $\text{AAB}_{t,t',\ell}$. The effect of changing $\text{AND}_{\log t'}$ to $\text{AB}_{t',\ell}$ keeps its rank and weight roughly the
494 same, while increasing its sparsity by a factor of ℓ and increasing its max-supp-entropy by
495 a factor of $\sqrt{\ell}$. We show, using our composition lemma (Lemma B.1), that the properties
496 of $\text{AD}_t \circ_{\text{target}} \text{AND}_{\log t'}$ and $\text{AD}_t \circ_{\text{target}} \text{AB}_{t',\ell}$ change in a similar fashion. Thus, a suitable
497 setting of the parameters t, t', ℓ yields functions on the k' -curve for all plots (Claim B.3).

498 3.2 Proof techniques for Theorem 1.6

499 We split our goal into two parts: constructing functions on the k -line when $\frac{\kappa}{\rho} \log \kappa \leq \kappa'' \leq \kappa$,
500 and constructing functions on the k'' -curve when $\kappa \leq \frac{\kappa}{\rho} \log \kappa$. To construct functions on
501 the k -line, we use the functions $\text{AD}_{t,t',a}$ constructed for the proof of Theorem 1.4, since
502 $k'(\text{AD}_{t,t',a}) = k''(\text{AD}_{t,t',a})$.

503 For constructing functions on the k'' -curve, we need to construct functions f such that

$$504 \quad \delta(f) = \Theta \left(\frac{r(f)}{k''(f) \log(k''(f)/r(f))} \right). \quad (15)$$

505 We will use a similar technique as in our construction of functions on the k' -curve in
506 Theorem 1.4. We start from the function $\text{AD}_{t,t'}$ at the intersection point. Note that $\text{AD}_{t,t'}$

507 satisfies Equation (15). We modify $\text{AD}_{t,t'}$ such that the rank, weight and max-rank-entropy
 508 changes very little but the sparsity increases by a multiplicative parameter 2^p . We achieve
 509 this by replacing a variable (say x) in $\text{AD}_{t,t'}$ with $x \cdot \text{AND}(y_1, \dots, y_p)$, where x and y_i s are all
 510 variables in $\text{AD}_{t,t'}$, but for any i , x and y_i do not appear in the same monomial (Claim B.4).
 511 The new function f still satisfies Equation (15). This places f on the k'' -curve in a plot
 512 corresponding to the same rank as that of $\text{AD}_{t,t'}$, but where the sparsity increases by a factor
 513 of 2^p . By suitably setting p , t and t' , we obtain functions on the k'' -curve for all plots. This
 514 proves the second bound in Theorem 1.6.

515 4 Conclusions

516 In this paper, for Boolean functions f , we study the relationship between weight and other
 517 Fourier-analytic measures namely rank, sparsity, max-supp-entropy and max-rank-entropy.
 518 For a threshold $t > 0$, Chang's lemma gives a lower bound on the weight of a Boolean
 519 function f in terms of $\dim\left(\left\{S \subseteq [n] : |\widehat{f}(S)| \geq \frac{1}{t}\right\}\right)$. We consider three natural thresholds t
 520 in Chang's lemma, namely $k(f)$, $k'(f)$ and $k''(f)$, yielding three lower bounds on weight in
 521 terms of these measures. We prove new lower bounds on weight in Theorems 1.3 and 1.5,
 522 and our bounds dominate all the above-mentioned bounds from Chang's lemma for a wide
 523 range of parameters.

524 When $\log k(f) = \Theta(r(f))$, the function $f = \text{AND}$ already shows that all the above lower
 525 bounds are tight. To consider all other feasible relationships between $k(f)$ and $r(f)$, we
 526 divide our investigation of these lower bounds into two different parts. In the first part,
 527 we vary over all feasible settings of $r(f)$, $k(f)$ and $k'(f)$, and construct functions that
 528 witness tightness of our lower bounds in Theorem 1.3 for nearly all such feasible settings
 529 (Theorem 1.4). In the second part, we vary over all feasible settings of $r(f)$, $k(f)$ and $k''(f)$,
 530 and construct functions that witness near-tightness of our lower bounds in Theorem 1.5 for
 531 nearly all such feasible settings (Theorem 1.6). These functions are constructed by carefully
 532 composing the Addressing function with suitable inner functions. We show a composition
 533 lemma (Lemma B.1), which relates the properties of the composed function with those of
 534 the inner functions; this allows us to come up with functions that match our lower bounds.

