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DETERMINING THE SHAPE OF A PATTERN CLASS FROM SAMPLED POINTS IN $\mathbb{R}^2$

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An important problem in pattern recognition is determining the shape of a pattern class from its sampled points. A procedure which provides multivalued shape has been suggested here for a pattern class in $\mathbb{R}^2$. The procedure can be viewed in two phases. Phase I deals with the decomposition of sample set into some groups of nearly rectangular shape. Phase II determines each of the sub-classes corresponding to the groups separately, aggregates them and obtains the multivalued shape of the pattern class. The effectiveness of the procedure has been demonstrated on some artificially generated data sets or pattern classes as well as on the Indian Telugu vowel speech data set. The convergence of the estimated set to the original set has been verified successfully using two different metrics between sets. One is the well-known Hausdorff metric. The other is a new metric which has been defined in the paper.

INDEX TERMS: Set estimation, fuzzy sets, geometric complexity, windows, accuracy factor, goodness of fit

1. INTRODUCTION

A. *Pattern Class*

The paper deals with finding the shape of a pattern class from a set of training samples in $\mathbb{R}^2$. In most of the real life problems (*e.g.*, Telugu Vowel data), pattern classes are bounded. Thus the pattern classes considered in this paper are all bounded. A definition of pattern class is given below for $\mathbb{R}^s$.

DEFINITION 1 A set $\mathbb{A} \subseteq \mathbb{R}^s$ is said to be a pattern class[1] if

i) $\mathbb{A}$ is path connected compact,
ii) $cl(Int(\mathbb{A})) = \mathbb{A}$, [*cl* means closure, *Int* means interior]
iii) $Int(\mathbb{A})$ is path connected and
iv) $\lambda(\delta\mathbb{A}) = 0$ where $\delta\mathbb{A} = \mathbb{A} \cap cl(\mathbb{A}^c)$ and λ is the Lebesgue measure in $\mathbb{R}^s$. [c means complement] ■

The above definition is given using topological and measure theoretic concepts. The relevance of the properties (i), (ii), (iii) and (iv) is provided in [1]. Let $\mathcal{A} = \{\mathbb{A} : \mathbb{A}$ satisfies (i), (ii), (iii) & (iv) above$\}$. $\mathcal{A}$ is the collection of all classes in $\mathbb{R}^s$. $\mathcal{A}$ is used in the Appendix in relation to a few results. Any $\mathbb{A} \in \mathcal{A}$ is referred as the pattern class or set in this paper. In all the subsequent sections, s is assumed to be 2 since the paper deals with two dimensional case alone.

Once the pattern class is known or determined, then the following steps may be incorporated, if necessary, for the future use.

1. The boundary of the class can be found out which may be used for classification purpose.
2. Storing the total pattern class may need much of memory space. There are various ways in which the storage utility can be increased.
   a) Only boundary information may be stored if there are no holes. If holes are present in the pattern class, then the information about the holes have also to be stored.
   b) If the shape of the pattern class is a regular one like a disc or a rectangle etc., one can store the centre and the radius or the lengths of the sides etc.
   c) More often than not, the shape is an irregular one but it approximates to a regular one. Then one can find the approximate regular shape and store it.
   d) A pattern class which can be visualized as a union of regular sets, can be stored using the information about the regular sets alone.

In conclusion, if the pattern class is determined then some salient features, if possible, about the class can be extracted which are useful in making decisions about a course of action to be taken (identification, classification and pattern description). Generally, in many problems, a few sampled points are known about the class. Hence finding the class and its shape from sampled points is extremely useful in real life.

B. Previous Works

There are various approaches described in literature for determining shape of a pattern class from sampled points.[1–6] These methods are mostly heuristic in nature and they provide an exact boundary or shape of the pattern class. One of the observations about these algorithms is that the boundary of the class contains some of the sample points which need not be true. It is necessary to extend the boundaries to some extent to handle the possible uncovered portions by the sampled points. The extended portions should have the following properties.

1) As the number of sample points increases, the extended portions should decrease.
2) The extended portions should have less possibility to be in the pattern class then the portions explicitly highlighted by the sampled points.

The second property leads to a multivalued boundary of a pattern class. The basic concept of one of the existing methods[1–6] is described below in short for illustration.

Edelsbrunner *et al.*[2] introduced the notion of the "α-shape" of a finite set of points, for arbitrary real α. This notion is a generalization of the convex hull. Given a set S of N points in a plane, the convex hull of S may be defined as the intersection of all closed half planes that contain all points of S. The α-hull of S is the complement of the union of all open discs of radius not less than $1/\alpha$ (for arbitrary negative real α) which contains no point of S. The shape of the set is determined using α-hulls. If the selection of α is proper then the shape of the set obtained by the method resembles the intuitive shape of the planar set. But this method does not provide multivalued shape of a pattern class.

