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Roughness of a Fuzzy Set

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ABSTRACT

An integration between the theories of fuzzy sets and rough sets has been attempted by providing a measure of roughness of a fuzzy set. Several properties of this new measure are established. Some of the possible applications for handling uncertainties in the field of pattern recognition are mentioned.

1. INTRODUCTION

The theory of fuzzy sets [1] provides an effective means of describing the behavior of systems which are too complex or too ill-defined to admit precise mathematical analysis by classical methods and tools. It has shown enormous promise in handling uncertainties to a reasonable extent, particularly in decision-making models under different kinds of risks, subjective judgment, vagueness, and ambiguity. Extensive application of this theory to various fields, e.g., expert systems, control systems, pattern recognition, and image processing, has already been well established.

More recently, the theory of rough sets [2] has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy, or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of and reasoning with vague

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and/or imprecise knowledge, data classification, data analysis, machine learning, and knowledge discovery [3, 4]. The theory is also proving to be of substantial importance in many areas of applications [4].

It may be noted that fuzzy set theory hinges on the notion of a membership function on the domain of discourse, assigning to each object a grade of belongingness in order to represent an imprecise concept. The focus of rough set theory is on the ambiguity caused by limited discernibility of objects in the domain of discourse. The idea is to approximate any concept (a crisp subset of the domain) by a pair of exact sets, called the lower and upper approximations. But concepts, in such a granular universe, may well be imprecise in the sense that, these may not be representable by crisp subsets. This leads to a direction, among others, in which the notions of rough sets and fuzzy sets can be integrated, the aim being to develop a model of uncertainty stronger than either. The present work may be considered as an attempt in this line.

In a partitioned domain of discourse, a measure of roughness of an ordinary set has been introduced by Pawlak [3]. This is extended here to give a measure of roughness of a fuzzy set defined in the partitioned domain, making use of the concept of a rough fuzzy set [5]. A preliminary study of the idea is presented in this paper.

The next section contains requisite notions of rough sets and fuzzy sets. In Section 3, definition of the measure and some consequences are put forth. An interpretation of this measure in the field of pattern recognition is given to conclude the paper.

2. PRELIMINARIES

We first define some basic concepts of rough set theory.

Let the domain U of discourse (also called *universe*) be a nonempty finite set, and R an equivalence relation on U. The pair $\langle U, R\rangle$ is called an *approximation space* [2]. Let $X_1, \ldots, X_n$ denote the equivalence classes in U due to R, i.e., $\{X_1, \ldots, X_n\}$ forms a partition of U.

To represent such a partition of U, one may use the idea of an *information system* [2]. An information system with U as universe would, formally, be a pair $\langle U, X\rangle$, where X is a set of *attributes*. Each attribute x can be understood as a total function $x: U \to V_x$, which associates to every object an attribute value. One can easily observe that every subset Y of the attribute set X induces an equivalence relation $IND(Y)$, called an *indiscernibility relation*, as follows:

$$IND(Y) \equiv \{(u,v) \in U^2 : x(u) = x(v), \quad \text{for each } x \in Y\}.$$

It may further be noticed that $IND(Y) = \bigcap_{x \in Y} IND(x)$.

Let us return to an arbitrary approximation space $\langle U, R \rangle$, giving rise to a partition $\{X_1, \ldots, X_n\}$ of U.

If $A \subseteq U$, the *lower approximation* $\underline{A}$ and *upper approximation* $\overline{A}$ of A in the approximation space $\langle U, R \rangle$ are respectively given as follows [2]:

$$\underline{A} \equiv \bigcup \{X_i \colon X_i \subseteq A\},$$

$$\overline{A} \equiv \bigcup \{X_i \colon X_i \cap A \neq \emptyset\}, \qquad i \in \{1, \ldots, n\}.$$

$\underline{A}$ ($\overline{A}$) is interpreted as the collection of those objects of the domain U that definitely (possibly) belong to A.

The triple $\langle U, R, A \rangle$ is called a *rough set* [6]. Equivalently, the pair $\langle \underline{A}, \overline{A} \rangle$ may be called a rough set.

$A(\subseteq U)$ is called *exact* (also called *definable*) in the approximation space $\langle U, R \rangle$ if and only if $\underline{A} = \overline{A}$.

$A, B(\subseteq U)$ are said to be *roughly equal* in the approximation space $\langle U, R \rangle$ if and only if $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$. The notion of rough equality indicates that, relative to the available information, one is unable to discern between the sets concerned. It is thus a kind of indiscernibility at the concept level.

