

Spectral Fuzzy Sets and Soft Thresholding

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ABSTRACT

A new concept of spectral fuzzy sets has been introduced for dealing with the uncertainty in subjective assignment of membership value to the elements of a fuzzy set. The concept is characterized by a set of membership functions reflecting an individual's (or several individuals') opinion. The uncertainty due to spectralness has been explained. To demonstrate the necessity and effectiveness of this concept, the problem of image segmentation is considered as an example. Here the uncertainty may arise from differences in opinion in selecting a threshold from an ambiguous flat-valleyed histogram and in selecting ill-defined region boundaries. It has been shown that the new concept can be considered as a useful tool for the management of such uncertainty since it allows the implementation of soft decisions on the basis of various opinions on membership functions resulting in a band of thresholds with their respective uncertainties. A hard (crisp) decision obviously corresponds to one with minimum uncertainty. Besides the introduction of a new concept of spectral fuzzy set, the work also shows a natural and appropriate way that the uncertainty arising from settling a definite degree of fuzziness (or from defining a precise membership function) may still lead to soft decisions when the image regions are ill-defined.

1. INTRODUCTION

Zadeh [1] first proposed a certain formal solution to the ancient problem that the classes of objects encountered in the real world do not always have precisely defined memberships. There are sets that do not have a rigid

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demarcating boundary for which a gradual transition from zero to unity membership can be assigned. These sets are named *fuzzy sets*, classes with a continuum of grades of membership. The theory of fuzzy sets in the areas of reasoning, pattern recognition, and image processing [2-6], by taking into account some uncertainties, imprecision, and vagueness inherent in patterns and in the statements and conclusions of human beings, proved its worth well (or at least equally well) over two-valued classical set theory. The theory is more general, natural, and flexible (elastic), and it provides approximate but effective results.

The flexibility of the theory is associated with the concept of its membership function. The grade of membership is not a probability. Basically it is a measure of the compatibility of an object with the concept represented by a fuzzy set. For example, 0.3 is the compatibility of someone with the set "tall" and it is not the probability that he is tall. In other words, the value 0.3 is related to the extent to which the concept represented by "tall" must be stretched to fit him. The higher the value of membership, the lesser will be the amount of stretching of the concept. In this sense, fuzziness is analogous to elasticity [7].

Since the grade of membership is both subjective and dependent on context, some difficulty of adjudging the membership value still remains. In fact, one of the controversial issues of fuzzy set theory application, which not only the critics but also the users feel, is the evaluation of membership functions. That is, how do we assess the membership of each element of a set, regarding which opinions can vary, giving rise to uncertainties? For example, having the opinions of 10 people on the membership of height between 5 feet and 6 feet in the set of "tall persons," there would be subtle to conspicuous differences among their opinions. Again, a person himself also sometimes varies in his opinion when given the task of evaluating the appropriate membership.

The discussion just mentioned shows that it is difficult to select an appropriate membership function. On the other hand, it would be very welcome if all opinions could be given their due respect and further operations could be done with all the membership values. So, the case of a set or a band (spectrum) of opinions on membership functions comes into the picture. An attempt at introducing a new set, called the *spectral fuzzy set* (or, Sp-fuzzy set, for short), where instead of a single unique membership function a set of functions reflecting various opinions on membership of elements is considered so that each membership grade is attached to one of these functions, is made in this paper.

It has been observed that human beings do not find it very difficult to assign memberships to elements that have either very low or very high possibility of belonging to that set. For example, opinions regarding the membership of a man whose height is 5 feet in the set of "tall men" may not vary widely

because it is a well-agreed-upon fact that he is not tall. On the other hand, if a man's height is 5 feet 6 inches, opinions regarding his "tallness membership" can vary widely. Let us take another example from real life. If someone is asked to give a full glass of water several times with the same glass, the variation in the amount of water will be less compared to the case when half a glass of water is asked for. Similar observations may be made when the task is performed by several people. It therefore appears that the variation in membership function assignment is high for the elements having fairly medium belongingness.

The degree of fuzziness (d_2) of a set as measured by the index of fuzziness [8], entropy [9], and index of crispness [10] provides, on a global level, an average amount of difficulty in deciding whether an element would be considered as a member of the set or not. Besides these, the sp-fuzzy set has one more ambiguity (or difficulty) d_1 regarding the assessment of membership value of an element. This may be called the "spectral index."

All these aspects have been dealt with in this paper. Finally, the problem of image thresholding has been considered, as an illustration, to demonstrate the necessity and effectiveness of the concept of spectral fuzzy sets. Consider, for example, the problem of object/background classification by defining a membership function (on the gray scale) characterizing a fuzzy set "object region." Here (especially when the regions are ill-defined), opinions may vary among individuals in defining such a function, i.e., in assigning membership value on a gray scale denoting the degree of belonging of a level to the object region. This difference in opinion leads to an uncertainty in selecting an object/background boundary by making the set "object region" a spectral fuzzy. Under this situation when several opinions are available and none can be discarded, an algorithm has been developed here that gives due respect to all the opinions, minimizes image ambiguity ($d = d_1 + d_2$) over the resulting band of membership functions, and then takes a collective soft decision by providing a set of thresholds along with their certainty values. A hard (crisp) decision may correspond to one with maximum certainty value. The concept of soft thresholding is natural and also appropriate to avoid committing ourselves to a specific thresholding when the regions in an image are ill-defined (fuzzy).

It should be mentioned here that the concept of spectral fuzzy set is not the same as that of probabilistic set [11] or interval-valued fuzzy set [12] or fuzzy set of type 2 [13] or ultrafuzzy set [7], which have also considered the difficulty in settling a definite degree of fuzziness (ambiguity) by relaxing the requirement for a precise membership function. Elements of a fuzzy set of type 2 have membership grades that are themselves type 1 (ordinary) fuzzy sets defined in the unit interval $[0, 1]$ rather than points in $[0, 1]$. Fuzzy sets of type n are also characterized by n -step recursively defined ambiguity. Interval valued fuzzy sets are characterized by membership grades that are closed

intervals (segments) in $[0, 1]$ rather than single numbers. An ultrafuzzy set is a type 2 fuzzy set that expresses membership grades as fuzzy numbers, e.g., "approximately 0.6," "close to 0.8", etc. These measures consider individual membership imprecision. In probabilistic set theory [11], the value of a membership function is regarded as a random variable to account for the membership variability among a set of individuals. A concept of probabilistic measure space is introduced there through a parameter space that plays an important role in dealing with the ambiguity and is expected to be adopted suitably according to each situation.

The spectral fuzzy sets, on the other hand, handle the aforesaid difficulty in settling a membership value of a definite degree of fuzziness by simply collecting various possible membership functions of the set reflecting an individual's (or several individuals') opinion on it. In other words, it is characterized with a set of membership functions (where each membership grade is attached to *one* of the membership functions) rather than with set-valued membership grades (where the source of membership grades is forgotten). The extent of opinion (spectrum) may be restricted by the bound functions of Murthy and Pal [14]. Variance of a probabilistic set (which indicates a disordered degree of judgement) is to some extent analogous to the measure "spectral index" (d_1) of the spectral fuzzy set.

2. DEFINITIONS

A fuzzy subset F of the universe of discourse ψ is defined as

$$F = \bigcup_j \mu_F(x_j)/x_j, \quad j = 1, 2, \dots, n \quad (1)$$

where F has a finite number of supports x_1, x_2, \dots, x_n . When the support of F is a continuum,

$$F = \int_{\psi} \mu_F(x)/x, \quad (2)$$

where the integral sign stands for the union of the fuzzy singletons.

$\mu_F: \mu \rightarrow [0, 1]$ is the compatibility or membership function of F and $\mu_F(x)$ denotes the degree of belonging of x to the set F .

A spectral fuzzy subset F having n supports is characterized by a set or a band (spectrum) of r membership functions (reflecting r opinions) and may be represented as

$$F = \bigcup_j \left[\bigcup_i \mu_i^j(x_j)/x_j \right], \quad x_j \in \psi, \quad i = 1, 2, \dots, r, \quad j = 1, 2, \dots, n. \quad (3)$$

r , the number of membership functions, may be called the cardinality of the opinion set. $\mu_i^j(x_j)$ denotes the degree of belonging of x_j to the set F according to i th membership function.

In case ψ is not finite,

$$F = \int_{\psi} \left[\bigcup_i \mu_i^j(x)/x \right], \quad x \in \psi, \quad i = 1, 2, \dots, r \quad (4)$$

2.1. HEIGHT OF F

$$hl(F) = \text{Sup}_j \left\{ \max_i (\mu_i^j) - \min_i (\mu_i^j) \right\} \quad (5)$$

where

$$\mu_i^j \text{ stands for } \mu_i^j(x_j), \quad j = 1, 2, \dots, n, \quad i = 1, 2, \dots, r.$$

2.2. WIDTH OF F

$$wd(F) = \text{Sup}_{\mu} \{ wd(\mu) \} \quad (6)$$

where

$$wd(\mu) = \text{Sup}(v) - \text{Inf}(v).$$

μ is a certain membership function. v is the ordered set of supporting elements having in their sets of opinions maximum membership $\geq \mu$ and minimum membership $\leq \mu$.