C. *Scope of This Paper*

This paper deals with a procedure for determining multivalued shape of a pattern class in the plane from a set of sampled points. The procedure can be viewed in two phases. The phase I is concerned with the decomposition of the sample set into groups of nearly rectangular shape. The decomposition is based on the boundary variations of the pattern class found in the sampled points. A modified window approach has been incorporated here to find the boundary variations of the sample set. The phase II determines each of the sub-classes corresponding to the groups separately, puts them together and finds the multivalued shape of the pattern class. The effectiveness of the procedure has been demonstrated on some artificially generated data sets as well as on a speech data. The convergence of the proposed algorithm has been verified using two different metrics.

The present work is a part of the investigation of the project "On Pattern Recognition and Uncertainty Management using Fuzzy Sets and Approximate Reasoning" being carried out in the E.C.S.U., Indian Statistical Institute, Calcutta. A multivalued recognition system[7] has already been formulated based on fuzzy set theory and approximate reasoning. The concept of decomposing a pattern class into subclasses of nearly rectangular shape, as described in this paper, has been incorporated successfully in the recognition system also.

In section II, some basic concepts along with the block diagram of the present work are discussed. Section III deals with the procedure for decomposing a training sample set. Section IV provides an approach to determine the multivalued boundary of a pattern class. Experimental results are provided in Section V. In Section VI, convergence of the proposed procedure is discussed. Section VII deals with the conclusion.

II. PROCEDURE FOR DETERMINING A PATTERN CLASS

A. *Some Concepts*

A method for determining the multivalued shape of a class from a set of sampled points is proposed in this paper. Some of the basic concepts used in developing this method are stated below.

a. Accuracy factor It has been argued in the previous section that the boundary of a pattern class obtained from its sample points should be extended to highlight the portions of the class possibly uncovered by the sample points. An accuracy factor (δ_N) based on the number of sampled points (N) is considered below for the said extension to manage the uncertainty. δ_N satisfies

$$\frac{1}{N^{0.49}} \leq \delta_N \leq \frac{1}{N^{0.33}} \tag{1}$$

so that as N increases, $\delta_N \to 0$ and $N\delta_N^2 \to \infty$. Since δ_N decreases with the increase of N, the accuracy of the obtained boundary also increases with the increase of N. Diagram 1 shows the allowed range of δ_N for different values of N. The inequality (1) has been considered because of the existing literature [Appendix-A].

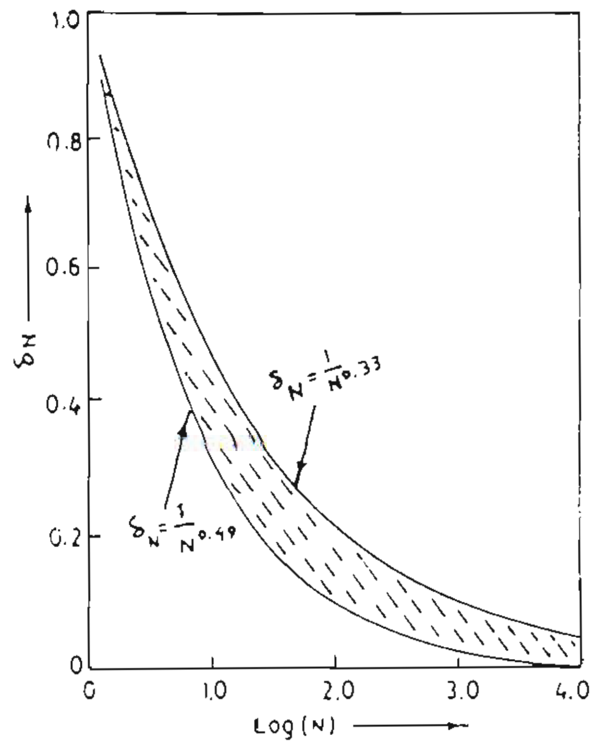


Figure 1 Allowed range of δ_N for different sample sizes (N).

b. Concept of boundary The procedure described here is for determining the multivalued shape of a pattern class in the plane from a set of training samples. The feature axes under consideration are referred as the first (F_1) and the second (F_2) axes respectively throughout the paper.