Roughness of a set A in the approximation space $\langle U, R \rangle$ is reflected by the ratio of the number of objects in its lower approximation to that in its upper approximation—the greater the value of the ratio, the lower the roughness. More explicitly, a *measure* ρ_A *of roughness* of A in $\langle U, R \rangle$ is defined thus [3]:

$$\rho_A \equiv 1 - \frac{|\underline{A}|}{|\overline{A}|},$$

where $|X|$ denotes the cardinality of a set X.

OBSERVATION 2.1.

(a) As $\underline{A} \subseteq A \subseteq \overline{A}$, $0 \leq \rho_A \leq 1$.
(b) By convention, when $A = \emptyset$, $\underline{A} = \emptyset = \overline{A}$ and $|\underline{A}|/|\overline{A}| = 1$, i.e., $\rho_A = 0$.
(c) $\rho_A = 0$ if and only if A is exact in $\langle U, R \rangle$, i.e., $\underline{A} = A = \overline{A}$.

We next come to the definition of a rough fuzzy set and allied notions [5], that shall form the basis of this work.

Let $\mathbf{A}\colon U \to [0, 1]$ be a fuzzy set in U [1], $\mathbf{A}(x)$, $x \in U$, giving the degree of membership of x in $\mathbf{A}$.

DEFINITION 2.1. The *lower and upper approximations of the fuzzy set* $\mathbf{A}$ in U, denoted $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$, respectively, are defined as fuzzy sets in U/R $(\equiv \{X_1, \ldots, X_n\})$, i.e., $\underline{\mathbf{A}}, \overline{\mathbf{A}}$: $U/R \to [0,1]$, such that

$$\underline{\mathbf{A}}(X_i) \equiv \inf_{x \in X_i} \mathbf{A}(x) \quad \text{and}$$

$$\overline{\mathbf{A}}(X_i) \equiv \sup_{x \in X_i} \mathbf{A}(x), \qquad i = 1, \ldots, n,$$

where inf denotes minimum and sup maximum.

$\langle \underline{\mathbf{A}}, \overline{\mathbf{A}} \rangle$ is called a *rough fuzzy set*. Equivalently, one may call the triple $\langle U, R, \mathbf{A} \rangle$ a rough fuzzy set.

OBSERVATION 2.2. When $\mathbf{A}$ is a crisp set, $\underline{\mathbf{A}}$, $\overline{\mathbf{A}}$ reduce respectively to the collection of equivalence classes constituting its lower and upper approximation in $\langle U, R \rangle$.

DEFINITION 2.2. Fuzzy sets $\underline{\mathscr{A}}, \overline{\mathscr{A}}$: $U \to [0,1]$ are defined as follows.

$$\underline{\mathscr{A}}(x) \equiv \underline{\mathbf{A}}(X_i) \quad \text{and} \quad \overline{\mathscr{A}}(x) \equiv \overline{\mathbf{A}}(X_i),$$

if $x \in X_i$, $i \in \{1, \ldots, n\}$.

OBSERVATION 2.3. $\underline{\mathscr{A}}$ and $\overline{\mathscr{A}}$ are fuzzy sets with constant membership on the equivalence classes of U. For any x in U, $\underline{\mathscr{A}}(x)$ $(\overline{\mathscr{A}}(x))$ can be viewed as the degree to which x definitely (possibly) belongs to the fuzzy set $\mathbf{A}$.

We state some consequences of the preceding definitions [5].

PROPOSITION 2.1.

(a) $\underline{\mathscr{A}} \subseteq \mathbf{A} \subseteq \overline{\mathscr{A}}$,
(b) $\overline{\mathbf{A} \cup \mathbf{B}} = \overline{\mathbf{A}} \cup \overline{\mathbf{B}}$,
(c) $\underline{\mathbf{A} \cap \mathbf{B}} = \underline{\mathbf{A}} \cap \underline{\mathbf{B}}$,
(d) $\underline{\mathbf{A}} \cup \underline{\mathbf{B}} \subseteq \underline{\mathbf{A} \cup \mathbf{B}}$,
(e) $\overline{\mathbf{A} \cap \mathbf{B}} \subseteq \overline{\mathbf{A}} \cap \overline{\mathbf{B}}$,
(f) $\overline{\mathbf{A}^c} = \underline{\mathbf{A}}^c$,
(g) $\underline{\mathbf{A}^c} = \overline{\mathbf{A}}^c$,
(h) $\overline{\overline{\mathscr{A}}} = \underline{\overline{\mathscr{A}}} = \overline{\mathbf{A}}$,
(i) $\underline{\underline{\mathscr{A}}} = \overline{\underline{\mathscr{A}}} = \underline{\mathbf{A}}$,

where

$$(\mathbf{A} \cup \mathbf{B})(x) \equiv \max(\mathbf{A}(x), \mathbf{B}(x)),$$

$$(\mathbf{A} \cap \mathbf{B})(x) \equiv \min(\mathbf{A}(x), \mathbf{B}(x)),$$

$$\mathbf{A}^c(x) \equiv 1 - \mathbf{A}(x), \quad \textit{for any } x \textit{ in } U, \quad \textit{and}$$

$$\mathbf{A} \subseteq \mathbf{B} \textit{ if and only if } \mathbf{A}(x) \leq \mathbf{B}(x), \quad \textit{for any } x \textit{ in } U.$$

