A Decade of the Z-Numbers

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***Abstract*—In this article, we present a study on the development in the theory and application of the Z-numbers since its inception in 2011. The review covers the formalization of Z-number-based mathematical operators, the role of Z-numbers in computing with words, decision-making, and trust modeling, application of Z-numbers in real-world problems such as multisensor data fusion, dynamic controller design, safety analytics, and natural language understanding, a brief comparison with conceptually similar paradigms, and some potential areas of future investigation. The paradigm currently has at least four extensions to its definition: multidimensional Z-numbers, parametric Z-numbers, hesitant-uncertain linguistic Z-numbers, and Z∗-numbers. The Z-numbers have also been used in conjunction with rough sets and granular computing for enhanced uncertainty handling. While this decade has seen a plethora of theoretical initiatives toward its growth, there remains a major work scope in the direction of practical realization of the paradigm. Some challenges yet unresolved are automated translation of (imprecise, sarcastic, and metaphorical) linguistic expressions to their Z-number forms, discernment of probability–possibility distributions to map real-world situations under consideration, analysis of linguistic equivalents of Z-operator results to intuitive human responses, the endogenous arousal of belief in intelligent agents, and analysis of biases embedded in expert-belief values that are primary inputs to Z-number-based expert systems. After a decade of the Z-numbers, the paradigm has proved to be of use in expert-input-based decision-making systems and initial value problems. Its applicability in high-risk, high-precision areas, such as deep-sea exploration and space science, remains unexplored.**

***Index Terms*—Computing with perceptions (CWP), computing with words (CWW), linear programming (LP), multicriteria decision-making (MCDM), multiobjective decision-making (MODM), trust modeling, Z-distance, Z-interpolation, Z-number arithmetic, Z-rule base, Z-similarity.**

I. Introduction: Zadeh's Z-Numbers

VOLUNTARY actions, in response to a situation (*S*) or a condition, are functions of decisions arising from analyses on the information (*I*) describing *S*. Thus, intuitively, the greater the reliability of *I*, the better is the decision made. In [1], Zadeh coined the Z-numbers as a formal summary of the reliability or confidence in the information conveyed by simple natural language statements arising out of a situation. He envisioned the paradigm to be of use in the modelling of everyday human decision-making across realms like "economics, decision analysis, risk assessment, prediction, anticipation, rule-based characterization of imprecise functions and relations, and biomedicine" [2].

> "Problems involving computation with Z-numbers is easy to state but far from easy to solve" [1].

This article presents a review of attempts made in the direction of Z-number-based mathematics, computation-operators, and application across domains. The following paragraphs (see Section I) highlight Zadeh's description of the paradigm. Section II enumerates analyses of computation methods, definition-extensions, and applications that have emerged over the years. Section III presents a brief comparison with conceptually similar techniques, and Section IV offers insights on where the paradigm stands today—providing pointers to potential future work directions. Table I, a supplement to this article, presents a succinct summary of all work on the Z-number paradigm.

Before describing the Z-number paradigm, we briefly present the concept of fuzzy numbers, as was defined by Zadeh. A fuzzy set $\boldsymbol{S}$ is called a fuzzy number if the membership function $\boldsymbol{\mu_s}$ is defined in the set of all real numbers $\boldsymbol{R}$ and satisfies the following conditions.

1) $\text{supp}(\boldsymbol{S}) = \{\boldsymbol{x_i} \in \boldsymbol{R}\text{: } \boldsymbol{\mu_s}(\boldsymbol{x_j}) > 0\}$ is finite where $i = 1,2,3,\ldots,\boldsymbol{n}$ and $\text{supp}(\boldsymbol{S})$ = support of $\boldsymbol{S}$.
2) $\boldsymbol{\mu_S}(\boldsymbol{x_j}) = 1 \ \ \forall \boldsymbol{j} \in [\boldsymbol{q}, \boldsymbol{r}]$ where $1 \leq \boldsymbol{q} \leq \boldsymbol{r} \leq \boldsymbol{n}$.
3) $\boldsymbol{\mu_S}(\boldsymbol{x_i}) \leq \boldsymbol{\mu_S}(\boldsymbol{x_j}) < 1 \ \forall \boldsymbol{i},\ \boldsymbol{j} \in [1,\ \boldsymbol{q}]$ and $(\boldsymbol{i} \leq \boldsymbol{j})$.
4) $1 > \boldsymbol{\mu_S}(\boldsymbol{x_i}) \geq \boldsymbol{\mu_S}(\boldsymbol{x_j}) \ \ \forall \boldsymbol{i},\ \boldsymbol{j} \in [\boldsymbol{r},\ \boldsymbol{n}]$ and $(\boldsymbol{i} \leq \boldsymbol{j})$.

Here, $\boldsymbol{\mu_S}(\boldsymbol{x}) : \boldsymbol{R} \rightarrow [0,\ 1]$ represents a π-type function [3], e.g., triangular or trapezoidal membership functions. In these functions, the membership value is first nondecreasing with $\boldsymbol{x}$, attains its maximum value of unity, and then becomes nonincreasing with $\boldsymbol{x}$. A fuzzy number, thus, indicates a fuzzy set having a π-type membership function and is characterized by the cardinality of the fuzzy set.

A Z-number [1], on the other hand, is an ordered pair of fuzzy numbers. While a fuzzy number is a representation of the total belongingness of its supporting points, a Z-number quantifies a linguistic variable using two membership values. Given a natural language statement, $\boldsymbol{Y}$, on subject $\boldsymbol{X}$, the "Z-number" of $\boldsymbol{Y}$ is a 2-tuple

$$Z\,(Y) = < A, B > \tag{1}$$

where

1) $\boldsymbol{A}$ equates to the restriction (constraint) on the values of $\boldsymbol{X}$ (a real-valued uncertain variable);

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2) $\boldsymbol{B}$ expresses the reliability (certainty) of $\boldsymbol{A}$; formalized as a fuzzy restriction on the value of the probability measure of $\boldsymbol{A}$;
3) typically, $\boldsymbol{A}$ and $\boldsymbol{B}$ are fuzzy-number representations of words or clauses in natural language;
4) values of $\boldsymbol{A}$ and $\boldsymbol{B}$ are precisiated through associations with membership functions $\boldsymbol{\mu_A}$, $\boldsymbol{\mu_B}$, respectively;
5) understandably, if $\boldsymbol{Y}$ is one of a series of statements on $\boldsymbol{X}, \boldsymbol{A}$ and $\boldsymbol{B}$ would derive from the general context and implied figure of speech, respectively;
6) $\boldsymbol{X}$ being a random variable, the term "reliability" (or "certainty") may be interpreted as the probability of the $<\boldsymbol{X}, \boldsymbol{A}>$ pairing. In other words, $\boldsymbol{B}$ may be interpreted as a response to the question: "How sure are you that $\boldsymbol{X}$ is $\boldsymbol{A}$?" $\boldsymbol{X}$ and $\boldsymbol{A}$ together represent a random fuzzy event on the real line $\boldsymbol{R}$, and the probability of this event can be expressed as

$$p = \int_{\boldsymbol{R}} \boldsymbol{\mu_A}(\boldsymbol{u})\, \boldsymbol{p_X}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}. \tag{2}$$

Where $\boldsymbol{u}$ is a value of $\boldsymbol{X}$ at a particular instance, $\boldsymbol{p_X}$ is the underlying (hidden) probability density of $\boldsymbol{X}$, and the restriction (known) on $\boldsymbol{p_X}$ is represented as

$$\int_{\boldsymbol{R}} \boldsymbol{\mu_A}(\boldsymbol{u})\, \boldsymbol{p_X}(\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \;\; is\ B. \tag{3}$$

The detailed derivation is provided in (4).

Some examples of Z-numbers in everyday sentences are as follows.

1) $\boldsymbol{Y_1}$ = It takes me about an hour to travel one way to my workplace.

Therefore, $\boldsymbol{X}$ = Time to reach workplace and $\boldsymbol{Z}$ = <about an hour, generally>.

2) $\boldsymbol{Y_2}$ = This picture is absolutely stunning!

Therefore, $\boldsymbol{X}$ = Impression of picture and $\boldsymbol{Z}$ = <stunning, absolutely>.