A typical training sample set is shown in a feature space in Fig. 2(a). To find the boundary variation of this set, four perpendicular directions (referred by the codes 1, 2, 3 and 4), as shown in Fig. 2(b), are considered. Fig. 2(c) shows the boundaries of the sample set in the coded directions 1 and 2, whereas the Fig. 2(d) shows the boundaries of the sample set in the coded directions 3 and 4. The complete boundary of the sample set can be obtained by combining these four boundaries. Depending on the situations, as described below, one axis will be considered as the base and the other will be considered as the height. In Fig. 2(c), the first feature (F_1) corresponds to the base and the second feature (F_2) corresponds to the height. Similarly in Fig. 2(d), the F_1 and F_2 axes correspond to the base and height axes respectively. The boundaries in the coded directions 1 and 2 of Fig. 2(c) are considered as the lower and upper boundaries respectively whereas the boundaries in the coded directions 3 and 4 of Fig. 2(d) are considered as the lower and upper boundaries respectively.

Formation of windows Let $(b_1, h_1), (b_2, h_2), \ldots, (b_N, h_N)$ be the sampled points in terms of base and height values. Initially from all the sampled points, the maximum (say, max_b) and minimum (say, min_b) of the base values are found. A base coverage factor, say ξ_b, is defined as

$$\xi_b = (max_b - min_b)\, \delta_N \tag{2}$$

where $(max_b - min_b)$ is the range of the base values and δ_N is the accuracy factor,

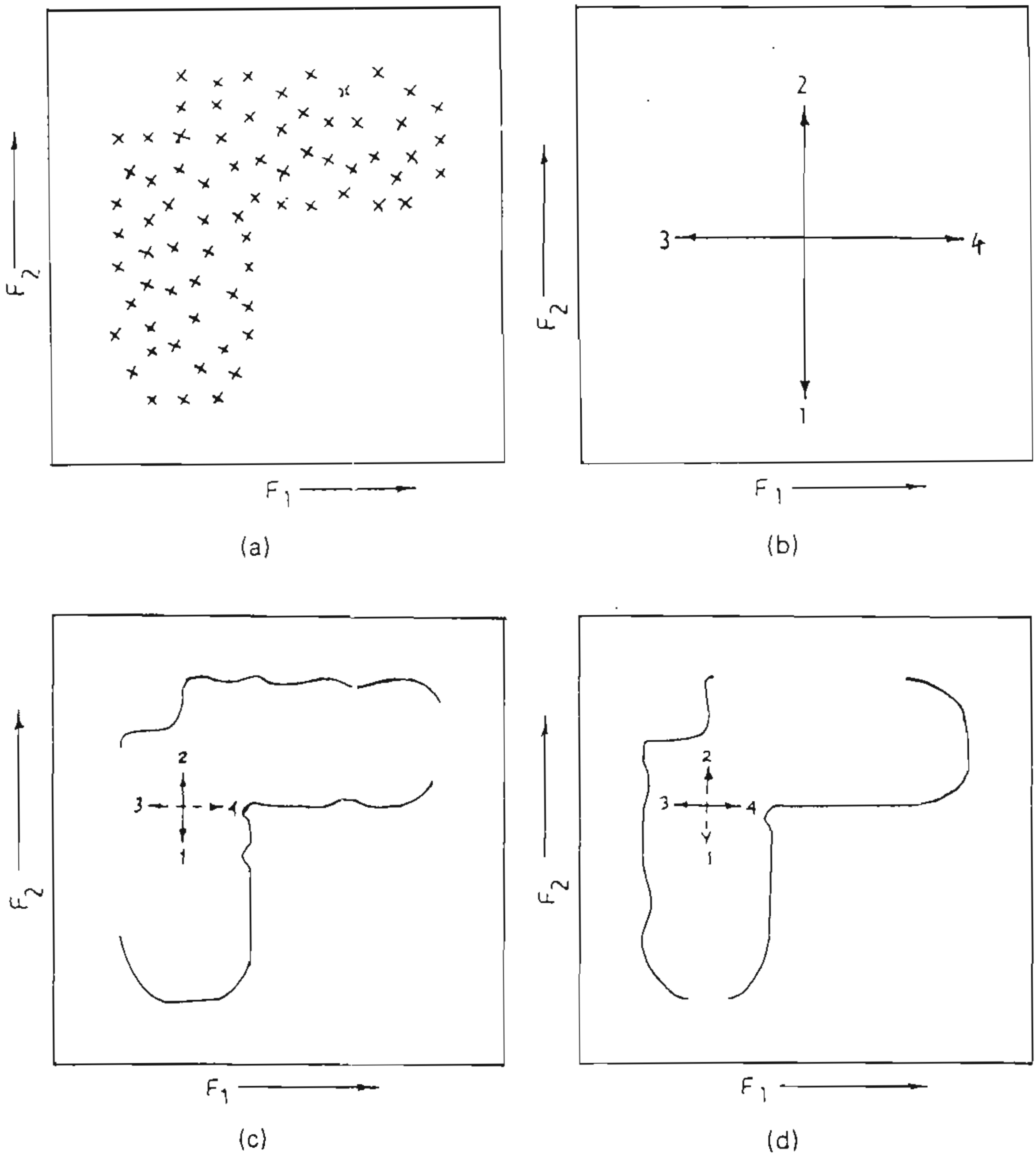


Figure 2(a)–(d) Illustrating the concept of boundary.