3. ROUGHNESS MEASURE OF A FUZZY SET

Let us consider parameters α, β, where $0 < \beta \leq \alpha \leq 1$, and the α-cut $\underline{\mathscr{A}}_\alpha$, β-cut $\overline{\mathscr{A}}_\beta$ of the fuzzy sets $\underline{\mathscr{A}}, \overline{\mathscr{A}}$, respectively, viz.,

$$\underline{\mathscr{A}}_\alpha \equiv \{x : \underline{\mathscr{A}}(x) \geq \alpha\} \quad \text{and}$$

$$\overline{\mathscr{A}}_\beta \equiv \{x : \overline{\mathscr{A}}(x) \geq \beta\}.$$

It may then be said that $\underline{\mathscr{A}}_\alpha$ ($\overline{\mathscr{A}}_\beta$) is the collection of objects in U with α (β) as the minimum degree of definite (possible) membership in the fuzzy set **A**. In other words, α, β act as thresholds of definiteness and possibility, respectively, in membership of the objects of U to **A**. We call $\underline{\mathscr{A}}_\alpha$ the *α-lower approximation* and $\overline{\mathscr{A}}_\beta$ the *β-upper approximation* of the fuzzy set **A** in $\langle U, R\rangle$.

Henceforth, we adhere to the above restriction on the parameters α and β, viz., $0 < \beta \leq \alpha \leq 1$.

OBSERVATION 3.1.

(a) $\underline{\mathscr{A}}_\alpha = \bigcup\{X_i : X_i \in \underline{\mathbf{A}}_\alpha\}$ and $\overline{\mathscr{A}}_\beta = \bigcup\{X_i : X_i \in \overline{\mathbf{A}}_\beta\}$, where $i \in \{1, \ldots, n\}$ and $\underline{\mathbf{A}}_\alpha$, $\overline{\mathbf{A}}_\beta$ are the α- and β-cuts, respectively, of the fuzzy sets $\underline{\mathbf{A}}$ and $\overline{\mathbf{A}}$ (cf. Definition 2.1). So, alternatively, $\underline{\mathscr{A}}_\alpha$ ($\overline{\mathscr{A}}_\beta$) may be looked upon as the union of those equivalence classes of U that have degree of membership in the lower (upper) approximation $\underline{\mathbf{A}}$ ($\overline{\mathbf{A}}$) of **A** at least α (β).

(b) $\underline{\mathscr{A}}_\alpha \subseteq \overline{\mathscr{A}}_\beta$ (using the fact that $\alpha \geq \beta$).

(c) It may be noticed that when **A** is a crisp set, $\underline{\mathscr{A}}_\alpha$ and $\overline{\mathscr{A}}_\beta$ reduce respectively to its lower and upper approximation in $\langle U, R\rangle$, for any choice of α, β.

We now propose a roughness measure of **A**.

DEFINITION 3.1. A *roughness measure* $\rho_{\mathbf{A}}^{\alpha,\beta}$ *of the fuzzy set* $\mathbf{A}$ *in* U *with respect to parameters* α, β, *where* $0 < \beta \le \alpha \le 1$, *and the approximation space* $\langle U, R\rangle$, *is defined thus*:

$$\rho_{\mathbf{A}}^{\alpha,\beta} \equiv 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|}.$$

OBSERVATION 3.2.

(a) $0 \le \rho_{\mathbf{A}}^{\alpha,\beta} \le 1$ (using Observation 3.1(b)).

(b) If β is kept fixed and α increased, $|\underline{\mathscr{A}}_\alpha|$ decreases and $\rho_{\mathbf{A}}^{\alpha,\beta}$ increases.

(c) If α is kept fixed and β increased, $|\overline{\mathscr{A}}_\beta|$ decreases and $\rho_{\mathbf{A}}^{\alpha,\beta}$ decreases.