When $\mu = 0.5$, v constitutes a band of cross-over points. For an ordinary fuzzy set, $hl(F) = wd(F) = 0$.

2.3. BASIC OPERATIONS

Let F and G be two spectral fuzzy subsets of ψ characterized by r membership functions (opinions). Let $S_F(x_j)$ and $S_G(x_j)$ represent the ordered set of r membership values of x_j (obtained from these functions) to the sets F and G , respectively. Then

$$S_F(x_j) = \langle \mu_F^i(x_j) \rangle, \quad i = 1, 2, \dots, r \quad (7)$$

$$S_G(x_j) = \langle \mu_G^i(x_j) \rangle, \quad i = 1, 2, \dots, r, x_j \in \psi. \quad (8)$$

$\mu_F^i(x_j)$ and $\mu_G^i(x_j)$ denote the i th (ordered) membership value of x_j to the sets F and G , respectively.

Union of F and G may be defined as

$$FUG = \bigcup_j \left[\bigcup_i \mu_{FUG}^i(x_j)/x_j \right], \quad x_j \in \psi \quad (9)$$

where

$$\mu_{FUG}^i(x_j) = \max(\mu_F^i(x_j), \mu_G^i(x_j)) \quad \text{for all } i, j. \quad (10)$$

Intersection of F and G may be defined as

$$F \cap G = \bigcup_j \left[\bigcup_i \mu_{F \cap G}^i(x_j)/x_j \right], \quad x_j \in \psi \quad (11)$$

where

$$\mu_{F \cap G}^i(x_j) = \min(\mu_F^i(x_j), \mu_G^i(x_j)) \quad \text{for all } i, j. \quad (12)$$

The complement of F is defined as

$$F' = \bigcup_j \left[\bigcup_i (1 - \mu_F^i(x_j))/x_j \right], \quad x_j \in \psi, \quad i = 1, 2, \dots, r \quad (13)$$

F is equal to G , i.e., $F = G$, iff

$$\mu_F^i(x_j) = \mu_G^i(x_j) \quad \text{for all } i, j. \quad (14)$$

G is a subset of F , i.e., $G \subseteq F$, iff

$$\mu_G^i(x_j) \leq \mu_F^i(x_j) \quad \text{for all } i, j. \quad (15)$$

The α th power of F is defined as

$$F^\alpha = \bigcup_j \left[\bigcup_i \left(\mu_F^i(x_j) \right)^\alpha / x_j \right], \quad x_j \in \psi, \quad i = 1, 2, \dots, r. \quad (16)$$

3. MEASURES OF AMBIGUITY OF SPECTRAL FUZZY SETS

The degree of fuzziness as measured by the terms *index of fuzziness*, *entropy*, and *index of crispness* [8–10] gives the average global level ambiguity or difficulty in deciding whether an element would be considered as a member of a set or not. Besides this ambiguity or difficulty, there is one more difficulty, in the case of spectral fuzzy sets, in assessing the membership value of an element. Therefore, a sp-fuzzy set is characterized by two types of ambiguity, namely, (1) the ambiguity is assigning a membership value to an element, and (2) the basic ambiguity expressed by assigning its membership value in that set.

Let d_1 and d_2 represent these difficulties. Then as d_1 reduces to zero (i.e., there is no difference in opinions), a sp-fuzzy set becomes a fuzzy set. If, in addition, d_2 decreases to zero, the fuzzy set further reduces to its crisp version (nearest ordinary set).

The functional nature of d_2 (if expressed by entropy or index of fuzziness or index of crispness) of a fuzzy set A characterized by a membership function $\mu_A(x)$, $x \in \psi$, is as follows:

- (i) $d_2(A)$ is maximum when $\mu_A(x) = 0.5$ for all x
- (ii) $d_2(A)$ is minimum when $\mu_A(x) = 0$ or 1 for all x
- (iii) $d_2(A) = d_2(A')$, A' being the complement set of A
- (iv) $d_2(A) \geq d_2(A^*)$ where A^* is a sharpened version of A such that

$$\begin{aligned} \mu_{A^*}(x) &\geq \mu_A(x) && \text{if } \mu_A(x) \geq 0.5 \\ &\leq \mu_A(x) && \text{if } \mu_A(x) \leq 0.5 \end{aligned} \quad (17)$$

A^* can be obtained by the INT operator, defined as

$$\begin{aligned}\mu_{\text{INT}}(x) &= 2\mu_A^2(x) \quad \text{if } 0.5 \geq \mu_A(x) \geq 0 \\ &= 1 - 2(1 - \mu_A(x))^2 \quad \text{if } 1 \geq \mu_A(x) \geq 0.5\end{aligned}\quad (18)$$

Regarding d_1 , it is assumed that the difference in opinions is low for those elements (supports) whose degree of inclusion or possibility of belonging to a set is subjectively very low or very high. The variation in opinion, on the other hand, is high for those supports whose degree of belongingness is fairly medium. This characteristic of d_1 seems to be intuitively appealing. For example, consider a sp-fuzzy set F , labeled "tall men" over the range 5 feet to 7 feet. The difference in opinions (or difficulty in assessing a membership) as expressed would be high around 5 feet 5 inches and would not be less than those around 5 feet or 7 feet in any way.

On the basis of this concept, let us define "Spectral Index" θ to give, in a global sense, a quantitative measure of the average differences (or disagreement) of opinion in assigning a membership value to a support. If F is a sp-fuzzy subset having n supports and r membership functions, $\theta(F)$ may be defined as

$$\theta(F) = d_1(F) = \frac{1}{n} \sum_j \theta_j, \quad j = 1, 2, \dots, n \quad (19)$$

$$\theta_j = \frac{S}{r_{c_2}} \left[\sum_{l=1}^{r-1} \sum_{i=l+1}^r |\mu'_i(x_j) - \mu'_l(x_j)| \right], \quad 1 \geq \theta_j \geq 0 \quad (20)$$

where

$$S = 1 / \frac{r}{2(r-1)}, \quad \text{if } r \text{ is even} \quad (21a)$$

$$= 1 / \frac{(r+1)}{2r}, \quad \text{if } r \text{ is odd.} \quad (21b)$$

The following points are noteworthy.

(i) The term S in the numerator of Equation (20) is included to make θ_j (average disagreement corresponding to j th support) lie between 0 and 1. It is to be noted that θ_j computes the difference between any two opinions in a set and then r_{c_2} averages over all such combinations.

(ii) $\theta_j = 0$ when $\mu_{F'}^i(x_j) = \mu_F^i(x_j)$ for all i, l and j ; i.e., for an ordinary fuzzy set with single membership function.

(iii) when r is even, θ_j is maximum ($= 1$) if there are $r/2$ unity values and $r/2$ zero values in the opinion set.

(iv) when r is odd, θ_j is maximum if the opinion set contains $(r + 1)/2$ unity values and $(r - 1)/2$ zero values or vice versa.

(v) $\theta(F) = \theta(F')$, F' being the complement set of F

(vi) Let us consider two ordered opinion sets with $r = 3$ for the j th supporting element as

$$S_F(x_i) = [0.5 \quad 0.49 \quad 0.48] \quad (22a)$$

and

$$S_G(x_i) = [0.51 \quad 0.5 \quad 0.49]. \quad (22b)$$

Here $\theta_j(F)$ and $\theta_j(G)$ have the same value. But if the contrast enhancement operator (INT) [Equation (18)] is applied to each element of these sets, it is found that $\theta_j(F)$ decreases whereas $\theta_j(G)$ increases. The INT operator makes a fuzzy set lead towards crisp. Therefore, though S_F and S_G are of equal spectral index, they are different at a higher (in the sense of crispness) stage. S_G , having an equal number of members on either side of the crossover point, possesses higher θ value, indicating its greater contrast as compared to S_F .

4. (DIS)SIMILARITY WITH OTHER CONCEPTS

There have been several attempts to deal with the difficulty in settling a definite degree of fuzziness (ambiguity) by relaxing requirement for a precise membership function and by making the distinction between grades of membership blurred. Various sets defined in this way include probabilistic sets [11], interval-valued fuzzy sets [12], type 2 fuzzy sets [13] and ultrafuzzy sets [7]. Their concepts are not the same as that of the spectral fuzzy sets described in this paper.