ξ_b is utilized to generate the windows from the sampled points such that the base coverage length of each window is at least ξ_b.

Similarly, the maximum (say, max_h) and the minimum (say, min_h) of height values are found and a height threshold factor, say ξ_h, is defined as

$$\xi_h = (max_h - min_h)\delta_N \tag{3}$$

where the first term $(max_h - min_h)$ is the range of the height values. ξ_h is used in deciding whether a training sample set is of nearly rectangular shape or not.

Now the training samples are arranged in ascending order according to the base values. The first window starts with the first sample point of the ordered training sample set and it includes all those samples one after another in ascending order until its base coverage length exceeds ξ_b. Assume that the first window ends with the κth sample. Then the second window will end with the $(\kappa + 1)th$ sample and to find the starting point of this window, we proceed backwards from κth sample until its base coverage length exceeds ξ_b. Similarly other windows are constructed by including one new sample point at the end and excluding some sample points from the beginning of the previous window such that the base coverage length would at least be ξ_b. The last window ends with the last sample point *i.e.*, sample point with the highest base value. Thus, some overlapping windows of sample points are generated utilizing the sample base values and the base coverage factor ξ_b.

The maximum and the minimum height sample values are found from each of the windows and these are taken to be the lower and upper boundary values respectively of that window. The combination of the upper boundary values highlights the upper boundary of the training sample set and combination of the lower boundary values provides the lower boundary of the training sample set.

c. Extension factor The uses of the extension factors have been stated earlier. The proposed method intends to find the multivalued boundary of a class. Hence, in order to find the possible uncovered portion of the class by the sample points, the boundaries are extended to some extent. The extended portions should have less possibility to be in the pattern class than the portions explicitly highlighted by the training samples.

Let ext_j $(j = 1, 2)$ be the extension factor of the pattern class corresponding to the jth feature. Let max_j and min_j be the maximum and minimum sample feature values corresponding to the jth feature respectively. Then the extension factor, ext_j, for the jth feature is defined as

$$ext_j = (max_j - min_j)\delta_N \; ; \quad j = 1, 2 \tag{4}$$

where δ_N is the accuracy factor. When the number (N) of sample points increases, the accuracy factor δ_N decreases and correspondingly the values of the extension factors ext_j's also decrease. Thus the extended portions also decrease and correspondingly the accuracy of the boundary increases.

B. Block Diagram

The block diagram of the proposed procedure for determining the multivalued shape of a class from a set of sample points is shown in Fig. 3. It consists of two parts, namely the Decomposition and the Fuzzy Processor. The Decomposition section deals with the decomposition of the sample set into some groups of nearly rectangular shape. The Fuzzy Processor determines each of the sub-classes corresponding to the groups separately and all these sub-classes are combined to find the multivalued shape of the pattern class.

The Decomposition section consists of three blocks as shown in Fig. 3. The Hole Detector block decomposes the training sample set with holes into groups to find the hole information. The Boundary Variation Calculator block considers the boundaries in four perpendicular directions [Fig. 2(b)] to find their boundary variation

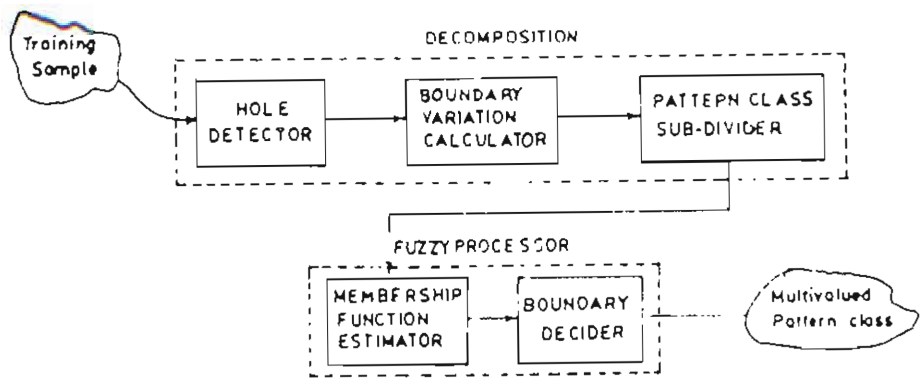


Figure 3 Block Diagram.

values. These boundary variation values are analyzed in the Pattern Class Sub-divider block to decompose (if necessary) the sample set into groups.