(d) If $\mathbf{A}$ is such that there is a member x in each equivalence class X_i $(i = 1,\ldots,n)$ with $\mathbf{A}(x) < \alpha$, then $\underline{\mathscr{A}}_\alpha = \varnothing$ and so $\rho_{\mathbf{A}}^{\alpha,\beta} = 1$.

(e) If $\mathbf{A}$ is a fuzzy set with constant membership on each equivalence class of U and $\alpha = \beta$, then $\underline{\mathscr{A}}_\alpha = \overline{\mathscr{A}}_\beta$, so that $\rho_{\mathbf{A}}^{\alpha,\beta} = 0$.

PROPOSITION 3.1. *If* $\mathbf{A}$ *is a constant fuzzy set, say* $\mathbf{A}(x) = \delta$, *for all* x *in* U, *then* $\rho_{\mathbf{A}}^{\alpha,\beta} = 0$ *with the exception when* $\beta < \delta < \alpha$, *in which case* $\rho_{\mathbf{A}}^{\alpha,\beta} = 1$.

Proof. $\underline{\mathbf{A}}(X_i) \equiv \inf_{x \in X_i} \mathbf{A}(x) = \delta = \sup_{x \in X_i} \mathbf{A}(x) \equiv \overline{\mathbf{A}}(X_i)$, $i = 1,\ldots,n$. Now if $\alpha, \beta > \delta$,

$$\underline{\mathscr{A}}_\alpha = \bigcup\{X_i : \underline{\mathbf{A}}(X_i) \ge \alpha\} = \varnothing = \bigcup\{X_i : \overline{\mathbf{A}}(X_i) \ge \beta\} = \overline{\mathscr{A}}_\beta, \qquad i \in \{1,\ldots,n\},$$

and if $\alpha, \beta \le \delta$, $\underline{\mathscr{A}}_\alpha = U = \overline{\mathscr{A}}_\beta$. So in both cases, $\rho_{\mathbf{A}}^{\alpha,\beta} = 0$.

When $\beta < \delta < \alpha$, $\underline{\mathscr{A}}_\alpha = \varnothing$ while $\overline{\mathscr{A}}_\beta = U$, so that $\rho_{\mathbf{A}}^{\alpha,\beta} = 1$. □

Let $\mathbf{A}, \mathbf{B}$ be fuzzy sets in U. If $\mathbf{A} \subseteq \mathbf{B}$, we cannot say in general whether $\rho_{\mathbf{A}}^{\alpha,\beta} \le \rho_{\mathbf{B}}^{\alpha,\beta}$ or $\rho_{\mathbf{B}}^{\alpha,\beta} \le \rho_{\mathbf{A}}^{\alpha,\beta}$. However, the following may be observed.

PROPOSITION 3.2. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\overline{\mathscr{A}}_\beta = \overline{\mathscr{B}}_\beta$, *then* $\rho_{\mathbf{B}}^{\alpha,\beta} \le \rho_{\mathbf{A}}^{\alpha,\beta}$.

Proof. If $\mathbf{A} \subseteq \mathbf{B}$, it is easy to show that for any α, β, $\underline{\mathscr{A}}_\alpha \subseteq \underline{\mathscr{B}}_\alpha$ and $\overline{\mathscr{A}}_\beta \subseteq \overline{\mathscr{B}}_\beta$. So $\overline{\mathscr{A}}_\beta = \overline{\mathscr{B}}_\beta$ implies that

$$\rho_{\mathbf{B}}^{\alpha,\beta} \equiv 1 - \frac{|\underline{\mathscr{B}}_\alpha|}{|\overline{\mathscr{B}}_\beta|} \le 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} \equiv \rho_{\mathbf{A}}^{\alpha,\beta}. \qquad \square$$

COROLLARY. *If* $\mathbf{A}(x) \geq 0.5$, *for all* x *in* U, $\beta \leq 0.5$, *and* $\mathbf{A} \subseteq \mathbf{B}$, *then* $\rho_{\mathbf{B}}^{\alpha,\beta} \leq \rho_{\mathbf{A}}^{\alpha,\beta}$. *If both* $\alpha, \beta \leq 0.5$, $\mathbf{A}(x) \geq 0.5$, *for all* x *in* U, *and* $\mathbf{A} \subseteq \mathbf{B}$, *then* $\rho_{\mathbf{A}}^{\alpha,\beta} = 0 = \rho_{\mathbf{B}}^{\alpha,\beta}$.