By definition, an ordinary fuzzy set is a type 1 fuzzy set and the elements of a type 2 fuzzy set have membership grades that themselves are type 1 fuzzy sets defined in the unit interval $[0, 1]$ rather than points in $[0, 1]$. For example, consider a type 2 fuzzy set "young" defined with a $(1 - S)$ type function over the crisp set of ages $X = \{5, 10, 20, 30, 40, 50, 60, 70\}$. Here the membership grades assigned to the elements $x \in X$ are type 1 fuzzy sets such as middle, high, not low, and so on defined on some set

$$Y = \{0, 0.1, \dots, 0.9, 1\} \subseteq [0, 1]. \quad (23)$$

Therefore, one may write the fuzzy grade of type 2 fuzzy set "young" as

$$\mu_{\text{young}}(x) = f(u_1)/u_1 + f(u_2)/u_2 + \cdots + f(u_m)/u_m,$$

$$u_i \in Y, \quad f(u_i) \in [0, 1] \quad (24)$$

and for $x = 30$

$$\mu_{\text{young}}(30) = \text{middle, say}$$

$$= 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7 \quad (25)$$

Fuzzy sets of type n ($n = 1, 2, \dots$) are characterized by n -step recursively defined ambiguity. Realistic meaning of the set declines as n increases.

An ultrafuzzy set is a type 2 fuzzy set that expresses membership grades as fuzzy numbers, e.g., $\mu_{\text{young}}(30) = \text{"approximately 0.6"}$ or "close of 0.6." An interval valued fuzzy set is characterized by membership grades that are segments (closed intervals) in $[0, 1]$ rather than single numbers, e.g., $\mu_{\text{young}}(30) = [0.5, 0.7]$. The length of a segment or interval that indicates an amount of imprecision is small/high for the least/most ambiguous (in the sense of class-belonging) elements.

The preceding measures consider individual membership imprecision. In probabilistic sets, the value of a membership function is essentially regarded as a random variable to account for the membership variability among a set of individuals. A probabilistic set on a total space X is characterized [11] by a defining function $\mu_A(x, \omega)$ that is a point (i.e., $x \in X$)-wise (B, B_C) -measurable function from a parameter space (Ω, B, P) to a characteristic space (Ω_C, B_C) . The parameter space is a probability space whose elements (depending on observers' subjectivity) represent a standard of judgement in dealing with the ambiguity, and it is expected to be adapted suitably according to each situation. The space (Ω_C, B_C) is a measurable space. It consists of all degrees of ambiguity and adopts $[0, 1]$, Borel set.

For example, $\mu_{\text{young}}(a)$ in the notion of a probabilistic set is a set of random numbers whose mean value is considered to be the same as the membership value of the corresponding (ordinary) fuzzy set, and variance indicates a disordered degree of judgement. The variance is maximum, say, around 30 (which are the most ambiguous elements) and minimum around 5 or 70 (i.e., least ambiguous elements). The choice of a suitable parameter space (and hence the probabilistic measure P) is therefore crucial in its application.

The spectral fuzzy set, on the other hand, is seen to handle the aforesaid difficulty in settling a definite degree of fuzziness in a set by simply incorporat-

ing a collection of individual opinions in its membership function. In other words, it is characterized by a set of membership functions (where each membership grade is attached to one of the membership functions) rather than with set-valued membership grades (where the origin of membership grades is forgotten). Also, no concept of probability is used here.

For example, a spectra fuzzy set “young” characterized by a set of four linear and nonlinear functions

$$\begin{aligned}\mu^1(x) &= 1 - 2\left(\frac{x-5}{55}\right)^2, & 32.5 \geq x \geq 5 \\ &= 2\left(\frac{x-60}{55}\right)^2, & 60 \geq x \geq 32.5 \\ &= 0, & x \geq 60\end{aligned}\quad (26)$$

$$\mu^2(x) = 1 - \left(\frac{x-5}{65}\right)^{1.5} \quad (27)$$

$$\begin{aligned}\mu^3(x) &= 1 - \left(\frac{x-5}{55}\right), & 60 \geq x \geq 5 \\ &= 0, & x \geq 60\end{aligned}\quad (28)$$

$$\mu^4(x) = 1 - (1/1 + e^{-(x-35)/10}) \quad (29)$$

has membership values at $x = 30$ as

$$\mu_{young}(30) = \{0.58, 0.76, 0.54, 0.62\} \quad (30)$$

where each membership grade is obtained from one of the aforesaid four membership functions. The spectral index θ at $x = 30$, representing the degree of difference in opinion regarding its membership grade, is to some extent analogous to the measure “variance” of a probabilistic set at that point.

The extent of opinion (spectrum) may be restricted by the bound functions of Murthy and Pal [14].

5. SOFT THRESHOLDING AND OBJECT BACKGROUND CLASSIFICATION

In this section we will be demonstrating the usefulness of spectral fuzzy sets as a tool for the management of uncertainty in the image thresholding problem. The problem of gray-level thresholding plays a key role in image processing and recognition. For example, in enhancing contrast in an image, we need to

select a proper threshold level so that some suitable nonlinear transformation can highlight a desirable set of pixel intensities compared to others. Similarly, in image segmentation, one needs proper histogram thresholding where the objective is to establish boundaries to partition the image space (crisply) into meaningful regions.

When the regions in an image are ill-defined (i.e., fuzzy), it is natural and also appropriate to avoid committing ourselves to a specific segmentation by allowing the segments to be fuzzy subsets of the image. Again, when the histogram contains a flat valley or some local minima in the valley, selection of a particular threshold becomes questionable as far as its certainty is concerned. In other words, it may then be appropriate to determine a (or a set of) threshold(s) associated with its (their) certainty value(s).

There have been some attempts in this regard that developed various tools and methodologies [4, 10, 15-18] using fuzzy set theory for object extraction, skeleton extraction, edge (contour) extraction, etc. Let us consider, for example, a recent approach to object extraction [16] by defining an S type membership function (on the gray scale) to characterize a fuzzy set "object region." The membership plane thus obtained from an image is initially varied by changing its cross-over point. An optimum plane, in the sense of minimizing some measure (e.g., entropy or index of fuzziness or compactness), is then extracted. The elements of the optimum plane denote the degree of belonging of various levels (pixels) to the set "object region," thus representing the segment (i.e., object) as a fuzzy subset of the image. For the purpose of crisp segmentation, the cross-over point (being the most ambiguous point) of the optimum plane may be taken as a threshold.

In the preceding example (especially when the regions are ill-defined), opinions may vary among individuals while defining a membership function, i.e., in assigning a membership grade to a gray level denoting the degree of its belonging to the object region. This difference in opinion leads to an uncertainty in selecting an optimum fuzzy segmented subset (and hence an object/background boundary) by making the set "object region" a spectral fuzzy.

Under this situation, when several opinions are available and none can be discarded, an algorithm has been developed here that gives due respect to all the opinions, minimizes image ambiguity $d = d_1 + d_2$ (Section 3) over the resulting band of membership functions, and then takes a collective soft decision by providing a *set of thresholds* along with their certainty values. Hard (crisp) decisions may correspond to ones with maximum certainty value, i.e., for which the difficulty in deciding it as a boundary level (objects or background) is minimum (maximum), taking into account various opinions.

This not only demonstrates a way to manage uncertainty arising from various opinions in defining a membership function, but it also provides a

useful method of extracting soft thresholds for ill-defined image regions, i.e., when the selection of a particular threshold may become questionable. Besides, it also indicates a way that the uncertainty that arises, in general, from blurring of the membership-grade (or from relaxation of the requirement for a precise membership function) can be managed in taking a decision. Again, although the problem of image segmentation is considered here, as an example, the same argument also holds good for other problems e.g., skeleton extraction and edge extraction of an image where incertitude arises from ill-defined region boundaries and various opinions on membership values.

Before explaining the algorithm, a definition of an image in the light of spectral fuzzy sets, various membership functions used, and a definition of the spectral index of an image are given in the following sections.

5.1. IMAGE DEFINITION

An image X of size $M \times N$ and L levels can be considered as an array of fuzzy singletons, where each singleton is characterized by a set of r membership grades denoting its degree of brightness relative to some brightness level l , $l = 0, 1, 2, \dots, L - 1$ by r opinions. In the notion of spectral fuzzy sets we may write (as an extension of the image definition based on fuzzy sets [4, 19])

$$X = \bigcup_m \bigcup_n \left[\bigcup_i \mu_x^i(x_{mn}) / (x_{mn}) \right],$$

$$m = 1, 2, \dots, M; \quad n = 1, 2, \dots, N; \quad i = 1, 2, \dots, r \quad (31)$$

Here $\mu_x^i(x_{mn})$ or μ_{mn}^i ($0 \leq \mu_{mn}^i \leq 1$) denotes the grade of possessing some property μ_{mn} (e.g., brightness, darkness, edginess, smoothness) by the (m, n) th pixel intensity x_{mn} according to the i th membership function (or opinion).