The theory of fuzzy sets has been used in the Fuzzy Processor section to extend the sample set and also to relate every point in the whole feature space to its possibility to be in the pattern class. The Membership Function Estimator block decides about the $\mathfrak{M}$ functions to represent each of the sub-classes corresponding to the groups. The Boundary Decider block determines each of the sub-classes separately, puts them together and finds the multivalued shape of a pattern class.

A short and over all description of the proposed procedure has been provided above. The following two sections *i.e.*, section *III* and *IV*, describe in detail the operations of different blocks of Fig. 3.

III. DECOMPOSITION

This section decomposes the training sample set into groups if the sample set is found to be not nearly rectangular in shape. Decomposition consists of three blocks, namely Hole Detector, Boundary Variation Calculator, and Pattern Class Sub-divider. The operations of these blocks are furnished below in detail.

A. Hole Detector

The intuitive idea behind holes of a pattern class can be put mathematically in the following way.

DEFINITION 2 A path connected and compact set $\mathbb{A}$ is said to have k holes if[8]

$$\mathbb{A}^c = \mathbb{B} \cup \mathbb{C}_1 \cup \mathbb{C}_2 \cup \ldots \cup \mathbb{C}_k \text{ such that}$$

i) $\mathbb{B}$ and $\mathbb{C}_i$ are path connected sets $\forall\ i = 1, 2, \ldots, k$,

ii) $\mathbb{B}$ is unbounded and $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_k$ are bounded.

iii) $\mathbb{B} \cup \mathbb{C}_{i1} \cup \mathbb{C}_{i2} \cup \ldots \cup \mathbb{C}_{ir}$ is a disconnected set for $1 \le i_1 < i_2 < \ldots < i_r \le k$, where $1 \le r \le k$.

iv) $\mathbb{C}_{i1} \cup \mathbb{C}_{i2} \cup \ldots \cup \mathbb{C}_{ir}$ is a disconnected set for $1 \le i_1 < i_2 < \ldots < i_r \le k$, where $2 \le r \le k$.

Then $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_k$, in the above definition, are said to be the holes of $\mathbb{A}$. ■

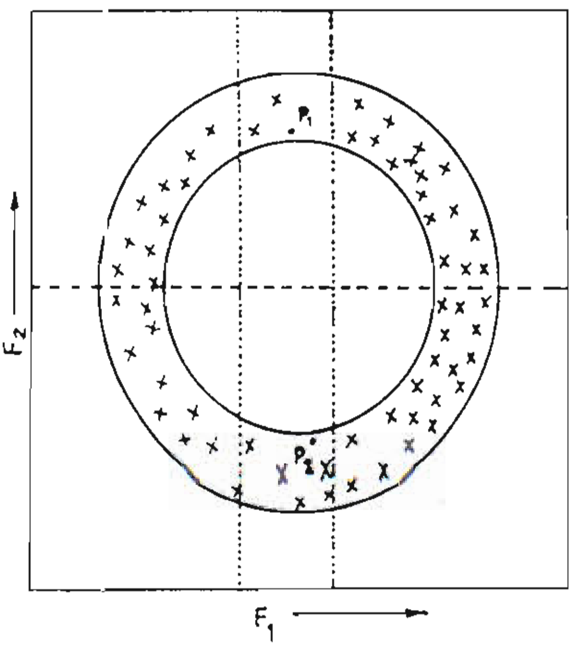


Figure 4 A typical pattern class with a hole.

The adopted procedure to find holes from a training sample set is discussed below.

Procedure The procedure considers the F_1 axis as the base and the F_2 axis as the height. Correspondingly the base coverage factor ξ_b and height threshold factor ξ_h are calculated by the equations (2) and (3). Windows are generated using the procedure described in section II. The sample points in each window are then arranged in ascending order according to the height sample values. If the difference of height values of any two consecutive sample points within a window exceeds ξ_h, then a hole is assumed to be present between the said sample pair. Let $\hbar'$ and $\hbar''$ be the height values corresponding to two such sample points. To illustrate this finding, a class with a hole is shown in Fig. 4 in which P_1 and P_2 are the sample points which satisfy the above condition and the corresponding window is shown using dotted lines.

To detect the hole, the sample set is decomposed into two groups according to whether the height values (*i.e.*, F_2 values) are less than $(\hbar'+\hbar'')/2$ or not. The decomposition leads to finding two groups where none of them possesses a hole. In Fig. 4, the line with dashes indicates the split. ■

The above routine is repeated until every group is found to be not containing any hole. When the sub-classes corresponding to the groups are combined in the Boundary Decider block [section IV], the holes will be excluded from the final shape of the pattern class. It is to be observed that this procedure detects not only the vertical holes but also the horizontal holes. Note that the proposed procedure can't detect the holes with width less than ξ_b and/or height less than ξ_h. Hence the detectable minimum hole size depends on ξ_b and ξ_h, which in turn depends on the sample size.