Proof. If $\beta \leq 0.5$, $\overline{\mathscr{A}}_\beta = U = \overline{\mathscr{B}}_\beta$, whence by the proposition, $\rho_{\mathbf{B}}^{\alpha,\beta} \leq \rho_{\mathbf{A}}^{\alpha,\beta}$. If $\alpha \leq 0.5$ also, $\underline{\mathscr{A}}_\alpha = U = \underline{\mathscr{B}}_\alpha$, so that $\rho_{\mathbf{A}}^{\alpha,\beta} = 0 = \rho_{\mathbf{B}}^{\alpha,\beta}$. □

OBSERVATION 3.3. One can find examples of fuzzy sets $\mathbf{A}, \mathbf{B}$ with $\mathbf{A}(x) \geq 0.5$, for any x in U, $\mathbf{A} \subseteq \mathbf{B}$, and such that $\rho_{\mathbf{B}}^{\alpha,\beta} < \rho_{\mathbf{A}}^{\alpha,\beta}$, where $\beta \leq 0.5$. However, it may be noticed that if $\mathbf{A}, \mathbf{B}$ are crisp sets satisfying the above conditions, $\rho_{\mathbf{A}}^{\alpha,\beta} = 0 = \rho_{\mathbf{B}}^{\alpha,\beta}$, for any α, β $(\alpha \geq \beta)$.

PROPOSITION 3.3. *If* $\mathbf{A} \subseteq \mathbf{B}$ *and* $\underline{\mathscr{A}}_\alpha = \underline{\mathscr{B}}_\alpha$, *then* $\rho_{\mathbf{A}}^{\alpha,\beta} \leq \rho_{\mathbf{B}}^{\alpha,\beta}$.

We now define the notion of rough equality of fuzzy sets.

DEFINITION 3.2. Fuzzy sets $\mathbf{A}, \mathbf{B}$: $U \to [0,1]$ are said to be *roughly equal* if and only if $\underline{\mathbf{A}} = \underline{\mathbf{B}}$ and $\overline{\mathbf{A}} = \overline{\mathbf{B}}$, where $\underline{\mathbf{A}}, \underline{\mathbf{B}}$ and $\overline{\mathbf{A}}, \overline{\mathbf{B}}$ are the lower and upper approximations of $\mathbf{A}$ and $\mathbf{B}$, respectively (cf. Definition 2.1).

OBSERVATION 3.4.

(a) It is easy to see that rough equality of fuzzy sets is an equivalence relation on the set of all fuzzy sets in U.

(b) Analogous to the crisp situation, one can say in this case that if fuzzy sets $\mathbf{A}, \mathbf{B}$: $U \to [0,1]$ are roughly equal, these are indiscernible in the context of the approximation space $\langle U, R \rangle$.

PROPOSITION 3.4. *If fuzzy sets* $\mathbf{A}, \mathbf{B}$: $U \to [0,1]$ *are roughly equal,* $\rho_{\mathbf{A}}^{\alpha,\beta} = \rho_{\mathbf{B}}^{\alpha,\beta}$, *for any choice of* α, β $(\alpha \geq \beta)$.

Proof. If $\mathbf{A}, \mathbf{B}$ are roughly equal, $\underline{\mathbf{A}} = \underline{\mathbf{B}}$ and $\overline{\mathbf{A}} = \overline{\mathbf{B}}$, by definition. So $\underline{\mathscr{A}} = \underline{\mathscr{B}}$ and $\overline{\mathscr{A}} = \overline{\mathscr{B}}$, whence for any α, β $(\alpha \geq \beta)$, $\rho_{\mathbf{A}}^{\alpha,\beta} = \rho_{\mathbf{B}}^{\alpha,\beta}$. □

Next we prove a relation between the roughness measures of fuzzy sets $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$, and $\mathbf{A} \cup \mathbf{B}$.

PROPOSITION 3.5.

$$\rho_{\mathbf{A}\cup\mathbf{B}}^{\alpha,\beta} |\overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta| \leq \rho_{\mathbf{A}}^{\alpha,\beta} |\overline{\mathscr{A}}_\beta| + \rho_{\mathbf{B}}^{\alpha,\beta} |\overline{\mathscr{B}}_\beta| - \rho_{\mathbf{A}\cap\mathbf{B}}^{\alpha,\beta} |\overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta|.$$

Proof. The following can easily be obtained from (b), (c), (d), (e), respectively, of Proposition 2.1.

(i) $\overline{\mathscr{A} \cup \mathscr{B}}_\beta = \overline{\mathscr{A}}_\beta \cup \overline{\mathscr{B}}_\beta$,
(ii) $\underline{\mathscr{A} \cap \mathscr{B}}_\alpha = \underline{\mathscr{A}}_\alpha \cap \underline{\mathscr{B}}_\alpha$,
(iii) $\underline{\mathscr{A}}_\alpha \cup \underline{\mathscr{B}}_\alpha \subseteq \underline{\mathscr{A} \cup \mathscr{B}}_\alpha$,
(iv) $\overline{\mathscr{A} \cap \mathscr{B}}_\beta \subseteq \overline{\mathscr{A}}_\beta \cap \overline{\mathscr{B}}_\beta$.