One may use either global or local information of an image in defining a membership function characterizing some property. For example, brightness or darkness properties can be defined in terms of a gray value of a pixel x_{mn} whereas edginess, smoothness, or textural properties need the neighborhood information of a pixel to define their membership functions. Similarly, positional or coordinate information is necessary, in addition to gray-level and neighborhood information, to characterize a dynamic property of an image.

Again, the aforesaid information can be used in a number of ways (in its various functional forms), depending on individual opinion and/or the problem at hand, to define a requisite membership function for an image property.

5.2. SPECTRAL MEMBERSHIP FUNCTIONS

Two kinds of spectral fuzzy membership functions are used here in the object extraction algorithm. These represent a set of r bright membership planes of an image and characterize thereby an "object region" as a spectral fuzzy set.

Let us assume that the object region corresponds to a higher gray level than the background region of an image X , and l is the gray level value of the (m, n) th pixel intensity x_{mn} . Then $\mu_x^l(x_{mn})$, denoting the degree of possessing brightness property or degree of belonging to the object region of x_{mn} (or of level l) according to the i th opinion, may be defined as

$$\mu_x^l(x_{mn}) = \mu_x^l(l) = \bigcup_i \left\{ S(l; a, b, c) + \pi(l; \Delta b, b) \alpha_i \right\} \quad (32)$$

where

$$\begin{aligned} \Delta b &= b - a \\ i &= 1, 2, \dots, r \\ S(x; a, b, c) &= 0, \quad x \leq a \\ &= 2\left(\frac{x-a}{c-a}\right)^2, \quad a \leq x \leq b \\ &= 1 - 2\left(\frac{x-c}{c-a}\right)^2, \quad b \leq x \leq c \\ &= 1, \quad x \geq c \end{aligned} \quad (33)$$

with

$$b = (a + c)/2, \text{ the cross-over point}$$

and

$$\begin{aligned} \pi(x; l, k) &= S\left(x; k-l, k-\frac{l}{2}, k\right), \quad x \leq k \\ &= 1 - S\left(x; k, k+\frac{l}{2}, k+l\right), \quad x \geq k. \end{aligned} \quad (34)$$

Equations (33) and (34) denote, respectively, the standard S function and the standard π function [20]. α_i is a positive or negative real number, $|\alpha_i| \leq 1$, and α_i is introduced to make $\mu_x^l(x_{mn})$ lie between zero and one. α_i therefore controls the height and width of the μ_x^l plane.

Equation (32) represents a set of r membership values (opinions) of x_{mn} regarding its degree of belonging to the object region. The difference in opinion as explained in Section 3 is maximum around b and decreases towards a and c . The standard S function is assumed here to have a variation (modulation) characterized by a standard π function.

The second version, on the other hand, consists of various forms of monotonically nondecreasing functions to represent a collection of opinions on the bright membership plane $\mu_x(x_{mn})$. In other words,

$$\begin{aligned} \mu_x(x_{mn}) &= \mu_x(\ell) \\ &= \{S(\ell; a, b, c), g_1(\ell), g_2(\ell)\}, \quad \ell = 0, 1, 2, \dots, L-1 \end{aligned} \quad (35)$$

where

$$g_1(\ell) = \left((\ell - a) / (c - a) \right)^\beta \quad (36)$$

and

$$g_2(\ell) = 1 - \left(1 - (\ell - a) / (c - a) \right)^\gamma \quad (37)$$

ℓ denotes the gray level value of x_{mn} . β and γ are positive constants and they control the height and width of the μ_x plane.

Figures 1 and 2 show the functional forms of Equations (32) and (35) for various parameter values with $a = 5$ and $c = 25$.

5.3. SPECTRAL INDEX OF AN IMAGE

The spectral index of an image X of L levels, M rows, and N columns may be defined from Equation (19) as

$$\theta(X) = \frac{1}{MN} \sum_m \sum_n \theta_{mn}, \quad m = 1, 2, \dots, M, \quad n = 1, 2, \dots, N \quad (38)$$

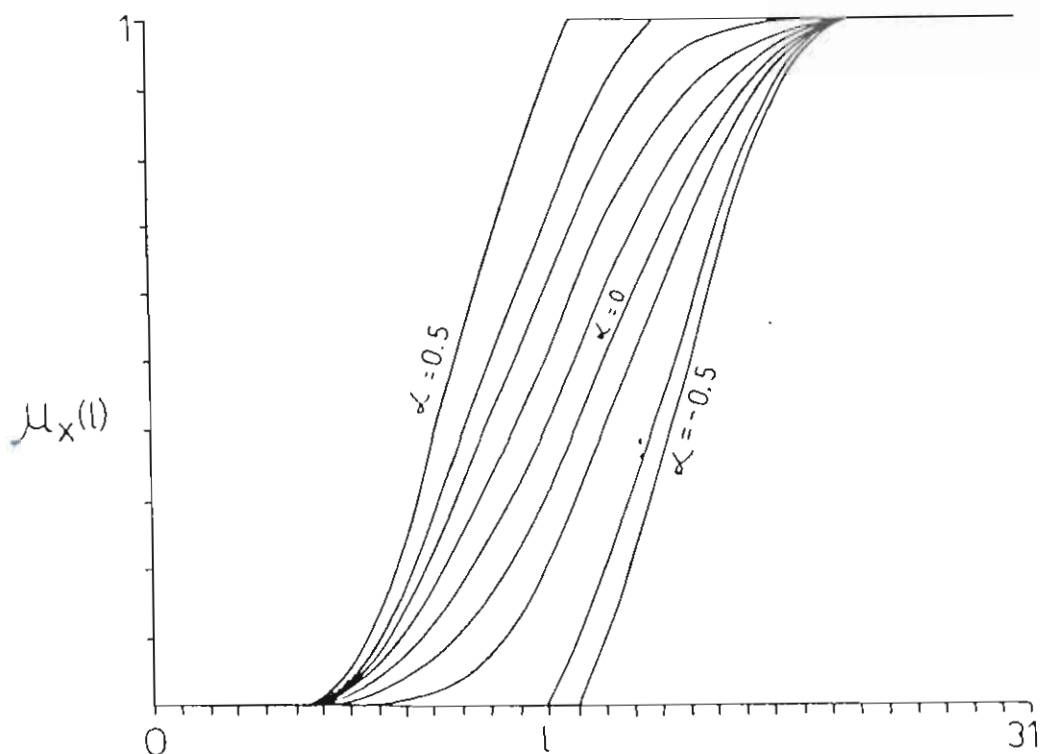


Fig. 1. Spectral membership function corresponding to Equation (32).

where

$$\theta_{mn} = \frac{S}{r c_2} \left[\sum_{k=1}^{r-1} \sum_{l=k+1}^r |\mu_k^A(x_{mn}) - \mu_l^A(x_{mn})| \right]$$

$$s = 1 / \frac{r}{2(r-1)}, \quad \text{if } r \text{ is even}$$

$$= 1 / \frac{(r+1)}{2r}, \quad \text{if } r \text{ is odd.} \quad (39)$$

If $h(l)$ denotes the number of occurrences of the level l in X , then Equation (39) can be written as

$$\theta(X) = \frac{1}{MN} \sum_l \theta(l) h(l) \quad (40)$$

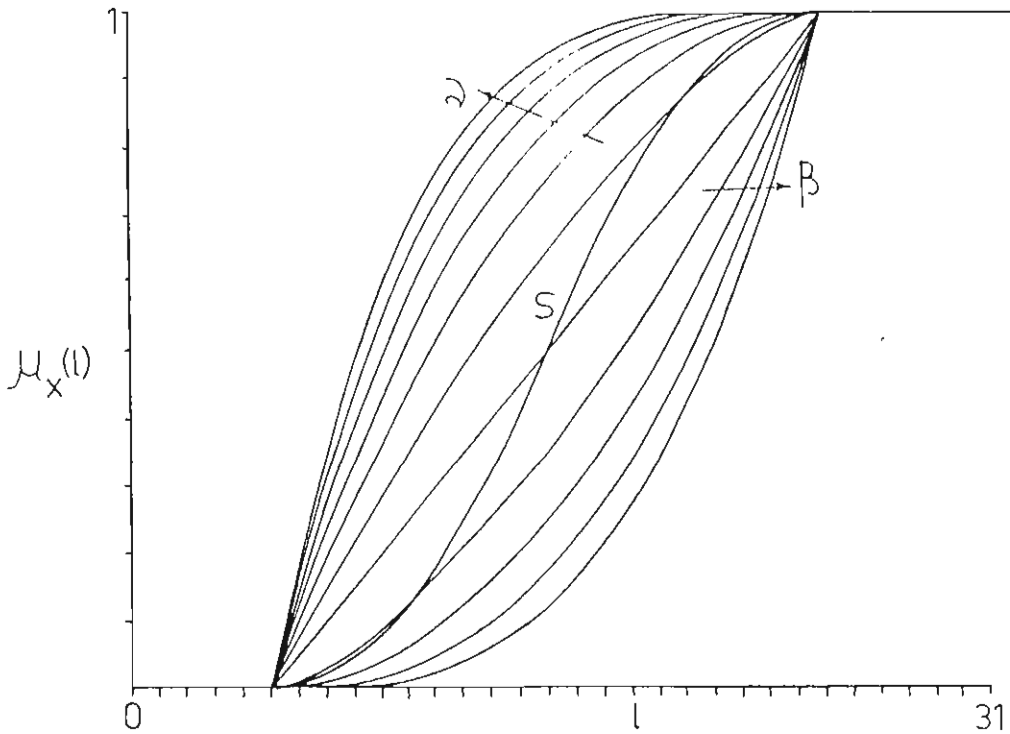


Fig. 2. Spectral membership function corresponding to Equation (35).