As is evident from these sentences, the complexities of the real world are far beyond what can be solely captured through interval or type-1 fuzzy-number representations. The $\boldsymbol{B}$ in Z-numbers has the potential of encapsulating an agent's "subjective sense" of confidence in any information. The formal structure of the Z-number thus supports the extension of type-2 fuzzy sets [4], [5] into level-2 computing with words (CWW) [6]–[8]. CWW stands for the precisiation of- and operations on- semantics of complete natural language sentences, instead of only their component adjectives and adverbs. Section II-B describes the application of Z-numbers in "integrating" [9], [10] affective computing and CWW.

Extending on the basic Z-numbers, the ordered 3-tuple $<\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}>$ is referred to as a "Z-valuation" and it is equivalent to the assignment statement "$\boldsymbol{X}$ is $<\boldsymbol{A}, \boldsymbol{B}>$." For example, the Z-valuation of Y_2 is <Impression of picture, stunning, absolutely> implying that "<Impression of the picture> is <stunning, absolutely>." Inclusion of X allows implicit grouping of Z-valuations into context-sensitive "Z-information" granules that find application as inputs to expert and decision-making systems. The Z-valuation $<\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}>$ is viewed as a generalized constraint on X and is defined by

$$\begin{aligned} &\text{Probability}(\boldsymbol{X} \text{ is } \boldsymbol{A})\text{is } \boldsymbol{B} \\ \text{or } \;\; &\boldsymbol{p} = \textstyle\int_{\boldsymbol{R}} \boldsymbol{\mu_A}(\boldsymbol{u})\, \boldsymbol{p_X}(\boldsymbol{u})\, \boldsymbol{du} \text{ is } \boldsymbol{B} \\ \text{or } &\boldsymbol{p} = \boldsymbol{\mu_B}\left(\textstyle\int_{\boldsymbol{R}} \boldsymbol{\mu_A}(\boldsymbol{u})\, \boldsymbol{p_X}(\boldsymbol{u})\, \boldsymbol{du}\right) \\ \text{subject to } &\textstyle\int_{\boldsymbol{R}} \boldsymbol{p_X}(\boldsymbol{u})\, \boldsymbol{du} = 1. \end{aligned} \tag{4}$$

In [1], Zadeh suggests the use of the extension principle to solve (3). However, this being computationally intensive, he contemplates over the use of typical probability distribution processes (e.g., Gaussian distribution, etc.) for X and approximation of A by its bandwidth A_b, as simplification measures. Besides arithmetic, algebraic, and logic operations, he points at the need for the design of methods to rank Z-numbers, a calculus of IF–THEN rules with Z-number valued components, measures of entropy, as well as countermeasures of information loss. The past decade has witnessed numerous solution-attempts in the directions set by Zadeh.

The following section presents a near-chronological study and our analyses of these attempts. Section II-A describes work in the area of Z-number mathematics (covering algebraic equations, basic operators in the discrete and continuous domains, differential equations, linear programming models, and uncertainty estimates). Section II-B focuses on the application of Z-numbers in linguistics. Section II-C presents the use of Z-numbers in trust-modeling and decision-making. Section II-D is dedicated to the application of Z-numbers to solve certain real-world challenges such as alloy selection, safety analytics, and psychological research.

Section III presents a brief comparison between the Z-numbers and conceptually similar paradigms. Section IV highlights major achievements in Z-number-based theoretical and applied research, and challenges and possible strategies toward their resolution. Despite its many advantages in the realms of CWW and decision-making, the primary challenges [9], [10] in the implementation of the Z-numbers are:

1) Mathematical abstraction of deep semantics of natural language expressions (inclusive of irregularities of human interactions, affects, metaphors, sarcasm, abstract, or non-precise utterances)—for parameter $\boldsymbol{A}$;
2) Emulation of endogenous arousal of certainty values, qualia [11], [12] and metacognition—for parameter $\boldsymbol{B}$;
3) Discernment of probability distributions and membership functions that would "completely" represent events under consideration—for Z-number-based operations.

II. Z-Numbers Over the Years: Methodologies and Applications

Since its coinage, there have been numerous attempts toward formalization and extension of the paradigm. Some of these focus on the development of the mathematical properties of Z-numbers, while others deal with the linguistic elements of the paradigm for modeling aspects of human behavior. This section highlights research across pure Z-number arithmetic and the role and application of Z-numbers in CWW and perceptions, decision-making, and real-world problem-solving. All descriptions are also presented in Table I, as a supplement to this article.

A. Z-Number Mathematics

1) Z-Number-Based Algebraic Equations: As has been illustrated in Section I, the Z-number paradigm was coined by Zadeh [1] as a mechanism for the summarization of the reliability or confidence in the information embedded in a natural language statement. If $\boldsymbol{Y}$ is a natural language statement on any subject $\boldsymbol{X}$, the Z-number equivalent of $\boldsymbol{Y}$ is represented as a 2-tuple [refer (1)] $\boldsymbol{Z(Y)} = <\boldsymbol{A}, \boldsymbol{B}>$, where $\boldsymbol{A}$ is a constraint on the value of $\boldsymbol{X}$ (real-valued random variable) and $\boldsymbol{B}$ represents the fuzzy restriction on the probability of $\boldsymbol{A}$. Zadeh also describes Z^+-numbers [1] as

$$Z^+ (Y) =< A, R > \tag{5}$$

where $\boldsymbol{R}$ represents the underlying probability distribution of $\boldsymbol{X}$ in the Z-valuation $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B})$, and consequently, a Z^+-valuation is expressed as $(\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{p_X})$. A Z^+-number evidently combines the possibility and probability distributions of $\boldsymbol{X}$, thereby associating itself with bimodal distributions. These distributions become compatible if the following condition holds:

$$\int_{\boldsymbol{R}} \boldsymbol{u} \boldsymbol{p_X} (\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} = \frac{\int_{\boldsymbol{R}} \boldsymbol{u} \boldsymbol{\mu_A} (\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}}{\int_{\boldsymbol{R}} \boldsymbol{\mu_A} (\boldsymbol{u})\, \mathrm{d}\boldsymbol{u}}. \tag{6}$$

The relation between a Z-number and a Z^+-number is

$$Z (A, B) = Z^+(A, \mu_A.p_X isB) \tag{7}$$

where $\mu_{\boldsymbol{A}}$ and $\boldsymbol{p_X}$ are the membership function of $\boldsymbol{A}$ and the probability distribution (density) of $\boldsymbol{X}$, respectively, and (.) signifies the scalar product between the two variables. The scalar product of $\mu_{\boldsymbol{A}}$ and $\boldsymbol{p_X}$ is the probability measure of $\boldsymbol{A}$ that can be represented as

$$\mu_A.p_X = \int_{\boldsymbol{R}} \boldsymbol{\mu_A} (\boldsymbol{u})\, \boldsymbol{p_X} (\boldsymbol{u})\, \mathrm{d}\boldsymbol{u} \tag{8}$$

where $\boldsymbol{u}$ is an instance of $\boldsymbol{X}$. A Z-number, thus, establishes a restriction on $\boldsymbol{p_X}$, expressed as "μ_A. $\boldsymbol{p_X}$ is $\boldsymbol{B}$," instead of providing the actual value of $\boldsymbol{p_X}$.

2) Basic Operations on Z-Numbers: Kang *et al*. in [13] describe the first mechanisms of handling Z-numbers as operands. The authors here suggest the conversion of Z-numbers of sentences—occurring in the areas of control, decision-making, and modeling—to their fuzzy number equivalents as the first step of Z-number-based operations. Here, component *B* is defuzzified to a numeric value *a*, which is multiplied with *A*, resulting in a fuzzy number. Although this procedure has low analytical and computational complexity, the "conversion" [14] of Z-numbers to fuzzy numbers involves loss of original information, reducing benefits that would have been achieved otherwise. Besides this, conversion to a different format and subsequent operations are not equivalent to handling Z-numbers in their fundamental form.