Now

$$\rho^{\alpha,\beta}_{\mathbf{A}\cup\mathbf{B}} \equiv 1-\frac{|\underline{\mathscr{A}\cup\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}\cup\mathscr{B}}_{\beta}|}=1-\frac{|\underline{\mathscr{A}\cup\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}|}\leq 1-\frac{|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}|}, \tag{A}$$

using (i) and (iii).

Also

$$\rho^{\alpha,\beta}_{\mathbf{A}\cap\mathbf{B}} \equiv 1-\frac{|\underline{\mathscr{A}\cap\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}\cap\mathscr{B}}_{\beta}|}=1-\frac{|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}\cap\mathscr{B}}_{\beta}|}\leq 1-\frac{|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}|}{|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}|}, \tag{B}$$

using (ii) and (iv).

As for any finite sets X, Y,

$$|X\cup Y|=|X|+|Y|-|X\cap Y|,$$

we have

$$\begin{aligned}\rho^{\alpha,\beta}_{\mathbf{A}\cup\mathbf{B}}|\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}| &\leq |\overline{\mathscr{A}}_{\beta}\cup\overline{\mathscr{B}}_{\beta}|-|\underline{\mathscr{A}}_{\alpha}\cup\underline{\mathscr{B}}_{\alpha}| \\ &= |\overline{\mathscr{A}}_{\beta}|+|\overline{\mathscr{B}}_{\beta}|-|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}|-|\underline{\mathscr{A}}_{\alpha}|-|\underline{\mathscr{B}}_{\alpha}|+|\underline{\mathscr{A}}_{\alpha}\cap\underline{\mathscr{B}}_{\alpha}| \\ &\leq |\overline{\mathscr{A}}_{\beta}|+|\overline{\mathscr{B}}_{\beta}|-|\underline{\mathscr{A}}_{\alpha}|-|\underline{\mathscr{B}}_{\alpha}|-\rho^{\alpha,\beta}_{\mathbf{A}\cap\mathbf{B}}|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}|,\end{aligned}$$

by (A) and (B).

Finally, using definitions of $\rho^{\alpha,\beta}_{\mathbf{A}}$ and $\rho^{\alpha,\beta}_{\mathbf{B}}$, we get

$$\rho^{\alpha,\beta}_{\mathbf{A}\cup\mathbf{B}}|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}|\leq \rho^{\alpha,\beta}_{\mathbf{A}}|\overline{\mathscr{A}}_{\beta}|+\rho^{\alpha,\beta}_{\mathbf{B}}|\overline{\mathscr{B}}_{\beta}|-\rho^{\alpha,\beta}_{\mathbf{A}\cap\mathbf{B}}|\overline{\mathscr{A}}_{\beta}\cap\overline{\mathscr{B}}_{\beta}|. \qquad \square$$

Let us now consider roughness measures of the fuzzy set **A** and its complement $\mathbf{A}^c$ in a special situation.

PROPOSITION 3.6. *If $\alpha=0.5=\beta$, and for no equivalence class X_i ($i\in\{1,\dots,n\}$) of U, $\underline{\mathbf{A}}(X_i)=0.5=\overline{\mathbf{A}}(X_i)$, then $\rho^{\alpha,\beta}_{\mathbf{A}}|\overline{\mathscr{A}}_{\beta}|=\rho^{\alpha,\beta}_{\mathbf{A}^c}|\underline{\mathscr{A}}^c_{\alpha}|$.*

Proof. By (f), (g) of Proposition 2.1, $\overline{\mathbf{A}^c} = \underline{\mathbf{A}}^c$ and $\underline{\mathbf{A}^c} = \overline{\mathbf{A}}^c$.

Now for any $i = 1, \ldots, n$, and $x \in X_i$, $x \in \underline{\mathscr{A}^c}_\alpha$ if and only if $\underline{\mathbf{A}^c}(X_i) \geq \alpha$, i.e., $\overline{\mathbf{A}}^c(X_i) \geq \alpha$. So $x \in \underline{\mathscr{A}^c}_\alpha$ if and only if $\overline{\mathbf{A}}(X_i) \leq 1 - \alpha = 0.5$, i.e., $\overline{\mathbf{A}}(X_i) < 0.5$ (using assumption).