535 We also construct functions for which our lower bounds are asymptotically stronger than
 536 the lower bounds obtained from Chang's lemma for all choices of threshold (see Claim 1.7).
 537 All functions that we construct in this work might be of independent interest.

538 **Open Problems.** Since our proof of Theorem 1.2 is a generalization of the proof of the
 539 upper bound $r(f) = O(\sqrt{k(f)} \log k(f))$ due to Sanyal [25], it sheds light on the presence of
 540 the $\log k$ factor in Sanyal's upper bound. This still leaves the following question open: do
 541 there exist Boolean functions f for which $r(f) = \omega(\sqrt{k(f)})$?

542 There are some ranges of parameters where we were not able to construct functions with
 543 upper bounds matching our lower bounds from Theorem 1.5. It will be interesting to see if
 544 our techniques can be extended to cover these ranges as well.

545 All thresholds t considered for Chang's lemma in this work satisfy $\dim(\{S \subseteq [n] : |\widehat{f}(S)| \geq$
 546 $\frac{1}{t}\}) = r(f)$. It is an interesting problem to obtain Chang's-lemma-type bounds for thresholds
 547 for which this dimension is strictly less than $r(f)$.

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611 76.

612 **A Preliminaries**

613 All logarithms in this paper are taken to be base 2. We use the notation $[n]$ to denote the
614 set $\{1, 2, \dots, n\}$. When necessary, we assume t is a power of 2. We use the notation 1^n
615 (respectively, $(-1)^n$) to denote the n -bit string $(1, 1, \dots, 1)$ (respectively, $(-1, -1, \dots, -1)$).

616 For a function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, its \mathbb{F}_2 -degree, denoted by $\deg_{\mathbb{F}_2}(f)$, is the degree
617 of its unique \mathbb{F}_2 -polynomial representation. Throughout this paper, we often identify subsets
618 of $[n]$ with their corresponding characteristic vectors in \mathbb{F}_2^n . Thus when we refer to linear
619 algebraic measures of a collection of subsets of $[n]$, we mean the measure on the corresponding
620 subset of \mathbb{F}_2^n (where \mathbb{F}_2^n is viewed as an \mathbb{F}_2 -vector space).

621 Throughout this paper, we assume that f is not a constant function or a parity or a
622 negative parity, unless mentioned otherwise.

623 **A.1 Fourier analysis of Boolean functions**

624 Consider the vector space of functions from $\{-1, 1\}^n$ to \mathbb{R} equipped with the following inner
625 product.

$$626 \quad \langle f, g \rangle := \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} f(x)g(x).$$

627 For a set $S \subseteq [n]$, define a *parity* function (which we also refer to as *characters*) $\chi_S :$
628 $\{-1, 1\}^n \rightarrow \{-1, 1\}$ by $\chi_S(x) = \prod_{i \in S} x_i$. The set of parity functions $\{\chi_S : S \subseteq [n]\}$ forms
629 an orthonormal basis for this vector space. Hence, every function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ has a
630 unique representation as

$$631 \quad f = \sum_{S \subseteq [n]} \widehat{f}(S) \chi_S,$$

632 where $\widehat{f}(S) = \langle f, \chi_S \rangle$ for all $S \subseteq [n]$. The coefficients $\{\widehat{f}(S) : S \subseteq [n]\}$ are called the
633 *Fourier coefficients* of f . Define the Fourier ℓ_1 -norm of a function $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ by
634 $\|\widehat{f}\|_1 := \sum_{S \subseteq [n]} |\widehat{f}(S)|$. The Fourier support of f , denoted by $\text{supp}(f)$, is defined as

$$635 \quad \text{supp}(f) = \left\{ S \subseteq [n] : \widehat{f}(S) \neq 0 \right\}.$$

636

637 **► Remark A.1.** In the literature, Fourier support is generally denoted by $\text{supp}(\widehat{f})$. For ease
638 of notation we drop the hat symbol above f . A similar convention has been adopted in the
639 remaining parts of the paper.