with

$$\theta(l) = \frac{S}{r c_2} \left[\sum_{k=1}^{r-1} \sum_{i=k+1}^r |\mu_k^A(l) - \mu_i^A(l)| \right],$$

$$l = 0, 1, 2, \dots, L-1, 1 \geq \theta(l) \geq 0 \quad (41)$$

$\theta(l)$ denotes, in a global sense, an average amount of disagreement (or difficulty) is assigning a membership value (unanimously) to a pixel for possessing some property.

In the next section we will be describing how Equations (32), (35), and (40) can be used to give due respect to various opinions on membership functions (when none can be discarded) and to make a collective soft decision by providing a set of thresholds along with their uncertainty values. Since we are interested only in demonstrating the usefulness of spectral fuzzy sets, i.e., to show how the ambiguity in membership evaluation can be managed in the image segmentation problem, we have considered here an algorithm of Pal and Rosenfeld [16] for histogram thresholding that uses only the index of fuzziness as a measure of grayness ambiguity. Other measures of grayness ambiguity

(e.g., entropy, index of crispness [9, 10]) and spatial ambiguity (e.g., compactness) can also be used in this regard. The mathematical framework of the algorithm is available in [18].

For the convenience of readers, the definition of the linear index of fuzziness $\delta(X)$ of an image X with respect to a fuzzy property $\mu_x(x_{mn})$ is given next [4, 16, 18, 19].

$$\delta(X) = \frac{2}{MN} \sum_m \sum_n \min |\mu_x(x_{mn}) - \mu_{\underline{x}}(x_{mn})| \quad (42)$$

$$= \frac{2}{MN} \sum_m \sum_n \min [\mu_x(x_{mn}), (1 - \mu_x(x_{mn}))],$$

$$m = 1, 2, \dots, M, \quad n = 1, 2, \dots, N, \quad 0 \leq \delta(X) \leq 1 \quad (43)$$

where $\mu_{\underline{x}}(x_{mn})$ denotes the nearest two-tone version of the $\mu_x(x_{mn})$ plane, i.e.,

$$\begin{aligned} \mu_{\underline{x}}(x_{mn}) &= 0 \quad \text{if } \mu_x(x_{mn}) \leq 0.5 \\ &= 1 \quad \text{if } \mu_x(x_{mn}) > 0.5. \end{aligned} \quad (44)$$

If $h(\ell)$ denotes the number of occurrences of the level ℓ in X , then Equation (43) can be written as

$$\delta(X) = \frac{2}{MN} \sum_{\ell} T(\ell) h(\ell), \quad \ell = 0, 1, 2, \dots, L-1 \quad (45)$$

with

$$T(\ell) = \min [\mu_x(\ell), (1 - \mu_x(\ell))]. \quad (46)$$

$\delta(X)$ denotes, in a global sense, an average amount of difficulty (ambiguity) in deciding whether a pixel x_{mn} possesses some property $\mu_x(x_{mn})$ or not. In case there are r opinions on the $\mu_x(x_{mn})$ plane, we may write

$$\delta(X) = \frac{2}{MNr} \sum_i \sum_{\ell} T'(\ell) h(\ell) \quad (47)$$

where

$$T^i(\ell) = \min\left[\mu_x^i(\ell), (1 - \mu_x^i(\ell))\right],$$

$$i = 0, 1, 2, \dots, r, \quad \ell = 0, 1, 2, \dots, L-1 \quad (48)$$

The algorithm for segmentation is described in the next section.

5.4. ALGORITHM

Input: An $M \times N$ image X with minimum and maximum gray levels ℓ_{\min} and ℓ_{\max} , respectively.

Step 1: Construct a spectral "bright image" membership plane $\{\mu_x^i(\ell), i = 1, 2, \dots, r\}$ with r opinions using Equation (32) or Equation (35) with $b = \ell_q$, some gray level, $\ell_{\min} \leq \ell, \ell_q \leq \ell_{\max}$ and $\Delta b = c - \ell_q = \ell_q - a$. Let us denote this μ_x plane by $\mu_x |_{\ell_q}$.

Step 2: Compute the total amount of ambiguity in the above plane with

$$d(X) |_{\ell_q} = [\theta(X) + \delta(X)]/2 \quad (49)$$

$$= \frac{1}{2MN} \sum_{\ell} \theta(\ell) h(\ell) + \frac{1}{MNr} \sum_i \sum_{\ell} T^i(\ell) h(\ell)$$

$$= \frac{1}{MN} \sum_{\ell} \left[\frac{\theta(\ell)}{2} + \frac{1}{r} \sum_i T^i(\ell) \right] h(\ell) \quad (50)$$

where $\theta(X)$ and $\delta(X)$ are defined by Equations (40) and (47), respectively. $d(X)$ determines the total amount of incertitude (difficulty) in assigning a unique brightness membership value to a pixel and in deciding whether it will be considered as bright or not. Note that $\theta(\ell)$ was absent and r was considered to be unity in the algorithm of Pal and Rosenfeld [16].

Step 3: Vary ℓ_q from ℓ_{\min} to ℓ_{\max} , compute $d(X)$ of various μ_x planes thus obtained and select the one corresponding to $\ell_q = \ell_i$, say, for which $d(X)$ is a minimum.

The spectral fuzzy subset μ_x corresponding to $\ell_q = \ell_i$, can be viewed as optimum in the sense of its minimum d value, because for any other selection of ℓ_q , the d value will be greater than this. This can be further verified from Equation (50), which shows that the minimum value of $d(X)$ would always correspond to the valley region of the histogram having number of occurrence $h(\ell)$ low.

The band of cross-over points corresponding to the optimum spectral fuzzy subsets $\mu_x |_{\gamma_q = \gamma_i}$ can be regarded as the range of possible thresholds. In other words, the set v corresponding to width $w_d(0.5)$ of $\mu_x |_{\gamma_q = \gamma_i}$ represents the output threshold levels arising out of r opinions. It is to be noted that the cardinality of v denoting the number of resulting thresholds is always less than or equal to r .

Output: Various threshold levels together with their d values computed (on their respective opinion sets on the $\mu_x |_{\gamma_q = \gamma_i}$ plane) as

$$d(\gamma) = \theta(\gamma) + \sum_i T'(\gamma), \quad i = 1, 2, \dots, r, \quad (51)$$

$\theta(\gamma)$ and $T'(\gamma)$ being defined by Equations (41) and (48). The value of d associated with a level denotes the degree of difficulty (easiness) in deciding unambiguously that it is a member of the object or background (boundary).

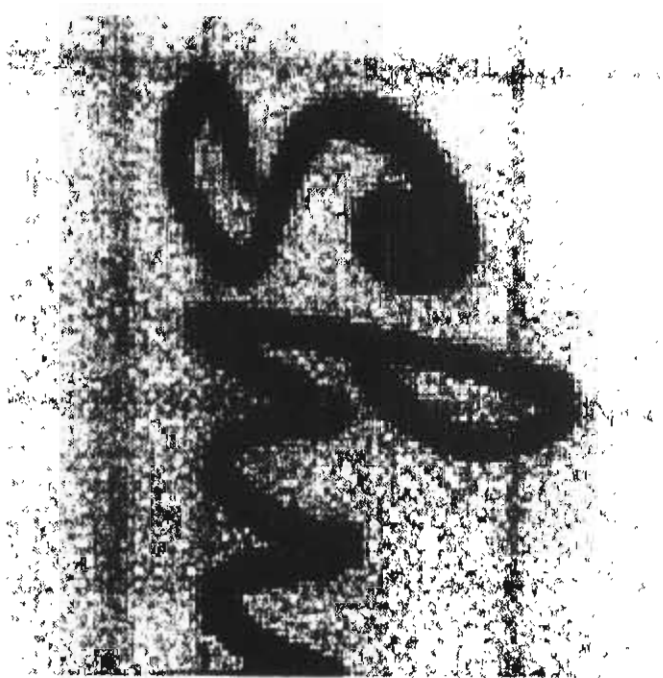
For the purpose of nonfuzzy (hard) segmentation, we may therefore select the one among the various output thresholds for which $d(\gamma)$ is maximum, i.e., the level for which the difficulty in deciding that it is a boundary level is minimum, taking into account various opinions.