On the other hand, $x \in \overline{\mathscr{A}}^c_\beta$ if and only if $\overline{\mathbf{A}}(X_i) < \beta = 0.5$. Thus $\underline{\mathscr{A}^c}_\alpha = \overline{\mathscr{A}}^c_\beta$. Similarly, $\overline{\mathscr{A}^c}_\beta = \underline{\mathscr{A}}^c_\alpha$. Now

$$\rho_{\mathbf{A}^c}^{\alpha,\beta} \equiv 1 - \frac{|\underline{\mathscr{A}^c}_\alpha|}{|\overline{\mathscr{A}^c}_\beta|} = 1 - \frac{|\overline{\mathscr{A}}^c_\beta|}{|\underline{\mathscr{A}}^c_\alpha|},$$

and so

$$\rho_{\mathbf{A}^c}^{\alpha,\beta} |\underline{\mathscr{A}}^c_\alpha| = |\underline{\mathscr{A}}^c_\alpha| - |\overline{\mathscr{A}}^c_\beta| = |\overline{\mathscr{A}}_\beta| - |\underline{\mathscr{A}}_\alpha| = \rho_{\mathbf{A}}^{\alpha,\beta} |\overline{\mathscr{A}}_\beta|. \qquad \square$$

OBSERVATION 3.5. If $\alpha = 0.5 = \beta$ and $\mathbf{A}(x) > 0.5$, for all x in U, $\rho_{\mathbf{A}}^{\alpha,\beta} = 0 = \rho_{\mathbf{A}^c}^{\alpha,\beta}$. If $\alpha = 0.5 = \beta$ and $\mathbf{A}(x) < 0.5$, for all x in U, then $\rho_{\mathbf{A}}^{\alpha,\beta} = 0 = \rho_{\mathbf{A}^c}^{\alpha,\beta}$, too.

Finally, we look at the effect on the roughness of a fuzzy set $\mathbf{A}$ in U, when the partition on U is made finer.

Let S be an equivalence relation on U finer than R, i.e,. $S \subseteq R$. We define as before, $\underline{\mathscr{A}}_\alpha$, $\overline{\mathscr{A}}_\beta$ in the approximation space $\langle U, R \rangle$ and the corresponding sets $\underline{\mathscr{A}}^f_\alpha$, $\overline{\mathscr{A}}^f_\beta$ in $\langle U, S \rangle$. Let us denote simply by $\rho_{\mathbf{A}}$ and $\rho^f_{\mathbf{A}}$, the roughness measures of $\mathbf{A}$ relative to α, β in $\langle U, R \rangle$ and $\langle U, S \rangle$, respectively.

PROPOSITION 3.7. $\rho^f_{\mathbf{A}} \leq \rho_{\mathbf{A}}$.

Proof. We claim that $\underline{\mathscr{A}}_\alpha \subseteq \underline{\mathscr{A}}^f_\alpha$ and $\overline{\mathscr{A}}^f_\beta \subseteq \overline{\mathscr{A}}_\beta$. Now S gives rise to a partition $\{Y_1, \ldots, Y_m\}$, say, of U such that for any $j \in \{1, \ldots, m\}$, there is $i \in \{1, \ldots, n\}$ with $Y_j \subseteq X_i$.

Let $x \in \underline{\mathscr{A}}_\alpha$ and $x \in X_i$, $i \in \{1, \ldots, n\}$. Now $x \in Y_j$, for some $j \in \{1, \ldots, m\}$ and $Y_j \subseteq X_i$. But $\inf_{y \in Y_j} \mathbf{A}(y) \geq \inf_{y \in X_i} \mathbf{A}(y) \geq \alpha$, so that $x \in \underline{\mathscr{A}}^f_\alpha$.

Again, let $x \in \overline{\mathscr{A}}^f_\beta$ and $x \in Y_j$, $j \in \{1, \ldots, m\}$. Then $x \in X_i$, for some $i \in \{1, \ldots, n\}$, and $Y_j \subseteq X_i$. But $\sup_{y \in X_i} \mathbf{A}(y) \geq \sup_{y \in Y_j} \mathbf{A}(y) \geq \beta$, so that $x \in \overline{\mathscr{A}}_\beta$. Thus

$$\rho^f_{\mathbf{A}} \equiv 1 - \frac{|\underline{\mathscr{A}}^f_\alpha|}{|\overline{\mathscr{A}}^f_\beta|} \leq 1 - \frac{|\underline{\mathscr{A}}_\alpha|}{|\overline{\mathscr{A}}_\beta|} \equiv \rho_{\mathbf{A}}. \qquad \square$$

We conclude this section by giving an example and trying to interpret the notions and results presented so far in its context.