640 Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ be any function. The Fourier sparsity of f , denoted by $k(f)$, is defined
641 as $k(f) = |\text{supp}(f)|$. For simplicity we assume that $k(f) \geq 2$ for all Boolean functions f
642 considered in this paper (unless explicitly mentioned otherwise). We often simply refer to
643 the Fourier sparsity as *sparsity*. For ease of notation, we sometimes abuse notation and say
644 that the elements of the Fourier support of f are the characters $\{\chi_S : S \subseteq [n], \widehat{f}(S) \neq 0\}$,
645 rather than the corresponding sets.

646 We require the following lemma (see, for example, [17]).

647 **► Lemma A.2 (Uncertainty Principle).** *Let $f : \{-1, 1\}^n \rightarrow \mathbb{R}$ be a polynomial and let U_n*
648 *denote the uniform distribution on $\{-1, 1\}^n$. Then,*

$$649 \quad \Pr_{x \sim U_n} [f(x) \neq 0] \geq \frac{1}{k(f)}.$$

650 We also require the following lemma relating the \mathbb{F}_2 -degree of a Boolean function and its
651 Fourier sparsity (see, for example, [3]).

652 ► **Lemma A.3.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function with $k(f) > 1$. Then,*

$$653 \quad \deg_{\mathbb{F}_2}(f) \leq \log k(f).$$

654 The next claim shows that $\deg_{\mathbb{F}_2}(f)$ does not change under a change of basis over the
655 Fourier domain.

656 ► **Claim A.4.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function and let $B \in \mathbb{F}_2^{n \times n}$ be an invertible
657 matrix. Define the function $f_B : \{-1, 1\}^n \rightarrow \mathbb{R}$ as*

$$658 \quad \widehat{f_B}(\alpha) = \widehat{f}(B\alpha) \quad \text{for all } \alpha \in \mathbb{F}_2^n,$$

659 Then f_B is Boolean valued and $\deg_{\mathbb{F}_2}(f_B) = \deg_{\mathbb{F}_2}(f)$.

660 The following corollary follows from [9, Theorem 13] and Lemma A.3.

661 ► **Corollary A.5.** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function, and let $\mathcal{S} \subseteq \text{supp}(f)$ be a basis
662 of $\text{span}(\text{supp}(f))$. Then,*

$$663 \quad \sum_{S \in \mathcal{S}} |\widehat{f}(S)| \leq 4 \log k(f).$$

664 We now define notions of restriction of a function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ to a subset
665 $A \subseteq \{-1, 1\}^n$.

666 ► **Definition A.6 (Restriction).** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ and $A \subseteq \{-1, 1\}^n$. The re-
667 striction of f to A is the function $f|_A : A \rightarrow \{-1, 1\}$ defined as $f|_A(x) = f(x)$ for all
668 $x \in A$.*

669 ► **Definition A.7 (Affine Restriction).** *Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$, let Γ be a set of parities
670 and $b \in \{-1, 1\}^\Gamma$ be an assignment to these parities. Define the function $f|_{(\Gamma, b)}$ to be the
671 restriction of f to the affine subspace obtained by fixing parities in Γ according to b . That is,*

$$672 \quad f|_{(\Gamma, b)} := f|_{\{x \in \{-1, 1\}^n : \chi_\gamma(x) = b_\gamma \text{ for all } \gamma \in \Gamma\}}.$$

673 A.2 Fourier expansions and properties of some standard functions

674 For any integer $n > 0$, define the function $\text{AND}_n : \{-1, 1\}^n \rightarrow \{-1, 1\}$ by $\text{AND}_n(x) = -1$ if
675 $x = (-1)^n$, and 1 otherwise. We drop the subscript n when it is clear from the context.