It is also to be mentioned here that the procedure decomposes a training sample set not only for a hole, but also for some particular type of concave boundaries. For example, sample sets having shapes like '*C*' or ']' or '∠' may also be decomposed. But this does not create any problem in the over all shape determining procedure, since the sample sets anyway had to be decomposed in the Pattern Class Sub-divider block.

B. Boundary Variation Calculator

This block tries to detect the geometric structure of a training sample set. The boundaries in four perpendicular directions [Fig. 2(b)] are considered here to find the boundary variation values. Our approach to construct the windows from a sample set and consequently to find the boundary values are described in section II. It may be recalled that the maximum height value and minimum height value of each window are considered as upper boundary value and lower boundary value of the window respectively. The description of the adopted approach to calculate the boundary variations is given below.

Calculation of boundary variations To describe the approach, let us consider the boundary variation values in a particular direction, say d. The procedure to obtain the boundary values has already been discussed earlier. Henceforth, it is assumed that there are m windows and their boundary values are $\mathcal{H}_i$, $i = 1, 2, \ldots, m$. A boundary variation factor, say $\mathcal{V}_d$, in direction d is defined as

$$\mathcal{V}_d = \left[\sum_{i=1}^{m-1} (\mathcal{H}_i - \mathcal{H}_{i+1})^2\right] \Big/ \xi_h^2 \tag{5}$$

where ξ_h is the height threshold factor for the considered direction d. Here the division factor ξ_h^2 has been used to make $\mathcal{V}_d$ unitless.

Now let max_H and min_H be the maximum and minimum of the $\mathcal{H}_i$'s ($i = 1, 2, \ldots, m$) respectively. If the difference of max_H and min_H does not exceed ξ_h *i.e.*,

$$\text{if} \quad (max_H - min_H) \leq \xi_h \tag{6}$$

then the sample set is assumed to be nearly rectangular in shape. In such a case, the variation factor $\mathcal{V}_d$ is assumed to be zero *i.e.*, make $\mathcal{V}_d = 0$. Otherwise the sample set is considered as decomposable in the considered direction d. ■

Initially, F_1 axis has been considered as the base to generate the windows and F_2 axis has been considered as the height to find the boundary variation factors $\mathcal{V}_1$ and $\mathcal{V}_2$ for the coded directions 1 and 2 respectively. Similarly, by reversing the roles of F_1 and F_2 axes above, the boundary variation factors $\mathcal{V}_3$ and $\mathcal{V}_4$ for the coded directions 3 and 4 are calculated. Thus the boundary variation factors for the considered four boundary directions are obtained in this block.

C. Pattern Class Sub-divider

This block analyzes the boundary variation factors to determine whether the training sample set is to be decomposed or not. If the sample set is to be decomposed then this block decomposes it into groups. Before actually dividing the sample set, the decision about the direction of decomposition is to be made. To decide this, it finds the direction in which the value of the variation factor is maximum. *That is*, the direction $D \in \{1, 2, 3, 4\}$ is found where $\mathcal{V}_D \geq \mathcal{V}_d$ for $d = 1, 2, 3, 4$. If $\mathcal{V}_D = 0$, then the sample set is assumed to be nearly rectangular in shape and it is not further decomposible.

Otherwise *i.e.*, if $\mathcal{V}_D > 0$ then it is assumed that the sample set is not nearly rectangular in shape and it is to be decomposed into groups. Now from the direction of decomposition (*i.e.*, D) the windows with their base and boundary values and the corresponding height threshold value ξ_h are recalled. The samples are then arranged in ascending order according to the base values.

For making a cluster of windows, the maximum boundary value is found. The starting window for the cluster is taken as the window whose boundary value is same as the maximum boundary value. The position of the starting window is noted. The following windows for the starting window are arranged one after another in the cluster until the difference between the boundary values of the current window and the starting window is less than or equal to ξ_h. Similarly the preceding windows are also put in the window cluster. The samples lying in the above window cluster are assigned to the first group of samples.

The above routine is repeated on the remaining windows until all the windows are exhausted. This leads to the formation of window clusters. Every window cluster results in a group of sample points. Thus the given training sample set is decomposed into a few groups of sample points.

The decomposition procedure is applied on these groups repeatedly until all the groups are found to be nearly rectangular in shape. It is to be observed that the higher the number of groups, the higher will be the accuracy of the shape of the pattern class.

IV. FUZZY PROCESSOR

The previous section dealt with the decomposition of training sample set into groups. The present section is devoted to obtain the sub-classes corresponding to these groups and combining them to get the estimated shape of the pattern class. This section consists of two blocks, namely the Membership Function Estimator and the Boundary Decider.