EXAMPLE. Let the domain U of discourse comprise students of a class in a school, and $X_1, \ldots, X_n$ denote n sections of the class. Let $\mathbf{A}$: $U \to [0,1]$ be a representation of the fuzzy concept "tall," i.e., $\mathbf{A}(x)$ is the degree of tallness of the student x in U.

Now those sections in which the minimum (maximum) membership value of students with respect to tallness is at least α (β) are chosen, and the students of these sections are taken to constitute the α-lower (β-upper) approximation $\underline{\mathscr{A}}_\alpha$ ($\overline{\mathscr{A}}_\beta$) of $\mathbf{A}$. $\underline{\mathscr{A}}_\alpha$ ($\overline{\mathscr{A}}_\beta$) could be viewed as the collection of students who are definitely (possibly) tall at least to the extent α (β).

Roughness $\rho_{\mathbf{A}}^{\alpha,\beta}$ of the concept "tall" relative to the thresholds α, β and the given information system (viz., the class U and its sections $X_1, \ldots, X_n$) is then determined by the fraction of possibly tall students who are definitely tall. We notice the following.

(i) The greater the fraction, the lower the roughness of the concept "tall."

(ii) When, in each section, minimum degree of tallness is α, roughness of the concept "tall" is zero.

(iii) If the minimum degree of tallness in each section is less than α, roughness is maximum, viz., unity.

(iv) Let us suppose that the degree $\mathbf{A}(x)$ of tallness of each student x in U is at least 0.5. We consider the dilated version $\mathbf{B}$ and concentrated version $\mathbf{C}$ of $\mathbf{A}$ representing the sets "more or less tall" and "very tall," respectively [1]. Let us further suppose that the degree of any student x being "very tall" is also at least 0.5. Now if we take $\beta \leq 0.5$, it may be concluded (by Proposition 3.2) that $\rho_{\mathbf{B}}^{\alpha,\beta} \leq \rho_{\mathbf{A}}^{\alpha,\beta} \leq \rho_{\mathbf{C}}^{\alpha,\beta}$, for any choice of $\alpha (\geq \beta)$.

(v) Let the sections of the class U be further divided and α, β kept fixed. Then one finds that, generally, both the number of sections and the overall number of students contributing to $\underline{\mathscr{A}}_\alpha$ increase. On the other hand, though the number of sections contributing to $\overline{\mathscr{A}}_\beta$ generally increases, the overall number of students in $\overline{\mathscr{A}}_\beta$ decreases (cf. Proposition 3.7). Hence roughness of the concept "tall" decreases.

4. CONCLUDING REMARKS

The measure ρ could find many applications in pattern recognition and image analysis problems, where U denotes a gray image or feature space, and $X_1, X_2, \ldots, X_n$ represent n regions. The fuzzy set $\mathbf{A}$ can be viewed to represent the ill-defined pattern classes or some imprecise image property such as brightness, darkness, edginess, smoothness, etc. Relative to thresholds α, β, roughness of such an imprecise property **A** can then be measured in terms of the ratio of the number of feature points definitely satisfying **A** to the number of feature points possibly satisfying **A**. Algorithms for image enhancement and segmentation, or seed point extraction of clustering, can be formulated as is done by fuzziness measures [7].

For example, fuzzy segmentation of an image X using the measure ρ can be obtained as follows. Define a fuzzy object region over the image space with membership plane $\mathbf{A} = \mu_X$ of constant bandwidth using a two-dimensional π-function. Vary the crossover point of a μ_X-plane and compute ρ for a fixed value of α and β. Find that μ_X^*-plane for which ρ is minimum. Such a μ_X^*-plane represents a fuzzy segmented version of the image (with lower and upper approximations as determined by α and β). Note that in conventional fuzzy segmentation, the uncertainty is handled in terms of only class membership of pixels in the μ_X^*-plane. Here, in addition, the lower and upper approximations of the μ_X^*-plane are taken into consideration for managing the uncertainty. This fuzzy segmentation can be refined by dilating and/or concentrating the μ_X^*-plane according to the corollary of Proposition 3.2, so that ρ is reduced further.

Instead of altering the μ-plane, one can also control the thresholds α and β (Observation 3.2) in order to change ρ. This criterion may be used for defining a quantitative measure of image enhancement or some other processing tasks.

The concept of splitting $X_1, X_2, \ldots, X_n$ is analogous to increasing the resolution of a digital image, and can well be utilized for subpixel classification problems and for detecting the boundaries of regions precisely.

It may be pointed out that in this paper, a roughness measure of a fuzzy set has been defined and its properties studied relative to a domain of discourse equipped with a crisp partition. A generalization of the idea, when the partition on the domain is fuzzy, is under study.

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