676 ► **Definition A.8 (Bent functions).** *A function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ is said to be a bent
677 function if $|\widehat{f}(S)| = |\widehat{f}(T)|$ for all $S, T \subseteq [n]$.*

678 ► **Definition A.9 (Indicator function).** *For any integer $n \geq 1$ and $b \in \{-1, 1\}^n$, define the
679 function $\mathbb{I}_b : \{-1, 1\}^n \rightarrow \{0, 1\}$ by*

$$680 \quad \mathbb{I}_b(x) = \begin{cases} 1 & x = b, \\ 0 & \text{otherwise.} \end{cases}$$

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681 ► **Definition A.10** (Addressing function). For any integer $t \geq 2$, define the Addressing function
 682 $\text{AD}_t : \{-1, 1\}^{\log t} \times \{-1, 1\}^t \rightarrow \{-1, 1\}$ by

$$683 \quad \text{AD}_t(x, y) = y_{\text{bin}(x)},$$

684 where $x \in \{-1, 1\}^{\log t}$ and $y \in \{-1, 1\}^t$, and $\text{bin}(x)$ denotes the integer in $[t]$ whose binary
 685 representation is given by x (where -1 's are viewed as 1 in the string x , and 1 's are viewed
 686 as 0). We refer to the x -variables as addressing variables, and the y -variables as target
 687 variables.

688 We next define a way of modifying the Addressing function that is of use to us. In this
 689 modification, we replace target variables by functions, each acting on disjoint variables.

690 ► **Definition A.11** (Composed addressing functions). Let $t \geq 2$, $\ell_1, \dots, \ell_t \geq 1$ be any integers.
 691 Let $g_i : \{-1, 1\}^{\ell_i} \rightarrow \{-1, 1\}$ be any functions for $i \in [t]$. Define the function $\text{AD}_t \circ_{\text{target}}$
 692 $(g_1, \dots, g_t) : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\ell_1 + \dots + \ell_t} \rightarrow \{-1, 1\}$ by

$$693 \quad \text{AD}_t \circ_{\text{target}}(g_1, \dots, g_t)(x, y_1, \dots, y_t) = \text{AD}_t(x, g_1(y_1), \dots, g_t(y_t)),$$

694 where $x \in \{-1, 1\}^{\log t}$ and $y_i \in \{-1, 1\}^{\ell_i}$ for all $i \in [t]$.

695 For any function $g : \{-1, 1\}^s \rightarrow \{-1, 1\}$, we use the notation $\text{AD}_t \circ_{\text{target}} g$ to denote the
 696 function $\text{AD}_t \circ_{\text{target}}(g, g, \dots, g) : \{-1, 1\}^{\log t} \times \{-1, 1\}^{ts} \rightarrow \{-1, 1\}$.

697 A.3 Fourier-analytic measures of Boolean functions

698 We now introduce a few Fourier-analytic measures on Boolean functions that we use through-
 699 out the rest of the paper, and state some important relationships between them. Recall that
 700 we use the notation $\dim(S)$ to denote the dimension of the span of the set S .

701 ► **Definition A.12** (Fourier rank). Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function. Define the
 702 Fourier rank of f , denoted $r(f)$, by

$$703 \quad r(f) = \dim(\text{supp}(f)).$$

704 We often refer to Fourier rank as simply *rank*. Sanyal [25] showed the following upper bound
 705 on the rank of Boolean functions in terms of their sparsity.

706 ► **Theorem A.13** ([25, Theorem 1.2]). Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function. Then

$$707 \quad r(f) = O(\sqrt{k(f)} \log k(f)).$$

708 We require the following observation which gives a simple upper bound on the rank of a
 709 Boolean function.

710 ► **Observation A.14.** Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function and Γ be a set of parities.
 711 If for all $b \in \{-1, 1\}^\Gamma$ the restricted function $f|_{(\Gamma, b)}$ is constant then $r(f) \leq |\Gamma|$.

712 Recall that for any function $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ and any real $t > 0$, we define $\mathcal{S}_t := \{S \subseteq$
 713 $[n] : |\widehat{f}(S)| \geq 1/t\}$ (we suppress the dependence of \mathcal{S}_t on f as the underlying function will be
 714 clear from context).