When there is no disagreement in opinion on the membership function, i.e., a single membership function is used to define the bright image plane, $\theta(\gamma) = 0$, and the crisp threshold corresponds to the cross-over point (which is the most ambiguous point) of the resulting optimum μ_x plane (as obtained with the algorithm of Pal and Rosenfeld [16]).

Since we are interested here (as mentioned before) mainly in demonstrating how the uncertainty that arises among individuals in defining a membership function can be managed in determining object/background thresholds and in showing a way of extracting a set of thresholds with their certainty values for ill-defined image regions (i.e., when the selection of a specific threshold may become questionable), we have adopted here a fuzzy set theoretic algorithm for histogram thresholding simply based on minimization of only grayness ambiguity on the bright membership plane. One may also use in this regard a similar algorithm based on minimization of geometrical ambiguity (or both grayness and geometrical ambiguity) on the brightness plane.

5.5. RESULTS

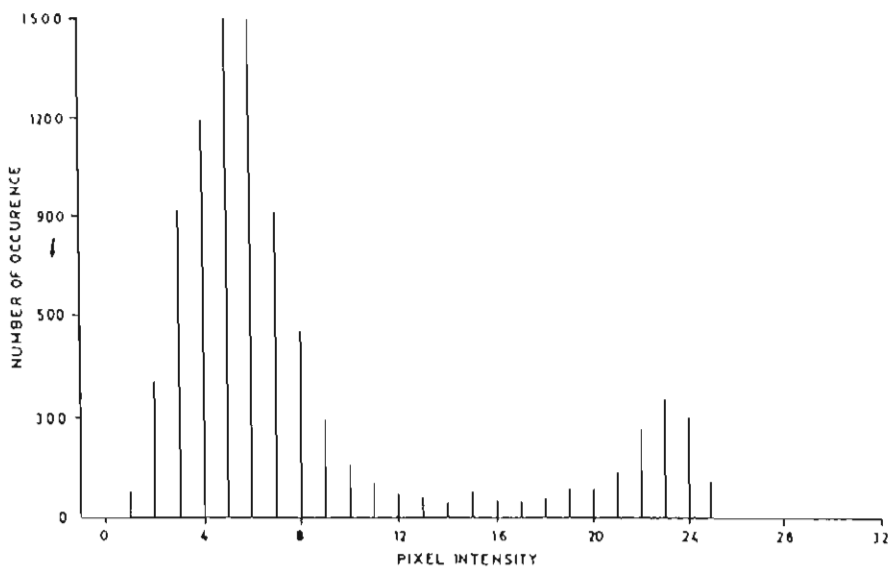
Figures 3(A), 4(A), and 5(A) show the images of "Shu," "Boy," and "Biplane" with levels 32 and dimensions 96×99 , 64×64 , and 64×64 , respectively. Their histograms (Figures 3(B), 4(B), and 5(B)) are seen to have



LEGEND



(a)



(b)

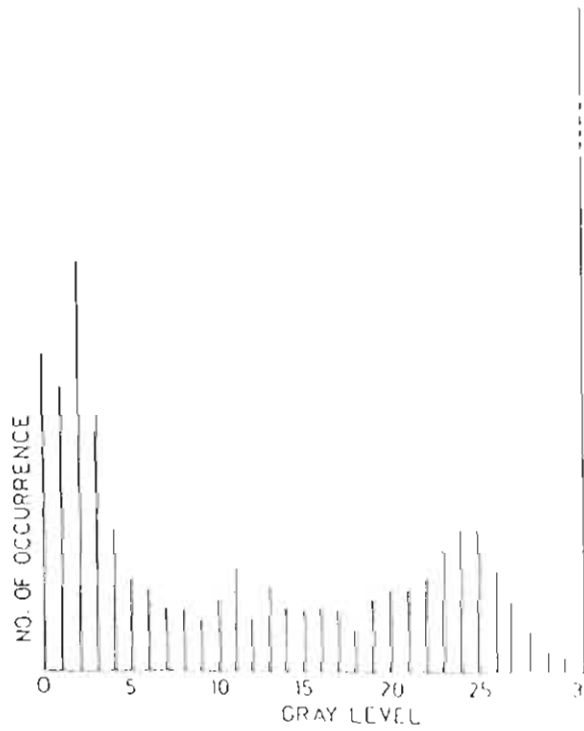
Fig. 3. (a) Input "shu" image. (b) Histogram.



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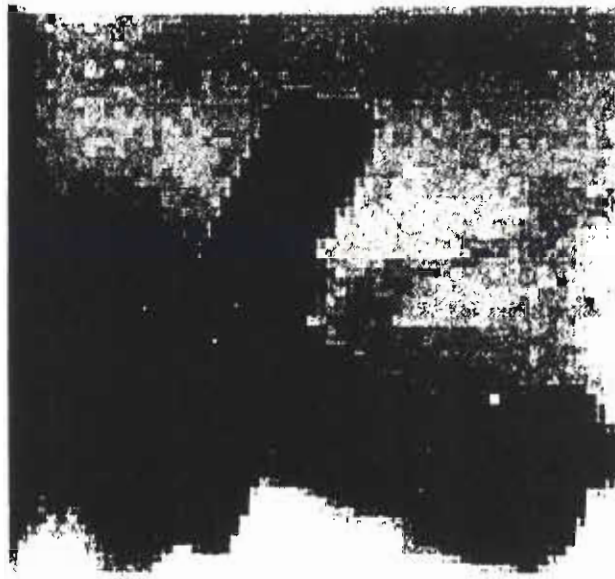


(a)



(b)

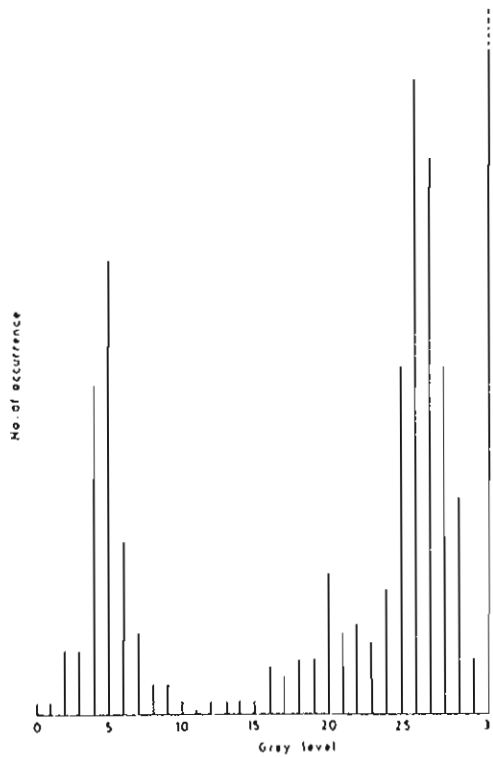
Fig. 4. (a) Input "boy" image (b) Histogram



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(a)



(b)

Fig. 5. (a) Input "biplane" image. (b) Histogram.