Articles [15] and [16] present theoretical studies of different interpretations of Z-valuations. The author here envisions a Z-valuation $<\boldsymbol{X}, \boldsymbol{A}, \boldsymbol{B}>$ as a linguistic summary, where $\boldsymbol{A}$ is the "summarizer" and $\boldsymbol{B}$ is the "quantity in agreement." He observes that a Z-valuation is capable of inducing a possibility distribution over the space of all probability distributions over the random variable $\boldsymbol{X}$. Besides question-answering and reasoning with respect to $\boldsymbol{X}$, the possibility distribution facilitates operations (leading to fused distributions as a function of the semantic relationship) between Z-valuations. He further notes that measures of "specificity" [17] of possibility distributions allow ranking of Z-valuations based on their information content. This ranking process, however, involves defuzzification and consequent loss of information. The author has also visualized Z-valuations as possibility distributions (type-2 fuzzy subset) over the space of compatible Dempster–Shafer (D–S) [18] belief structures. In each of these representations, the primary challenge lies in translating a subjective natural language expression ($\boldsymbol{E}$) to its underlying probability distribution in order to arrive at its Z-valuation ($\boldsymbol{Z}(\boldsymbol{E})$) without affecting the information content of $\boldsymbol{E}$.

Articles [18] and [19] describe the formalization of Z-number arithmetic for complex Z-information processing. The authors here rightly observe that Z-number arithmetic involves a "synergy of probabilistic and fuzzy arithmetic"—necessitating generalization of the extension principle [1] to fuse probabilistic and possibilistic restrictions [14], [15].

Aliev *et al*. [19] focus on discrete Z-number arithmetic, motivated by factors such as: 1) human beings use discrete sets of linguistic terms to describe real-world phenomenon; 2) discrete fuzzy arithmetic and discrete probability distributions are computationally less intensive than their continuous counterparts; and 3) opting for a universal view of uncertainty modeling, instead of fixed probability distributions that constrain modeling of real-world phenomena. Formulae for addition, standard subtraction, square, square-root, and ranking of discrete Z-numbers are described here. In [20], Aliev *et al*. extend all aforementioned discrete Z-number arithmetic operations into those for continuous Z-numbers. A continuous Z-number is an ordered pair $\boldsymbol{Z} = (\boldsymbol{A}, \boldsymbol{B})$, where $\boldsymbol{A}$ is a continuous fuzzy number playing the role of a fuzzy constraint on values that a random variable $\boldsymbol{X}$ may take (i.e., $\boldsymbol{X}$ is $\boldsymbol{A}$), and $\boldsymbol{B}$ is a continuous fuzzy number with a membership function $\boldsymbol{\mu_B}$: [0, 1] $\rightarrow$ [0, 1] acting as a fuzzy constraint on the probability measure of $\boldsymbol{A}$ (i.e., $p(\boldsymbol{A})$ is $\boldsymbol{B}$). Computations with continuous Z-numbers are envisaged to be of use in handling imprecise and partially reliable information of real-world problems concerning decision analysis, optimization, economics, management, and forecasting. A key feature of the operations described in [18] and [19] is that none involve transformations of Z-numbers to their fuzzy-number correspondents, thereby addressing the loss of information issue of the earlier techniques. Aliev *et al.* [20] describe an ISO 9126 standard software for the simulation of discrete Z-number arithmetic.

Definitions in [19] and [20], however, draw on Zadeh's extension principle [1] and, thereby, suffer from increasing entropy when applied on uncertain granules of information. As a mechanism to counter the predicament, Aliev *et al.* [21] describe a general framework for the formalization of Z-number arithmetic operators based on horizontal membership functions. Processes for addition, subtraction, multiplication, division, ranking, square, square-root, and function-limits of Z-numbers have been defined here. Practical application of these mathematical frameworks necessitates analyses of equivalence of Z-number representations of problem-specific "words and phrases."

Jiang *et al.* [22] present a method for evaluating generalized fuzzy numbers using concepts of centroid, degree of fuzziness, and their spread. This technique is then used to rank Z-numbers, where the parameters *A* and *B* [refer (1)] are treated as independent generalized fuzzy numbers. Fuzzy restrictions on *A* are prioritized, and neither parameter is defuzzified.

Articles [23] and [24] deal with the definitions of specificity of Z-numbers, comparisons between specificity measures, and operations thereof. The specificity [17] is the degree to which a fuzzy subset points to one and only one element as its member. The specificity of a Z-number is a function of the specificity of each of its fuzzy components $\boldsymbol{A}$ and $\boldsymbol{B}$—indicating the degree to which a fuzzy vector, with components $\boldsymbol{A}$ and $\boldsymbol{B}$, designates a unique Z-number. Some questions that are worth contemplating over, at this juncture, and serve as pointers to future research, are: 1) can we use the specificity of two Z-numbers to indicate their synonymity; and 2) would Z-numbers of the same expressions—but in different domains—have the same specificity?

In [25] and [26], Aliev *et al*. present definitions for and applications of the Hausdorff distance and the Euclidean distance between two discrete Z-numbers for the formulation of Z-valued IF–THEN rule-based interpolation for approximate reasoning. Sensitivity analysis of the procedure described here shows results as functions of the qualitative inputs. The authors claim that the approach is universal in nature and could be effective in decision making, system analysis, forecasting, and control in ecology, production, economics, and other such areas. The universality, however, remains to be validated, and these measures do not utilize the combined effect of the randomness and the fuzziness of the Z-parameters, leading to information loss. The linguistic interpretation of these distance measures is one of many areas of future research.

Kang *et al*. [27] formalize a measure of the total utility of a Z-number as a function of its constraint and reliability parameters. The definition is universal across all forms of probability distributions and membership functions and is envisioned to be a better tool for defuzzification (with minimal information loss) and ranking of Z-numbers in multicriteria environments. Das *et al.* [3] describe another weighted Z-similarity measure that is able to retain the association between the Z-parameters, thereby addressing the information loss issue in other methods. Shen *et al*. [28] describe a novel comprehensive Z-distance measure that takes into account the similarity between the probability distributions (underlying parameter $\boldsymbol{A}$) and the reliability measures (representing parameter $\boldsymbol{B}$) describing the Z-numbers in question. This distance measure is used in the formulation of a Z-VIKOR algorithm for multicriteria decision-making (MCDM). The authors test their algorithm to solve a government-assessment problem. A formal analysis of these all-inclusive measures would be an intriguing study.

Dubois *et al*. [29] provide a number of alternative interpretation/representation schemes for Z-numbers. In one of the approaches, a Z-number is described as a weighted family of "crisp" Z-numbers in which $\boldsymbol{A}$ denotes a crisp interval (i.e., $\boldsymbol{A} = [\boldsymbol{a}^-, \boldsymbol{a}^+]$) on the real line and $\boldsymbol{B}$ is a probability interval $[\boldsymbol{b}^-, \boldsymbol{b}^+]$. While in another approach, $\boldsymbol{A}$ is defined as a crisp interval and $\boldsymbol{B}$ as a fuzzy interval. In yet another definition, both $\boldsymbol{A}$ and $\boldsymbol{B}$ are described as fuzzy intervals—where one schematic uses a combination of the mixed modes described earlier. In another form, a Z-number is converted into a pair of possibility distributions forming a generalized p-box, where the probability of each fuzzy event is constrained by upper and lower bounds. Computations using such Z-numbers reduce uncertainty propagation with random intervals. The authors observe that no valid solution is obtained when only one of $\boldsymbol{A}, \boldsymbol{B}$ is fuzzy, thereby reinstating that it is not suitable to assign fuzzy probabilities to crisp events and vice versa—a problem in most Z-number-based techniques described in the earlier sections.

3) Z-Number-Based Differential Equations: In [30], Mazandarani *et al*. introduce Z^+-number-based operations based on semigranular functions, followed by concepts of limit, continuity, differentiability, integration, and Laplace transforms of Z-number-valued functions. The authors further describe a framework for Z-differential equations that can outperform fuzzy differential equations in decision-making in uncertain environments. Z-differential equations are represented as bimodal differential equations by combining fuzzy differential equations and random differential equations. The effectiveness of Z-differential equations in the diagnosis and treatment of Hydrocephalus is demonstrated. Hydrocephalus is a brain disease due to the irregularities in the cerebrospinal fluid flow, causing ventricular dilation and brain compression.

4) Linear Programming Problem: Authors in [31] propose a differential evolution (DE) method for the solution of Z-number-based linear programming (Z-LP) problems. The parameters and decision variables are in the form of Z-number expressions that bear signatures of the uncertain real-world, and the discrete Z-number arithmetic operators of [19] are used to arrive at required solutions. While, theoretically, fuzzy-LP and Z-LP solutions stand in the same league, the latter technique is in sync with the real world.