715 ▶ **Definition A.15.** Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function. Define the Fourier max-
716 sup-entropy of f , denoted $k'(f)$, by

$$717 \quad k'(f) := \operatorname{argmin}_t \{\mathcal{S}_t = \operatorname{supp}(f)\}.$$

718 Equivalently,

$$719 \quad k'(f) := \max_{S \in \operatorname{supp}(f)} \left\{ \frac{1}{|\widehat{f}(S)|} \right\}.$$

720 Define the Fourier max-rank-entropy of f , denoted $k''(f)$, by

$$721 \quad k''(f) := \operatorname{argmin}_t \{\dim(\mathcal{S}_t) = r(f)\}.$$

722 We often refer to the Fourier max-supp-entropy and Fourier max-rank-entropy as simply
723 max-supp-entropy and max-rank-entropy, respectively.

724 ▶ **Lemma A.16** (Relationships between parameters). Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any
725 function. Then the following inequalities hold.

- 726 1. $\log k(f) \leq r(f) = O(\sqrt{k(f)} \log k(f))$.
- 727 2. $\sqrt{k(f)} \leq k'(f) \leq k(f)/2$.
- 728 3. $\max \left\{ \sqrt{r(f)}, r(f)/(4 \log k(f)) \right\} \leq k''(f) \leq k'(f)$.

729 ▷ **Claim A.17.** Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ a function with $k(f) \geq 2$. Then

$$730 \quad \delta(f) = \Omega \left(\frac{k(f)}{k'(f)^2} \right).$$

731 ▷ **Claim A.18.** Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be any function. Then

$$732 \quad \|\widehat{f}\|_1 \leq 3\sqrt{k(f)\delta(f)}.$$

733 We require the following observation about the rank, sparsity, max-supp-entropy, max-
734 rank-entropy and weight of the addressing function, AD_t , which follows immediately from
735 definitions and first principles. We omit its proof.

736 ▶ **Observation A.19.** Let $t \geq 2$ be any positive integer. Then the rank, sparsity, max-supp-
737 entropy, max-rank-entropy and weight of AD_t are $(t + \log t)$, t^2 , t , t and $1/2$, respectively.

738 **B** Upper bound proofs

739 The following lemma is a useful tool for our upper bounds. We refer the reader to [6, Section
740 6.2.1] in the full version of our paper for a proof.

741 ▶ **Lemma B.1** (Composition lemma). Let $t \geq 2, m \geq 1$ be any positive integers, and let
742 $g : \{-1, 1\}^m \rightarrow \{-1, 1\}$ be a non-constant function such that there exists a non-empty set
743 $S \subseteq [m]$ with $0 \neq |\widehat{g}(S)| \leq |\widehat{g}(\emptyset)|$. Let $f : \{-1, 1\}^{\log t + mt} \rightarrow \{-1, 1\}$ be defined as

$$744 \quad f = \operatorname{AD}_t \circ_{\text{target}} g.$$

745 Then

$$746 \quad r(f) = t \cdot r(g) + \log t, \tag{16}$$

$$747 \quad k(f) = 1 + t^2(k(g) - 1), \tag{17}$$

$$748 \quad k'(f) = t \cdot k'(g), \tag{18}$$

$$749 \quad k''(f) = t \cdot k''(g), \tag{19}$$

$$750 \quad \delta(f) = \delta(g). \tag{20}$$

751

752 **B.1 Setting parameters in our constructed functions**

753 In this section we state the main claims that go into proving Theorems 1.4 and 1.6. Recall
 754 that these theorems require us to exhibit functions which achieve certain bounds. Claims B.2
 755 and B.3 correspond to the bounds in Theorem 1.4. Claims B.4 and B.5 correspond to the
 756 bounds in Theorem 1.6. All functions referred to below are informally defined in Section 3.
 757 See the full version of our paper [6, Section 6.1] for formal definitions and [6, Sections 6.2,
 758 6.3] for proofs of these claims.