The concept of membership functions in the light of Fuzzy set theory is brought here to represent each of the sub-classes corresponding to the groups separately. Membership functions have also been used to relate every point in the whole feature space to its possibility to be in the pattern class. The Membership Function Estimator block finds the appropriate membership functions to represent each of the sub-classes corresponding to the groups. The Boundary Decider block determines each of the sub-classes separately, puts them together and obtains the multivalued shape of the pattern class.

A. Membership Function Estimator

For any pattern class, the possibility of being a member of the class is maximum for all those points lying in the central portion of the class. As the distances of the points from the points of central portion increase, the possibilities decrease and ultimately go to zero. Any function having the above property may be considered as the representative membership function for a (sub) pattern class. As the $\mathfrak{R}$ function is well established to dictate this property,[9] it is considered here as the representative membership function.

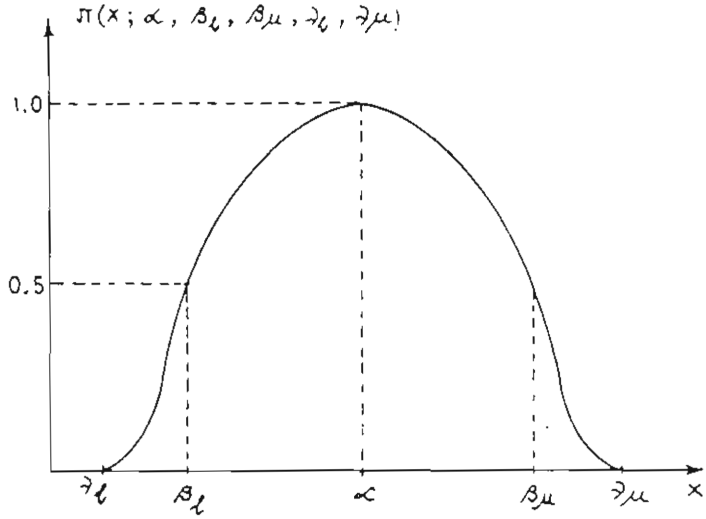


Figure 5 Pie (Π) function.

Thus, the sub-classes corresponding to the groups are characterized by different Π functions across different axes of the form $\Pi(x; \alpha_{kj}, \beta_{lkj}, \beta_{ukj}, \gamma_{lkj}, \gamma_{ukj})$ where k indicates the group number ($k = 1, 2, \ldots, no$, no denotes the number of groups); j indicates the axis number ($j = 1, 2$); α_{kj} is the peak value where the membership value is 1.0; β_{lkj} and β_{ukj} are the lower and upper most ambiguous points where the membership values are 0.5; γ_{lkj} and γ_{ukj} are the lower and upper end points beyond which the membership value are zero. The functional form of such a Π function[9] is stated below:

$$\Pi(x; \alpha, \beta_l, \beta_u, \gamma_l, \gamma_u) = \begin{cases} S(x; \gamma_l, \beta_l, \alpha) & \text{for} \quad x \le \alpha \\ 1 - S(x; \alpha, \beta_u, \gamma_u) & \text{for} \quad x \ge \alpha \end{cases} \tag{7}$$

where

$$S(x; a, b, c) = \begin{cases} 0 & \text{for} \quad x \le a \\ \frac{1}{2}\left(\frac{x - a}{b - a}\right)^2 & \text{for} \quad a < x \le b \\ 1 - \frac{1}{2}\left(\frac{x - c}{b - c}\right)^2 & \text{for} \quad b < x \le c \\ 1 & \text{for} \quad x > c. \end{cases} \tag{8}$$

The structure of such a Π function is shown in Fig. 5. Given a particular group of samples and a particular feature, the domain of the Π function is $[\gamma_l, \gamma_u]$. It is assumed that the extended portions are $[\gamma_l, \beta_l]$ and $[\beta_u, \gamma_u]$ and the highlighted portion by the sample set is $[\beta_l, \beta_u]$. These extended portions ultimately provide the multivalued shape.

Determination of membership functions To determine the membership functions (which are taken as $\Re$ functions), the parameters of them corresponding to various sample groups are to be evaluated. Here each of the sample groups is considered separately. Let max_{kj} and min_{kj} be the maximum and minimum of the training sample set respectively corresponding to *jth* feature ($j = 1, 2$) and *kth* sample group ($k = 1, 2, \ldots, no$). Then the parameters of the $\Re$ function corresponding to *jth* feature and *kth* sub-class (*i.e.*, *kth* sample group) are assigned as follows:

$$\alpha_{kj} = (max_{kj} + min_{kj})/2;$$

$$\beta_{lkj} = min_{kj}; \qquad \beta_{ukj} = max_{kj};$$

$$\gamma_{lkj} = min_{kj} - ext_j; \qquad \gamma_{ukj} = max_{kj} + ext_j;$$

$$j = 1, 2; \tag{9}$$

where ext_j is the extension factor for the *jth* feature [Eq. 4].