5) Z-Number Uncertainty: Kang *et al.* [32] describe a new uncertainty measure, for a fuzzy set ($\boldsymbol{F}$), as functions of the fuzziness and range (or cardinality) of $\boldsymbol{F}$. This definition is then extended to the context of the uncertainty of discrete and continuous Z-numbers, where the uncertainty derives from the fuzziness of the Z-number parameters ($\boldsymbol{A}$ and $\boldsymbol{B}$). Just as is highlighted for the formulations for Z-number specificity and Z-number-based rule bases, in the earlier sections, some practical challenges here are the mapping of real-world phenomena (described as natural language expressions) to appropriate Z-number representations and the translation of Z-number operation results to context-specific natural language expressions.

The following section describes endeavors in the direction of Z-numbers in natural language processing (NLP). Drawing from our observations in the preceding paragraphs, a potential area of future research is the convergence of Z-number arithmetic for NLP.

B. Z-Numbers in CWW and NLP

Articles [9], [10] analyze features and challenges of the Z-number paradigm in terms of its conceivable role in the development and implementation of CWW [7], [33]. The authors here contemplate a future where an artificial generally intelligent (AGI) [34], [35] agent can identify its endogenous "beliefs" for parameter $\boldsymbol{B}$ as a function of its intuition and experiences to instantiate Z-numbers in real-time to solve real-world problems. An approximate procedure of natural language understanding

(NLU) using Z-numbers [36], [37] has been framed for the purpose.

In [38], Banerjee *et al.* propose an augmentation of the Z-numbers for modeling dynamic real-world events. In addition to the basic Z-number factors, parameters to encapsulate the context (*C*), time-frame of activated memories (*T*), and consequent affective responses (*AG*)—in response to any event—form the Z∗-numbers. These numbers are semantically richer than the Z-numbers and can be envisioned as operands of "thoughts-in-action" in embodied [39] systems. An approximate procedure for Z∗-number-based NLU in an AGI framework has been formalized in [36] and [40]. Practical realization of the paradigm necessitates formalization of models of contemplation and comprehension and their translation into Z∗-based procedures (involving Z∗-based algebra and/or logic).

The Z-numbers do not explicitly encode the context of sentences. These numbers are, thus, unidimensional, i.e., are capable of precisiating only simple sentences particular to a context. Besides the Z∗-numbers, the multidimensional Z-numbers—proposed by Shen *et al*. [41]—are a mechanism for summarization of meanings embedded in the compound and complex sentences (***S***) on a subject (*X*). A multidimensional Z-number is represented as $((\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}), \boldsymbol{B})$, where $\boldsymbol{A_i}$ represents the *i*th restriction (dimension or descriptor) of a random variable (***X***), $\boldsymbol{A} = (\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots \boldsymbol{A_n})$ is the restriction vector of ***X***, and ***B*** equates to the restriction on the probability measure of ***A***. Operators and an algorithm for ranking multidimensional Z-numbers are defined in [41]. While Z∗-numbers and multidimensional Z-numbers are an improvement over the conjunction method of handling compound sentences described in [9] and [10], the automated translation of linguistic expressions to richer constructs remains unsolved. Formulation of measures of specificity of operation results involving these paradigms is one of the many salient areas of future investigation.

C. Z-Numbers in Decision-Making and Trust Modeling

In [42], Kang *et al*. extend their approach, described in [13], to arrive at approximate solutions for MCDM with *Z*-numbers. In the suggested framework, criteria weights and values of alternative conditions are in the form of Z-numbers. The procedure begins with the construction of a fuzzy decision matrix of Z-numbers of the evaluation-criteria weights and values of the given alternatives. These Z-numbers are converted to their fuzzy numeric equivalent, and the consequent decision-matrix is normalized. The numeric Z-numbers are then transformed into their crisp forms [13], and the priority of each alternative—as a function of the criteria weights—is computed. An example of vehicle selection, based on their price, journey-time, and comfort, using the proposed algorithm has been demonstrated. This procedure, although computationally inexpensive, suffers from the problem of information loss, as has been discussed earlier in the mechanisms that involve defuzzification.

Azadeh *et al*. [43] present a Z-number-based analytical hierarchy process (AHP) of decision-making. The algorithm begins with the top–down construction of the hierarchical structure of the criteria and subcriteria of the problem, followed by the alternatives. The criteria and alternatives are in the form of Z-numbers of expert inputs. Criteria of the lower levels are compared to those in the higher level, and pairwise comparisons set the criteria in the same level. The pairwise comparison matrix is defuzzified into crisp matrices, and the normalization of the geometric mean technique is applied to compute the weights of the criteria from the fuzzy pairwise comparison matrices. Once the weight of criteria, subcriteria are evaluated and multiplied to obtain a global weight of subcriteria, an overall score for the alternatives is set, and a decision is made accordingly. This technique also suffers from information loss because of the defuzzification of the Z-number inputs. The authors have tested their proposed method through simulation of a university selection problem.

The best worst method (BWM) [44] of MCDM has been found to lead to less inconsistent results and requires lesser pairwise comparisons than the AHP. In the BWM, the best and the worst criteria are first identified by decision makers (DM). The best and worst alternatives, with respect to these criteria, are then selected. For every criterion, all the alternatives are individually compared with the best and worst options. The final scores of the alternatives are derived as functions of the weights of different sets of criteria and alternatives. Over the years, BWM has been extended to the rough and fuzzy domains. A Z-based BWM for MCDM is described in [45]. The algorithm uses defuzzified Z-inputs in the fuzzy-BWM [46] method. Although the technique suffers from loss of information, as is the case with all the other aforesaid defuzzified Z-number procedures, it demonstrates greater consistency than the BWM and fuzzy-BWM. The authors have tested the proposed method in solving a supplier-development problem. The lower inconsistency is attributed to the consideration of vagueness, uncertainty, and reliability of the DM in the Z-numbers (parameter ***B***). One of the many possible future research directions is the development of techniques to handle missing expert inputs in the decision matrix, incorporating mechanisms to handle expert biases, and translation of complex linguistic expressions to their Z-number formats. Decision-making using crisp Z-number descriptors is described in [47]. (Refer Table I in the Supplementary Material.)

Aliev *et al*. [48] describe the use of Z-information in decision-making problems, where the probabilities of states of nature and outcomes of alternatives are presented as Z-numbers. The proposed approach proceeds in two stages. First is the conversion of Z-numbers to their fuzzy-number equivalents [13]. Second is the application of these values to a fuzzy utility function to arrive at the best alternative. The procedure is computationally rigorous, and the loss of information, induced by defuzzification, is an issue.

Bakar *et al.* [49]–[51] describe other two-stage Z-number ranking mechanisms for decision making. Some (e.g., [50]) use an intuitive vectorial centroid (for trapezoidal fuzzy numbers) approach to defuzzification followed by the application of fuzzy-TOPSIS [52] to rank the Z-alternatives, while others (e.g., [46]) rank the Z-inputs based on their Jaccard similarity. All these algorithms suffer from defuzzification-induced information loss, do not build semantic correlations between the Z-number parameters, and their practical rigor remains to be established.

Bayesian decision techniques are widely used across diverse domains for estimation, prediction, and decision support. In [53], Marhamati *et al*. attempt the modeling of expert knowledge—expressed in natural language—through Z-numbers and their subsequent application to Bayesian decision support systems. An advantage gained here is that not all the conditional probabilities (or likelihoods) need to be known in advance, as opposed to conventional Bayesian techniques. The unknown values can be approximated from the known likelihoods. The proposed algorithm has also been implemented in the CWShell [54] and its results are consistent with conservative crisp Bayesian approaches.

Authors in [55] improve upon the decision-making schemes of [48] and [49] by formalizing a Z-number-based multiattribute decision-making technique that does not defuzzify Z-numbers. In a Z-information valued decision environment, the characteristic vector of every alternative is a set of Z-numbers (assigned by an expert or a group of experts) per attribute. Instead of translating the Z-number inputs to their fuzzy number equivalents, the authors here conceptualize an ideal Z-point (or centroid) and a negative Z-point such that distances between the characteristic vector of each alternative and these Z-points can be evaluated. This leads to computation of the degree of membership of every alternative to the positive ideal solution. The alternative with the highest degree of membership is the required result. This methodology allows ranking of more than two Z-numbers (unlike those in [49]–[51]). It has been used to simulate the solution of a web services selection problem.