759 \triangleright **Claim B.2.** For all $\rho, \kappa, \kappa' \in \mathbb{N}$ such that κ is sufficiently large, for all $\epsilon > 0$ such that
 760 $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ and $\frac{\kappa \log \kappa}{\rho} \leq \kappa' \leq \kappa$, for $t = \frac{2\rho}{\log \kappa}$, $t' = \frac{\kappa \log^2 \kappa}{\rho^2}$ and $a = \frac{2\kappa' \log \kappa}{\rho}$,

- 761 ■ $\Omega(\epsilon\rho) = r(\text{AD}_{t,t',a}) = O(\rho)$.
- 762 ■ $k(\text{AD}_{t,t',a}) = \Theta(\kappa)$.
- 763 ■ $k'(\text{AD}_{t,t',a}) = \Theta(\kappa')$.
- 764 ■ $\delta(\text{AD}_{t,t',a}) = \Theta\left(\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa}\right)^2\right)$.

765 \triangleright **Claim B.3.** For all $\rho, \kappa, \kappa' \in \mathbb{N}$ such that κ is sufficiently large, for all constants $\epsilon > 0$, such
 766 that $\kappa^{1/2} \leq \kappa' \leq (\kappa \log \kappa)/\rho$ and $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ for $t = \frac{2\rho}{\log \kappa}$, $t' = \frac{4\kappa'^2}{\kappa}$ and $\ell = 2 \left(\frac{\kappa \log \kappa}{\kappa' \rho}\right)^2$,

- 767 ■ $\Omega(\epsilon\rho) = r(\text{AAB}_{t,t',\ell}) = O(\rho)$.
- 768 ■ $k(\text{AAB}_{t,t',\ell}) = \Theta(\kappa)$.
- 769 ■ $k'(\text{AAB}_{t,t',\ell}) = \Theta(\kappa')$.
- 770 ■ $\delta(f) = O\left(\frac{\kappa}{\kappa'^2}\right)$.

771 \triangleright **Claim B.4.** For all $\rho, \kappa, \kappa'' \in \mathbb{N}$ such that κ is sufficiently large, for all constants $\epsilon > 0$ such
 772 that $\log \kappa \leq \rho \leq \kappa^{1/2-\epsilon}$, $e\rho \leq \kappa'' \leq \frac{\kappa \log \kappa}{\rho}$, for $t = \frac{2\rho}{\log(\kappa''/\rho)}$, $t' = \frac{\kappa''}{\rho} \log(\kappa''/\rho)$, $p = \log\left(\frac{4\kappa}{\kappa''}\right)$,

- 773 ■ $r(\text{mAD}_{t,t',p}) = \Theta(\rho)$.
- 774 ■ $\Omega(\kappa) = k(\text{mAD}_{t,t',p}) = O(\kappa/\epsilon)$.
- 775 ■ $k''(\text{mAD}_{t,t',p}) = \Theta(\kappa'')$.
- 776 ■ $\delta(\text{mAD}_{t,t',p}) = \frac{\rho}{\kappa'' \log(\kappa''/\rho)}$.

777 \triangleright **Claim B.5.** For all $\rho, \kappa, \kappa' \in \mathbb{N}$ such that κ is sufficiently large, for all $\epsilon > 0$ such that
 778 $\log \kappa \leq \rho \leq \kappa^{\frac{1}{2}-\epsilon}$ and $\frac{\kappa \log \kappa}{\rho} \leq \kappa' \leq \kappa$, there exists a constant $c \geq 1$ such that the following
 779 holds for $t = \frac{2\rho}{\log \kappa}$, $t' = \frac{c\kappa \log^2 \kappa}{\rho^2}$ and $a = \frac{2c\kappa' \log \kappa}{\rho}$.

- 780 ■ $\Omega(\epsilon\rho) = r(\text{AD}_{t,t',a}) = O(\rho)$.
- 781 ■ $k(\text{AD}_{t,t',a}) = \Theta(\kappa)$.
- 782 ■ $k''(\text{AD}_{t,t',a}) = \Theta(\kappa')$.
- 783 ■ $\delta(\text{AD}_{t,t',a}) = \Theta\left(\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa}\right)^2\right)$.