TABLE 1
Soft Thresholds for Object Extraction

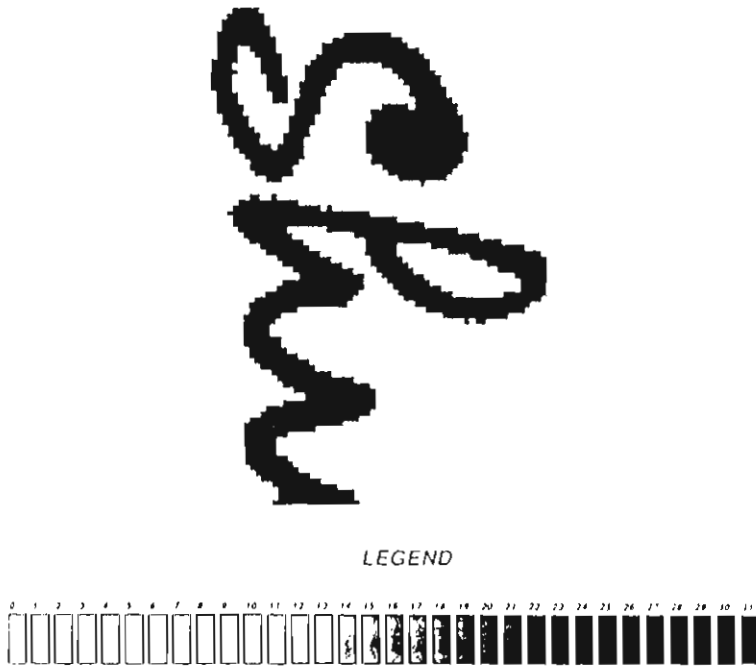
$\alpha = 0.2, 0.1, 0, -0.1, -0.2, -0.3$				$\alpha = 0.5, 0, -0.2, -0.3, -0.5$				$\alpha = 0.5, 0.3, 0.2, 0, -0.5$			
{TH}				{TH}				{TH}			
Shu	Boy	Bip	$d(/)$	Shu	Boy	Bip	$d(/)$	Shu	Boy	Bip	$d(/)$
16	8	12	2.10	15	7	11	0.69	14	8	11	1.48
17	9	13	1.86	16	8	12	1.00	15	9	12	1.00
				17	9	13	1.48	16	10	13	0.69

flat valleys in general and several local minima, particularly for the Boy histogram. Tables 1 and 2 show the various sets of thresholds obtained from object-background classification when different membership functions (opinions) corresponding to Equations (32) and (35) with $\Delta b = 4$ were considered to extract the brightness membership plane. The amount of $d(/)$ value associated with each possible threshold is also listed in the tables. Note that the $d(/)$ values shown (by Equation 51) are not normalized.

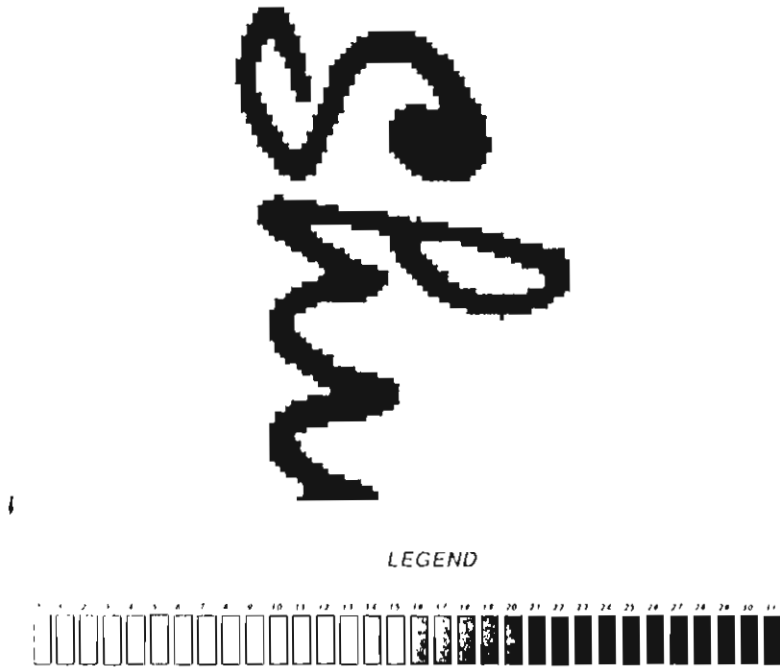
Figures 6-8 show the thresholded versions of input images corresponding to those detected levels. The set of thresholded images thus obtained for an input by taking a collective decision (on various opinions) are seen to have not much difference among themselves. Single (hard) thresholds produced by various nonfuzzy approaches [21, 22] for object extraction of these images were seen to correspond to one of those obtained here. The incorporation of the concept of spectral fuzzy set thus makes the selection soft by resulting in a band of possible thresholds (instead of a single threshold) with their respective ambiguity (uncertainty) values.

TABLE 2
Soft Thresholds for Object Extraction

$\beta = 1.0, 2.0, 3.0$ $\gamma = 1.5, 2.0$				$\beta = 1.0, 2.0, 3.0, 3.5$ $\gamma = 1.5, 2.0$				$\beta = 1.0, 2.0, 3.0, 3.5, 4.0$ $\gamma = 1.5, 2.0$			
{TH}				{TH}				{TH}			
Shu	Boy	Bip	$d(/)$	Shu	Boy	Bip	$d(/)$	Shu	Boy	Bip	$d(/)$
14	8	11	1.73	14	8	11	1.77	14	8	11	1.79
15	9	12	1.98	15	9	12	2.07	15	9	12	2.13
16	10	13	1.66	16	10	13	1.85	16	10	13	2.01
17	11	14	1.42	17	11	14	1.79	17	11	14	2.10



(a)



(b)

Fig. 6. Thresholded version of "shu" image. (a) Threshold = 14. (b) Threshold = 15.



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(c)

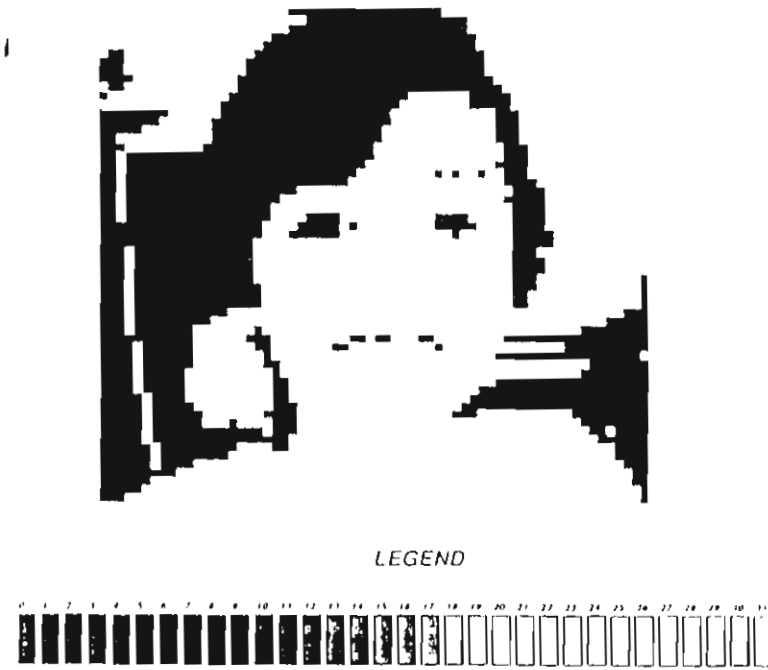


LEGEND

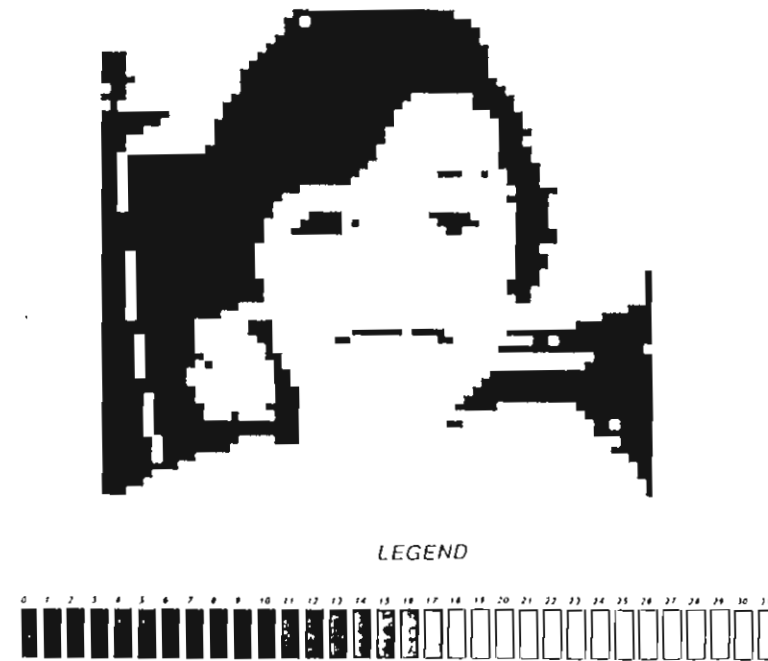


(d)

Fig. 6. (continued) (c) Threshold = 16. (d) Threshold = 17.



(a)



(b)

Fig. 7. Thresholded version of "boy" image. (a) Threshold = 7. (b) Threshold = 8.



LEGEND



(c)



LEGEND



(d)

Fig. 7. (continued) (c) Threshold = 9. (d) Threshold = 10.