Peng *et al.* [56] present a novel multicriteria group decision-making (MCGDM) process that uses the normal cloud model in conjunction with the Z-numbers. The normal cloud model [57] derives from probability and fuzzy set theories to analyze and encase the randomness and fuzziness of qualitative concepts. This facilitates interchangeable precise conversion between qualitative concepts and quantitative values without overt loss and distortion in information. This feature of the model makes it an ideal platform for the use of Z-number expressions to summarize real-world conditions. The authors have formulated Z-based basic, power-aggregation, and distance-operators for a normal cloud model, leading to an enhanced multiobjective optimization algorithm. An air pollution potential evaluation problem has been simulated to test the strength of the proposed method. This is another attempt at the use of Z-numbers in their original format instead of defuzzifying them. The technique, however, requires values for all unknowns and operations are computationally expensive.

Most Z-number-based decision-making algorithms assume that the experts or DM are rational. However, in the real world, they are affected by factors such as basic character, risk preference, knowledge level, psychological state, interactions, and external environmental factors. As such, DMs are bounded rational rather than absolutely rational. In [58], Peng *et al.* define "hesitant uncertain linguistic Z-numbers" (HULZNs)—a special form or a subclass of Z-numbers that utilizes an interval linguistic value to describe the fuzzy restriction (parameter $\boldsymbol{A}$) instead of a single linguistic value and grades of belief values (parameter $\boldsymbol{B}$) on the restriction. This captures the hesitancy of the experts (or DM) in their inputs for the criteria in decision-making problems. The authors combine the VIKOR [59]–[61] method (for ranking options from a set of alternatives with conflicting criteria), the HULZNs (for representation of criteria evaluation by the experts), and power aggregation operators (for expert-information fusion) into a novel HULZN-distance-based multicriteria decision algorithm to deal with real-world events, in which the criteria-weights are incompletely known. The proposed method presents precise results in comparison to existing techniques when tested to solve an ERP selection problem. The HULZN technique does not suffer from defuzzification-induced information loss, rather captures greater degrees of uncertainty and is less computation-intensive compared to contemporary Z-based MCDM algorithms [56]. Results of the algorithm are, however, dependent on the linguistic scale functions, risk preferences and biases of the experts. Another Z-Ranking method can be found in [62] and is briefly described in Table I (Supplementary Material), where Qiu *et al*. introduce a "g-difference" of continuous Z-numbers to compute the derivative of a Z-number function. A generalized centroid-based technique to rank Z-numbers and optimality conditions for Z-optimization problems is provided as well.

In [63], Wang *et al.* attempt to improve upon the HULZ technique in [58]—by combining "linguistic Z-numbers," TODIM [64] (to model DM behavioral properties and preferences), and the Choquet integral [65] (to assimilate nonadditive linguistic Z-numbers)—to incorporate the inherent subjectivity of the DMs into MCDM. Operators for the processing, assimilation, comparison, and distance between linguistic Z-numbers to arrive at the best solution have been defined as well. Just as with [58], operation results here depend on the semantics of linguistic scale functions and DM preferences. A comparison between these techniques, as well as ways to handle DM biases—toward trustworthy Z-number-based decision-making systems—are much required.

Article presented in [66] redefines different methods [64], [67] using Z-numbers and also uses them in their fuzzy versions [52], [68] (refer Table I in Supplementary Material) toward Z-number-based TODIM and TOPSIS algorithms for MCDM. These procedures use defuzzified Z-numbers as inputs to the fuzzy-TODIM and fuzzy-TOPSIS algorithms. Results so far are at par with conventional crisp equivalents of these algorithms. In-depth analysis of the information loss, tests of applicability of these methods across problem domains, and comparison with other Z-number-based decision algorithms are future areas of work.

Trust modeling is a fundamental operation in business intelligence. It helps build and ensure business reputation, consumer confidence, fair trading, and long-lasting mutual relationships. One of the primary goals of trust modeling is to be able to predict future trust values of the interacting parties accurately. In recommendation-based trust computation, an agent would find it easier to express the amount of trust that it has in another agent as a linguistic expression rather than a real trust value. Furthermore, the reliability of the agent's judgment is a crucial factor in decision making. Thus, the Z-numbers could be of use in the precisiation of the agent's trust in sync with the degree of the agent's reliability. In [69], Azadeh *et al*. propose the use of Z-numbers for <*trust-value*, *reliability*> representation and the application of an artificial neural network (ANN) to predict future trust values. The authors generate large amounts of synthetic time-series data to mimic real-world dynamics of trust values of a trusted entity—for a short term of 12 months and a medium term of 21 months—and use that to validate their methodology. Information loss through Z-number defuzzification remains a challenge.

In [70], a novel algorithm to rank Z-numbers, using possibility degrees and weight acquisition, for solving MCDM problems is described. The "possibility degree" of Z-numbers is determined by combining the possibility degrees of the Z-number parameters using an adjustable risk preference parameter. The article also studies the outranking relations of Z-numbers using the developed ranking algorithm. Finally, an extended Preference Ranking Organization Method for Enrichment Evaluation II based on the possibility degree of Z-numbers is developed for the MCDM problem under Z-number evaluation.

The article by Shen *et al*. [71] defines two similarity measures for different Z-numbers. One of the measures deals with random information and another with the fuzzy information between the Z-numbers in question. The random information in a Z-number is determined using $\boldsymbol{X}$ (the subject), $\boldsymbol{A}$ (fuzzy restriction on the values of X), and the probability measure of $\boldsymbol{A}$. The similarity measure between the fuzzy information of two Z-numbers consists of two parts, i.e., one for the values of $\boldsymbol{A}$ corresponding to those Z-numbers and another for the values of $\boldsymbol{B}$ (fuzzy restriction on the value of probability measure of A). Finally, it introduces a new idea, viz., smallest enclosing circle, based on the suggested similarity measures to solve MCDM problems. The effectiveness of the developed method is illustrated on regional circular economy development plan selection problems.

Investigations in [72] deal with the MCGDM problem using information fusion and information measures. The authors define a concept, viz., uncertain Z-numbers, where a fuzzy restriction interval $[\boldsymbol{A}^{-}, \boldsymbol{A}^{+}]$ is used instead of a particular fuzzy restriction $\boldsymbol{A}$. The uncertain Z-number, ZU, is thus defined as

$$ZU = \left(\left[A^{-}, A^{+}\right], B\right) \quad (9)$$

where $\boldsymbol{B}$ equates to the original concept of Z-numbers. A method named "Z-trapezium-normal clouds (ZTNCs)" is introduced to quantify the uncertainty. Different operations such as power aggregation, distance, and likelihood measures of ZTNCs are defined. Finally, a method for solving MCGDM problems is formulated by combining the power aggregation operators and likelihood measures. The developed method has been successfully applied to energy evaluation problems in nonfossil energy sources.

In [73], the emphasis is on ranking continuous Z-numbers. The article introduces a Z-multi-objective evolutionary algorithm (ZMOEA) to solve Z-multi-objective programming. First, a Z-number is converted to a regular fuzzy number. Then, a relationship between two fuzzy numbers, based on the lower limit of the possibility degree, is defined. Drawing from this relationship, a novel ZMOEA is formulated to solve optimization problems.

D. Application of Z-Numbers to Solve Real-World Problems

The Z-number paradigm is one of many well-known mechanisms of summarization of natural statements pertaining to specific events and/or decision criteria. These statements are sourced from experts or observers. An expert's input, being a function of his/her expertise and perception, is imprecise, vague, and subjective. A Z-number encapsulates at least two levels of real-world uncertainty (an expert's comment ($\boldsymbol{A}$) and his/her certainty/hesitancy ($\boldsymbol{B}$) with regards to $\boldsymbol{A}$) and thus finds extensive use in the representation of expert inputs to decision-making systems. This section describes some areas where the Z-numbers have proved to be of use. The transfer or extension of all mechanisms presented here to model other real-world problems is much required.

> "Informativeness is a desideratum when a Z-number is a basis for a decision. A basic question is: When is the informativeness of a Z-number sufficient to serve as a basis for an intelligent decision?" [1]

1) Psychological Research: In [74], Aliev *et al*. describe a Z-number-based fuzzy approach for modeling the effect of Pilates exercises on motivation, attention, anxiety, and educational achievement. The psychological inputs were collected through questionnaires prepared in accordance with internationally recognized instruments: Academic Motivation Scale, Test of Attention (D2 Test), and Spielberger's Anxiety Test completed by students. The GPA of students was used as the measure of educational achievement. Student inputs were translated into Z-numbers, and a Z-IF–THEN rule base—connecting educational achievement and the psychological parameters—was used for reasoning and Z-interpolation [1], [55] of the input to the appropriate assessment level. The analysis depicts no significant variance between the mean of results obtained through conventional statistical methods and Z-number-based modeling. The standard deviations of the outputs, estimated by statistical method and Z-number-based modeling, however, show significant differences. The authors adjudge the less variance in the Z-number-based modeling method as a derivative of the uncertainty controlling ability of the paradigm. Aliev *et al.* [25] demonstrate a similar Z-distance-based IF–THEN rule attempt at the modeling of the relationship between student motivation, attention, anxiety, and educational achievement.

2) Alloy Selection: In [75], the authors use an extended version (through the conjunction of fuzzy-Pareto-optimality and pessimism-optimism-degree-based comparison of Z-numbers, envisioning improved results) of the Z-number-based multiattribute decision-making technique in [55] for the selection of alloys for aeronautical purposes. The experts provided inputs for criteria such as strength level, plastic deformation degree, and tensile strength.

3) Marketing Mix Problem: In [76], Alizadeh *et al*. demonstrate the use of the Z-number-based linear programming technique in [31] to solve a marketing mix problem, where the DM criteria-inputs for six production processes aiming at maximizing profit, quality, and worker satisfaction are translated into Z-number operands.

4) Dynamic Control Systems: Abiyev [77] proposes a Z-number distance-based fuzzy-rule interpolation technique for the design of a dynamic plant controller. Comparison of the transient response characteristics obtained through conventional statistical procedures and the Z-number mechanism demonstrates the suitability of the latter in dynamic-systems design.

Abiyev *et al.* [78] describe the use of a similar Z-rule base and interpolation technique to design the control system of an omnidirectional soccer-playing robot. Resultant Z-numbers were defuzzified into precise locations for the robots. The technique is at par with basic fuzzy and type-2 fuzzy controllers.

5) Safety Analytics: Safety analytics deals with the descriptive, predictive, and prescriptive components of modern safety management systems. It involves the creation of a safety-rule database and the development of algorithms, tools, and techniques for effective decision-making. The success of safety analytics depends on the quality of training data, extraction of reliable information, and uncertainty-handling mechanisms for decision-making. Safety data are often collected from experts and is, therefore, a potential application area of the Z-number paradigm. Das *et al.* [3] propose a Z-similarity-based bow-tie importance measure (Z-BEC) for identifying the most critical event responsible for a particular accident scenario. Garg *et al.* [79] describe a granulized rough-set-based Z-number for enhanced representation and handling of experts' safety perception. Zamri *et al.* [80] detail a Z-TOPSIS procedure for the evaluation of causes of accidents at construction sites.

Jiang *et al.* [81] describe a model for failure mode and effect analysis (FMEA) based on Z-numbers. The authors construct a Z-valuation structure, where the assessments of risk factors are expressed by Z-numbers. The fuzzy weighted mean method is used to integrate the Z-valuations of risk factors and synthesize the integrated valuations of the experts in the FMEA team into a fuzzy number. Finally, Z-risk priority numbers are calculated by a modified method of ranking fuzzy numbers to rank failure modes. The primary advantage of this approach is that the risk assessments are based on Z-numbers, which contain the reliability of the uncertain valuations. Further, in comparison with the traditional fuzzy FMEA methodologies, this method overcomes setbacks arising from defining a large number of membership functions and IF–THEN rules, which are time-consuming and difficult. The extension of this technique to model similar real-world problems is indeed a promising area of future work.

6) Investment Risk Analysis: A Z-number-based regret theory method is proposed in [82] to determine the utility, rejoice, and regret values of Z-information. In general, the regret theory method comprises the utility function, utility value, and regret–rejoice function. The key step of this method is to calculate the utility value by combining the utility function with the probability density function of variables. However, the probability density function is unknown in many problems, and many studies have assumed that the variables follow a specific distribution (e.g., uniform distribution) so that the probability density function can be obtained. To overcome this limitation, a Z-number-based regret theory method is proposed to integrate the probability density function with the utility function to calculate the utility value.

7) Multicriteria Game Model: Studies in [83] emphasize the development of a game model to compare different strategies adopted by various players. The authors describe a novel concept of asymmetric normal Z-value (ANZ) to handle the bimodal restriction in Z-numbers. An innovative concordance/discordance index and outranking relations of ANZs are also presented. Finally, a model is developed that used the outranking relations of various strategies by multiple players based on several criteria and circumstances. The effectiveness of the presented model is demonstrated on an enterprise market competition problem. The study is suitable for the two-person nonconstant sum game and needs to be extended to other games, particularly n-person games.

In [84], an algorithm for analyzing strategies for evolutionary games using Z-number is proposed. Here, Z-number is used to simulate human behavior of competition and cooperation more accurately and flexibly.

8) Medical Diagnosis: The article by Wu *et al*. [85] focuses on handling uncertainty in medical diagnoses using the concept of Z-numbers. The authors propose a method for ranking fuzzy numbers and for transforming Z-numbers into basic probability assignments. The estimated probabilities of Z-number inputs are combined by Dempster's combination rule and are utilized for multisource differential diagnosis. The efficacy of the method has also been tested on risk analysis.

9) Data Envelopment Analysis (DEA): Azadeh and Kokabi [86] emphasize the modification of different existing DEA models to make them suitable for accepting Z-numbers to facilitate the handling of vague and incomplete data. The developed model (Z-DEA) can be used as a fuzzy DEA model when there is no ambiguity in the information and the experts are confident about their opinion. Similarly, the model becomes a conventional one when the inputs and outputs are crisp. In this study, first, the Z-DEA model is converted into a possibilistic linear programming model, and then it is converted into crisp parametric linear programming by applying the α-cut method. Z-DEA is applied to a portfolio selection problem in an Information Systems/Information Technology project to address various issues such as uncertainty and reliability.

10) Social Media Evaluation: The study in [87] focuses on the evaluation and selection of online social media content for public communication. The method developed here uses fuzzy AHP, Z-numbers, and fuzzy multiattributive border approximation area comparison for superior utilization of available channels of communication via the Internet, thereby improving the participation of citizens in public administration.

11) Z-Clusters: Aliev *et al*. [88] emphasize challenges arising from the uncertainty implicitly embedded in real-world data—it is not possible to find out the true (unbiased and absolute) probability distribution of such data. The authors address this problem by constructing Z-number valued IF–THEN rules from the data by using general type-2 (GT2) fuzzy clusters. The defined clustering method exploits the objective function of fuzzy C means and evolutionary algorithms to generate fuzzy sets of fuzzy clusters. The concept of rule construction draws from the relationship between type-2 fuzzy sets and Z-numbers. GT2 fuzzy numbers are generated using projections of GT2 fuzzy clusters. The fuzzy constraint $\boldsymbol{A}$ ($\boldsymbol{Z} = (\boldsymbol{A}, \boldsymbol{B})$) is computed as a centroid of the uncertainty footprint of a GT2 fuzzy number. The probability distribution corresponding to each type-2 fuzzy set is computed, and for each probability distribution, its membership to a fuzzy set of these distributions is determined. This is used as the value of $\boldsymbol{B}$—a fuzzy restriction on the reliability (probability measure) of $\boldsymbol{A}$. The developed method for IF–THEN rule generation in order to extract important information has been successfully applied on Parkinson's disease diagnosis.