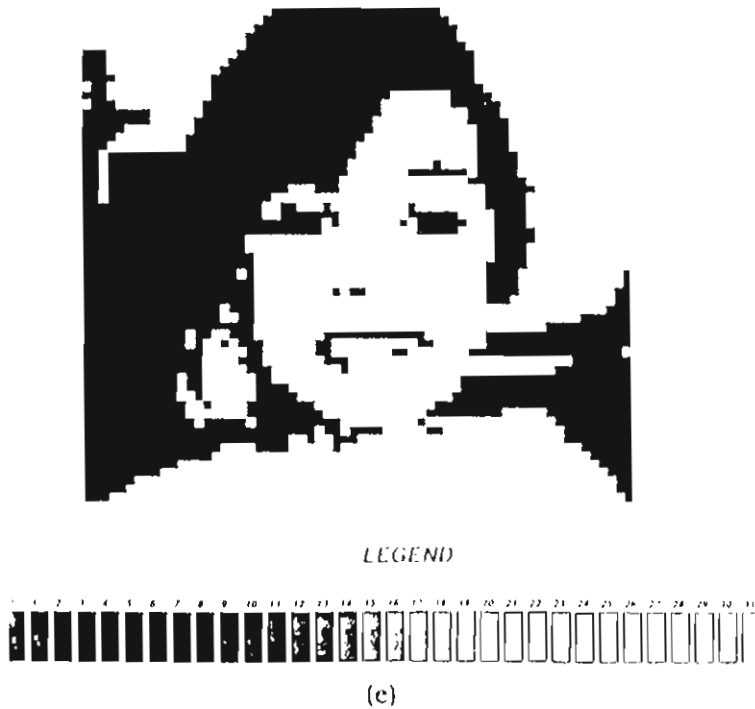


Fig. 7. (continued) (c) Threshold = 11.

As expected, the threshold having maximum d -value depends greatly on the choices of α for Equation (32), and of β and γ for Equation (35). When α makes the sp-membership functions consists of more low/high valued opinions as compared to Zadeh's standard function, the most preferable (certain) threshold is seen (2nd and 3rd parts of Table 1) to move towards the right/left of the gray scale. The case with Equation (35) is similar. The range of thresholds depends to some extent on the degree of difference in opinion (c.g., minimum for part 1 of Table 1 and maximum for Table 2).

Again, since $|v| \leq r$, the results that would have been obtained by considering the membership functions individually may not necessarily be the same as those obtained here, because the later one involves collective decision.

6. DISCUSSION

A new concept of spectral fuzzy set has been introduced for the management of uncertainty in the subjective assignment of membership functions characterizing a fuzzy set (or of membership values to the elements of a fuzzy set). It is characterized by a set of individual opinions on a membership function (where each membership grade is attached to one of the membership functions) rather than with set-valued membership grades (where the origin of

membership grades is forgotten). The way it differs from type 2 fuzzy set, ultra fuzzy set, interval valued fuzzy set, and probabilistic set is explained. The uncertainty due to spectralness of the membership function has been defined. It is to be mentioned here that the extent of the height and width of spectral

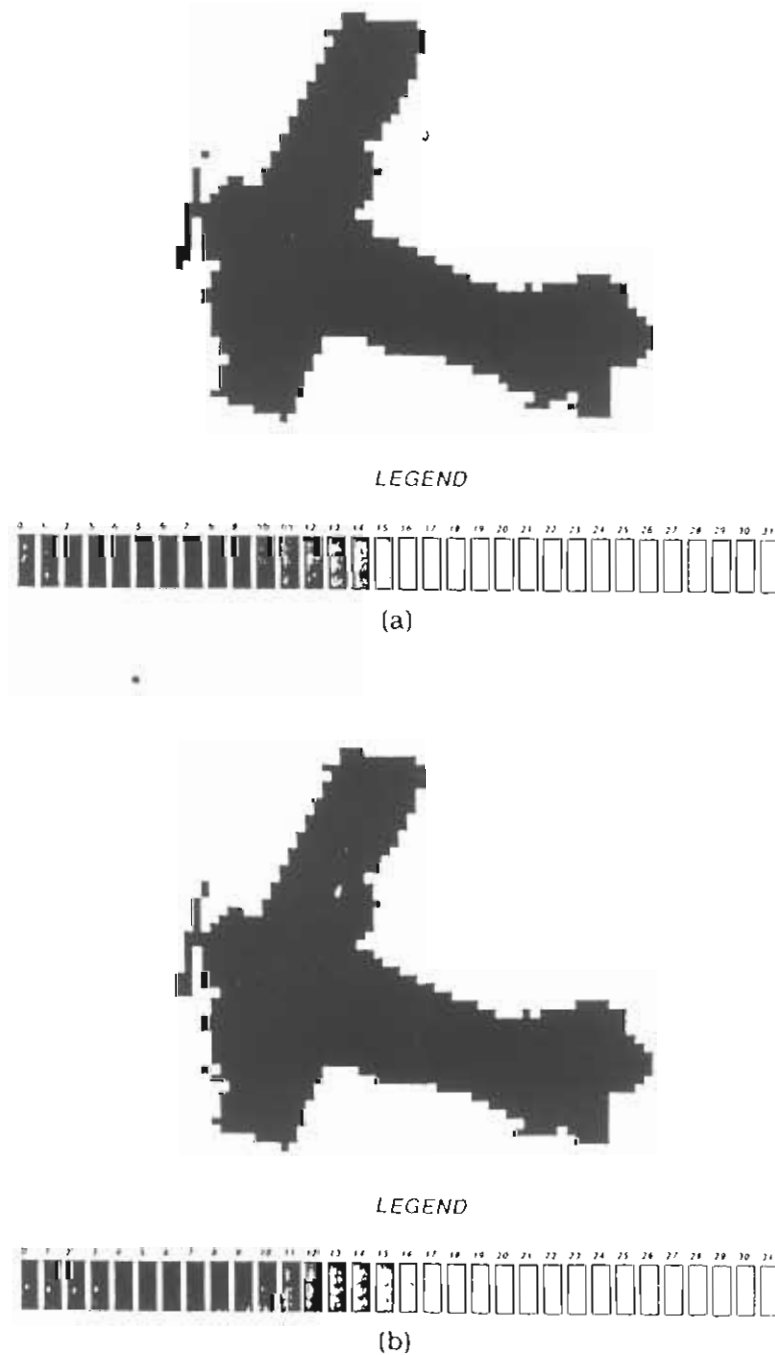


Fig. 8. Thresholded version of "biplane" Image. (a) Threshold = 11. (b) Threshold = 12.

membership function may be restricted by the bound functions formulated recently by Murthy and Pal [14, 18].

Finally, its necessity and appropriateness for the image processing/analysis problem has been demonstrated. It has been found that the sp-fuzzy set can be

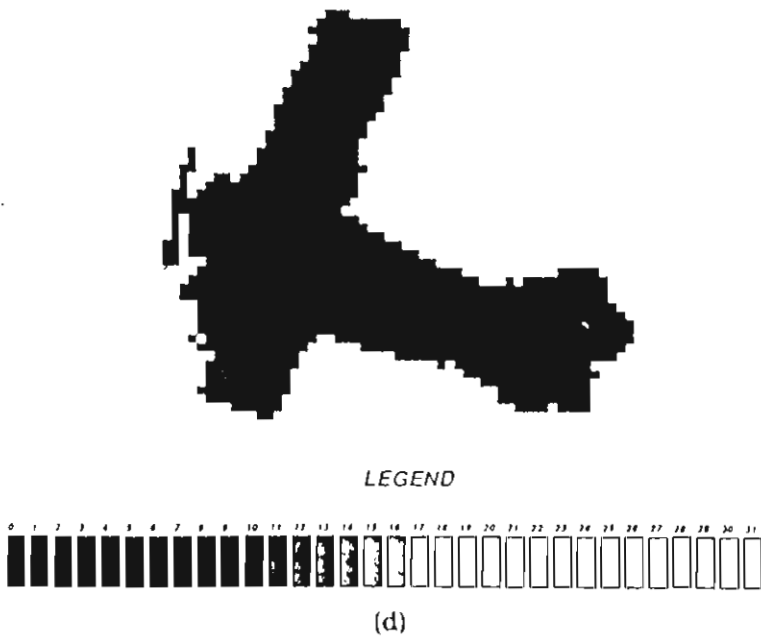


Fig. 8. (continued) (c) Threshold = 13. (d) Threshold = 14.

considered as a helpful tool for the management of uncertainty in selection of a threshold from an ambiguous flat valley by taking a soft decision and resulting in a band of thresholds (instead of a single threshold) with their respective certainties.

Although we have adopted here an algorithm for object extraction that considers only grayness ambiguity on the bright membership plane, one may also use geometrical ambiguity or textural ambiguity in demonstrating the usefulness of spectral fuzzy sets. Furthermore, the problems of edge detection and skeleton extraction (where incertitude arises from ill-defined regions and various opinions on membership values), and any expert system type application (where difference in experts' opinions leads to an uncertainty) may also be handled with this framework.

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