12) Sensor Data Fusion and Fault Diagnosis: The article in [89] presents a novel method—using Z-numbers and D–S [18] evidence theory—to model and process uncertainties in a multisensor data-fusion system. While the Z-numbers encapsulate the fuzziness and reliability of the sensor data, the D–S evidence theory is used to fuse the uncertain information in the Z-numbers. The membership function of the characteristic information of the variables (component $\boldsymbol{A}$ of a Z-number) is modeled using a Gaussian distribution function. The fuzzy reliability (component $\boldsymbol{B}$ of the Z-number) is a function of the divergence of the characteristics, over time, of the sensor data. A base fault model of the system is generated by analyzing the properties of typical failures. This model and the Z-numbers of the test data are matched to produce individual mass functions of reliability, which are then fused using the D–S evidence theory. This result is analyzed to generate a diagnostic decision for the system under consideration. Key advantages of the proposed method are that it provides robustness to the sensor data, and the complementary multisensor information aids reduction of the uncertainty in fault recognition, thereby enhancing the reliability of fault detection. Determination of the base fault model and mass functions are open areas of research. Applicability of this methodology, in conjunction with the multimodal data structures in [90] for machine-mind frameworks [37], is a potential area of future investigation.

13) Some Other Applications: Articles in [91] and [92] demonstrate the use of the multiattribute decision-making technique in [55] for the assessment of suppliers and memorandums of understanding, respectively. Chatterjee and Kar [93] describe the application of an extended version (consideration of subjective and objective weights as functions of parameters $\boldsymbol{B}$ and $\boldsymbol{A}$, respectively) of the decision-making algorithm in [42] to rank renewable energy sources in India. Forghani *et al.* [94] report the application of a Z-TOPSIS approach (additional use of PCA and linear programming) to supplier selection in the pharmaceutical supply chain. Pal *et al.* [95] describe the application of Z-numbers in the preparation of scene-difference-description-summaries for surveillance.

In [96], the authors use the idea of Zadeh's Z^{+}-number [1] to define a set, viz., $\boldsymbol{S}^{+}$—in which an element or pattern is represented by a pair of membership values and the level of confidence for that pair. While one of the memberships represents the degree of belongingness of a pattern to a class/set, the other implies its class of origin. The concept of $\boldsymbol{S}^{+}$ is further used to define an entropy measure (SPEM) that can be used to measure the degree of confidence in the computed entropy value. The SPEM is used to determine the relevance and the level of confidence of a miRNA to identify its significance for drug resistance in cancer patients.

III. Comparison of Z-Numbers With Some Similar Concepts

Having presented the various interpretations, applications, and extensions to the Z-number paradigm over the past decade, this section briefly elucidates upon differences between some other existing paradigms that bear conceptual similarities with the Z-numbers. Just as the Z-numbers, the paradigms presented here were all designed to handle complex real-world uncertainties—moving in the direction of generalized fuzzy set representations—where variables defining random events are described using more than one membership value.

1) *Spectral fuzzy set:* In [97], Pal *et al.* developed the "spectral fuzzy set" to handle difficulties in assigning a particular membership value to signify a certain degree of fuzziness. Spectral fuzzy sets are characterized by a set of membership functions—each representing opinions and, consequently, leading to a membership value. Spectral fuzzy sets have found application in image processing.

Formally, a spectral fuzzy set $\boldsymbol{F}$, in the universe of discourse $\boldsymbol{U}$ and having $\boldsymbol{n}$ supports, is characterized by a set of $\boldsymbol{r}$ membership functions (reflecting $\boldsymbol{r}$ opinions) and is represented as

$$F = \bigcup_j \left[\bigcup_i \boldsymbol{\mu}_{\boldsymbol{F}}^{\boldsymbol{i}}\left(\boldsymbol{x_j}\right)/\boldsymbol{x_j}\right], x_j \in \mathrm{U} \quad (10)$$

where $\boldsymbol{i} = 1, 2, \ldots, \boldsymbol{r}, \boldsymbol{j} = 1, 2, \ldots, \boldsymbol{n},$ and $\boldsymbol{\mu}_{\boldsymbol{F}}^{\boldsymbol{i}}(\boldsymbol{x_j})$ represents the degree of belongingness of $\boldsymbol{x_j}$ to the set $\boldsymbol{F}$ as per the $\boldsymbol{i}$th membership function.

While a spectral fuzzy set captures diverse opinions for a set of values $(\boldsymbol{x_j} \in \boldsymbol{U})$ on the subject ($\boldsymbol{F}$), a set of Z-numbers (or Z-information) on $\boldsymbol{F}$ would capture a multitude of opinions in conjunction with individual confidences or beliefs in the opinion.

A possible interpretation of spectral fuzzy sets—through the lens of the Z-numbers—could be where each membership function is a mathematical representation of a particular degree of confidence (viz., "most likely," "likely," "probably," etc.) pertaining to the context $\boldsymbol{F}$, and every $\boldsymbol{\mu}_{\boldsymbol{F}}^{\boldsymbol{i}}(\boldsymbol{x_j})$ represents the confidence on $\boldsymbol{x_j}$ for $\boldsymbol{F}$. This interpretation of the spectral numbers is restrictive since it specifies the number of supports in a universe of discourse and the number of confidence values possible, and herein lies the difference with the Z-numbers.

2) *Intuitionistic fuzzy set*: Defined by Atanassov [98], intuitionistic fuzzy sets are used to represent the nonmembership values of the elements in a particular set. Intuitionistic fuzzy sets have found wide application in pattern recognition, predictions, and decision-making.

An intuitionistic fuzzy set $\boldsymbol{A}$ in the universe of discourse $\boldsymbol{U}$ is defined as

$$A = \left\{\left\langle \boldsymbol{x}, \boldsymbol{\mu_I}\left(\boldsymbol{x}\right), \boldsymbol{\nu_I}\left(\boldsymbol{x}\right)\right\rangle \middle| \, \boldsymbol{x} \in \boldsymbol{U}\right\} \quad (11)$$

$$0 \leq \boldsymbol{u_I}\left(\boldsymbol{x}\right) + \boldsymbol{\nu_I}\left(\boldsymbol{x}\right) \leq 1 \;\; \forall \boldsymbol{x} \in \boldsymbol{U} \quad (12)$$

where $\boldsymbol{\mu_I} : \boldsymbol{U} \rightarrow [0, 1]$ and $\boldsymbol{\nu_I} : \boldsymbol{U} \rightarrow [0, 1]$ represent the degrees of membership and nonmembership, respectively, of the element $\boldsymbol{x}$. The primary difference between an intuitionistic fuzzy set and a Z-number is that while the former can be interpreted as a set of affirmative and dissenting opinions of an individual on a particular subject ($\boldsymbol{X}$), the latter inherently captures multiple (affirmative and dissenting) opinions on $\boldsymbol{X}$ in conjunction with individual confidences per opinion—these opinions could belong to the same or multiple individuals.

3) *Type-2 fuzzy set:* Type-2 fuzzy sets are an extension of the ordinary fuzzy set developed by Mendel and John [99] and Zadeh [100]. Here, the membership of an element in the set is described as a fuzzy number in [0, 1]. The concept applies to situations where the membership values

are uncertain or fuzzy (identified via a "footprint of uncertainty"). A type-2 fuzzy set reduces to a type-1 fuzzy set when the uncertainty over membership values vanishes. Type-2 fuzzy sets have found extensive use in CWW and computer vision.

A type-2 fuzzy set ($\tilde{\boldsymbol{A}}$) in the universe of discourse $\boldsymbol{U}$ is defined as

$$\tilde{\boldsymbol{A}} = \left\{ \left((\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}, \boldsymbol{u})\right) \middle| \quad \forall \boldsymbol{x} \in \boldsymbol{U} \quad \forall\, \boldsymbol{u} \in \boldsymbol{J}_{\boldsymbol{x}} \subseteq [0,1] \right\} \tag{13}$$

where $\boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}, \boldsymbol{u})$ represents the type-2 fuzzy membership function such that $\boldsymbol{u} \in \boldsymbol{J}_{\boldsymbol{x}} \subseteq [0, 1]$ and $0 \leq \boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}, \boldsymbol{u}) \leq 1$. $\tilde{\boldsymbol{A}}$ can also be represented as

$$\tilde{\boldsymbol{A}} = \int_{\boldsymbol{x} \in \boldsymbol{U}} \int_{\boldsymbol{u} \in \boldsymbol{J}_{\boldsymbol{x}}} \boldsymbol{u}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}, \boldsymbol{u}) / (\boldsymbol{x}, \boldsymbol{u}) \tag{14}$$

where $\boldsymbol{u} \in \boldsymbol{J}_{\boldsymbol{x}} \subseteq [0, 1]$ and $\int$ indicates the fuzzy union operator. The type-2 fuzzy membership function $\boldsymbol{\mu}_{\tilde{\boldsymbol{A}}}(\boldsymbol{x}, \boldsymbol{u})$ may be conceptually equated to a representation of confidence in $(\boldsymbol{x}, \boldsymbol{u})$. By virtue of the embedded multilevel uncertainty in its definition, we envision that the type-2 fuzzy set could serve as the platform for the realization of the Z-numbers. This and the potential extension of the Z-numbers to type-2 and type-n Z-numbers is an intriguing area of future investigation (as has also been indicated in Section IV-B).

To summarize discussions in this section, the type-2 fuzzy set, the spectral fuzzy set, and the intuitionistic fuzzy set bear conceptual similarities with the Z-numbers in terms of their endeavors in the encapsulation of real-world uncertainties. These concepts, however, structurally differ in their capacities of representation of these uncertainties. Given their individual strengths, an extension of each of these paradigms to their Z-number variants is a potential area of future research.

IV. Where We Stand

A. Challenges and Issues

The Z-numbers, as defined by Zadeh in [1], are precisiations of simple natural language statements. These encase the information (A) embedded in a statement and the agent's sense of reliability in A. This article presents a review of the advancements and applications made of the Z-numbers since its coinage. Theoretical advancements include a definition of discrete and continuous Z-arithmetic operators, Z-similarity measures, linear programming techniques, multiattribute and multiobjective decision algorithms, and strategies for computations with words and perceptions. Each of these operators and methods aims for the application of the Z-numbers in solving complex, real-world problems. Some of these methodologies are computationally rigorous and suffer from loss of information because of defuzzification of Z-inputs and nonutilization of the correlation between the Z-parameters, while others are as good as conventional statistical approaches with the added advantage of improved uncertainty representation and handling. These techniques have been used to simulate solutions to areas such as psychological modeling, alloy selection, supplier selection, dynamic controller design, safety analytics, multisensory data fusion, initial value problems (such as population biology and drug control), surveillance, and NLU.

This decade has also provided at least four extensions to the paradigm—the HULZNs, parametric Z-numbers, multidimensional Z-numbers, and the Z∗-numbers. The linguistic and parametric Z-numbers target decision-making problems, and the multidimensional and Z∗-numbers aim for enhanced computing with perceptions. The Z-numbers have also been used in conjunction with ANN, rough sets, and granules for improved real-world uncertainty handling.

It might be right to conclude that all advancements, so far, have been theoretical in nature. Practical application of the Z-numbers necessitates working on the resolution of certain issues that have—and continue to—afflict the paradigm. Some of the many challenges are as follows.

1) Automated translation of natural language expressions to Z-numbers and vice versa: As we progress into the era of "strong AI" or AGI systems, techniques for the automated conversion of natural language expressions (sourced from humans or other intelligent agents) to their Z-number equivalents (and vice versa) for decision-making would be required. This necessitates machines to be capable of understanding natural language sentences—inclusive of idiosyncrasies (irregularities, affects, metaphors, incompleteness, sarcasm, etc.) of human expressions.
2) Design of Z-entropy measures in sync with the representation of "deep semantics" of source linguistic expressions: This point is related to the emulation of "surprise" or "curiosity" in intelligent agents, as proportional to the information (or entropy) embedded in natural language expressions—sentences that are sources of the Z-number operands in decision-making operations. Intuitively, high Z-entropy values would signify lesser known or novel information and consequently trigger "surprise" in the intelligent agent and the "urge" to explore an event and add to the knowledge repository, instead of exploiting what is known.
3) Analyses of the linguistic equivalence between results of Z-operators and intuitive human responses: Merely operating upon Z-numbers is inconsequential unless the operation results can be mapped to equivalent natural language expressions. This is particularly crucial in human–machine interaction scenarios where results of machine-processes (Z-number operations) need to be effectively communicated to human beings.
4) Procedures for the identification of probability–possibility distributions and membership functions that completely map the event/problem under consideration: Drawing from studies presented in Section II, this point focuses on the need to design probability distributions that are devoid of all biases toward trustworthy Z-number representations. This is particularly relevant in application scenarios (see Section II-D) that depend on expert inputs as information sources. An area of future investigation, in this regard, is the machine drawing upon its own experiences to arrive at probability distributions and membership functions for events.
5) A study of intra- and interdomain Z-number synonymity: Mechanisms to consider the context of Z-numbers to account for Z-number specificity of similar

expressions—but with different connotations—within and across domains. This further extrapolates to the analyses of the sufficiency of Z-number parameters as fundamental features or determinants of the meaning of statements possible in any event.

Besides these issues of general interest, some other questions worth investigating are the: 1) arousal of the sense of belief/confidence in artificial expert-agents toward endogenous instantiation of Z-numbers; 2) use of Z-numbers in multimodal/mutisensory environments; and 3) the possibility of application of the paradigm to high-precision and high-risk domains such as space science, deep-sea exploration, differential diagnosis, bioinformatics, and climate informatics, to name a few. Challenges enumerated in this section pave the way for pointers to future areas of work—some of which are presented in the following section.

B. Future Works

Some particular technical insights into investigations of relevance in the times to come, which can also be used to address the aforesaid issues, are the following.

1) Methods for the estimation and reduction of Z-based operational complexity. While Tadayon *et al*. [101] describe techniques for the same, the pursuit of optimal Z-operations—across domains—is an intriguing area of research. [Refer Table I in the Supplementary Material for a brief description of [101].]
2) Z-number-based deep learning, where the Z-number parameters not only form weights of the neural network paths but are also instantiated/updated from network-learning processes. Mitra and Pal [102] present a logical feed-forward multilayer perceptron network using fuzzy sets across stages. Intuitively, substituting the fuzzy sets with Z-numbers in such networks would serve as a scaffold for the realization of Z-number-based classification, rule-generation for decision-making, and deep learning.
3) Z-number-based generation and updating of knowledge networks at real time, where $<\boldsymbol{X}, \boldsymbol{A}>$ would symbolize nodes of the network and $\boldsymbol{B}$ would equate to the confidence in the $<\boldsymbol{X}, \boldsymbol{A}>$ pair. Conceptually, $<\boldsymbol{X}, \boldsymbol{A}>$ would represent an experience, and the value of $\boldsymbol{B}$ would be a function of reinforcement of $<\boldsymbol{X}, \boldsymbol{A}>$ which would further influence activations thereof. A cluster of such $<\boldsymbol{X}, \boldsymbol{A}>$ links could also be extrapolated to Z-information sets or concept clusters on $\boldsymbol{X}$.
4) Z-number-based reinforcement learning models—the Z-entropy concept—that briefly touched upon in the earlier section could contribute to active reinforcement learning schemes; direct utility estimation techniques [103] could be used for automated probability distribution estimations and subsequent passive Z-number-based reinforcement learning of real-world events.
5) Defining sets of Z-numbers where the membership of an element $\boldsymbol{X}$ in set $\boldsymbol{Y}$ can be represented as $\mu_{\boldsymbol{Y}}(\boldsymbol{X}) = <\boldsymbol{A}, \boldsymbol{B}>$: $\boldsymbol{A}$ denoting the membership of belongingness of $\boldsymbol{X}$ in set $\boldsymbol{Y}$ and $\boldsymbol{B}$ denoting the expression of certainty of $\boldsymbol{A}$; formalized as a fuzzy restriction on the value of the probability measure of $\boldsymbol{A}$. Drawing cues from level-2 fuzzy sets [6], sets of Z-numbers could be extended to form level-2 Z-numbers—incorporating a further layer of the uncertainty of beliefs or confidences into the definition.
6) Analyses and comparison between the performance of Z-sets and type-2 fuzzy sets toward choosing superior technology for improved decision-making systems in the light of the ever-changing uncertain real world.
7) Z-number-based evolutionary computing strategies to model the endogenous arousal and evolution of thoughts and beliefs in AGI systems. This would involve not only the translation of emotional/affect cues embedded in natural language into numeric approximations for subsequent operations but also mechanisms to process synonyms and metaphorical expressions of beliefs.

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