B. Boundary Decider

In the previous block, membership functions corresponding to each sample group along each feature axis are calculated. Using these functions, each of the sub-classes corresponding to the groups is estimated and they are finally combined to get the estimated shape of the pattern class. All planar points in the feature space are labelled with their degrees of possibilities to be in the class. To show the shape of a pattern class in the plane, the feature space is divided into small rectangles and these are referred as the pixels. The sizes of all the pixels are same and these are made as small as possible such that each pixel can be distinguished from others in the output. Thus all pixels are labelled in terms of their possibility values to be in the pattern class. The zero possibility value pixels are taken to be outside the class. The method to obtain these possibility values is described below.

Procedure Let (x_1, x_2) be a typical feature value of such a pixel. Suppose m_{kj} denotes the membership value of the pixel corresponding to *kth* ($k = 1, 2, \ldots, no$) sub-class (i.e., *kth* sample group) and *jth* ($j = 1, 2$) feature. m_{kj} is calculated from the corresponding $\Re$ function as

$$m_{kj} = \Re(x_j; \alpha_{kj}, \beta_{lkj}, \beta_{ukj}, \gamma_{lkj}, \gamma_{ukj})$$

$$j = 1, 2; \qquad k = 1, 2, \ldots, no. \tag{10}$$

The combined membership of the pixel corresponding to *kth* sub-class (i.e., *kth* sample group), say μ_k, is defined as the geometric mean of m_{k1} and m_{k2}. *That is*,

$$\mu_k = (m_{k1} * m_{k2})^{1/2}, \qquad k = 1, 2, \ldots, no \tag{11}$$

Now, the possibility, say θ, of the pixel to be in the estimated pattern class is defined as the maximum of the membership values of that pixel for the sub-classes. *That is*,

$$\theta = \underset{k=1,2,\ldots,no}{Maximum}\{\mu_k\} \tag{12}$$

Let τ be the number of sub-classes for which the combined membership values (*i.e.*, μ_k's $k = 1, 2, \ldots, no$) are non-zero. To incorporate the effect of the neighbouring sub-classes with non-zero membership values in the estimated pattern class, the value of θ is increased to $\theta^{1/\tau}$ for $\tau > 1$. *That is*, when the membership values of the pixel (μ_k's) are non-zero for two or more sub-classes, then it indicates that the said pixel lies in those sub-classes, which in turn increases the possibility for the pixel to be in the finally obtained pattern class. ■

A method has been described above to find the possibility value (θ) of a pixel to be in the pattern class. Note that $0 \le \theta \le 1$. If the value of θ is zero, then the pixel is considered to lie outside the pattern class. Otherwise the pixel belongs to the pattern class with the possibility θ.

To obtain the complete shape of the pattern class in the feature space, the above routine is carried out for all the pixels in the feature domain. Thus every pixel is labelled with its possibility value to be in the pattern class. Thus the multivalued shape of a pattern class is obtained.

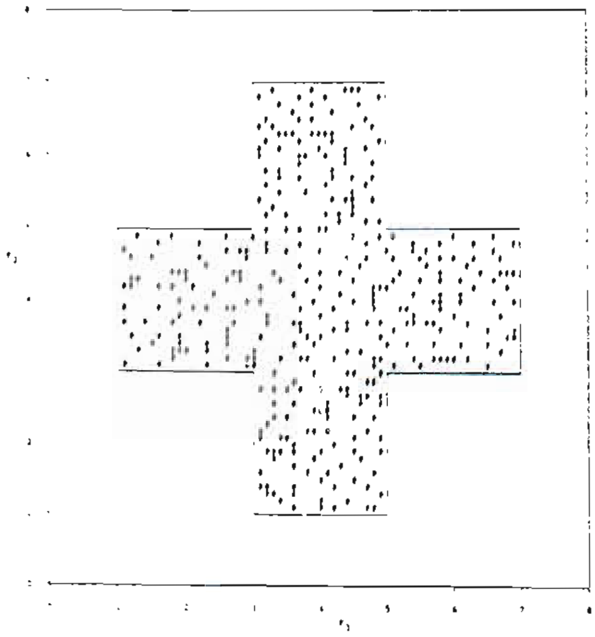
It is to be mentioned here that the proposed procedure is intuitive to a great extent. In the next section, the implementation and the usefulness of the suggested procedure are discussed.

V. IMPLEMENTATION AND RESULTS

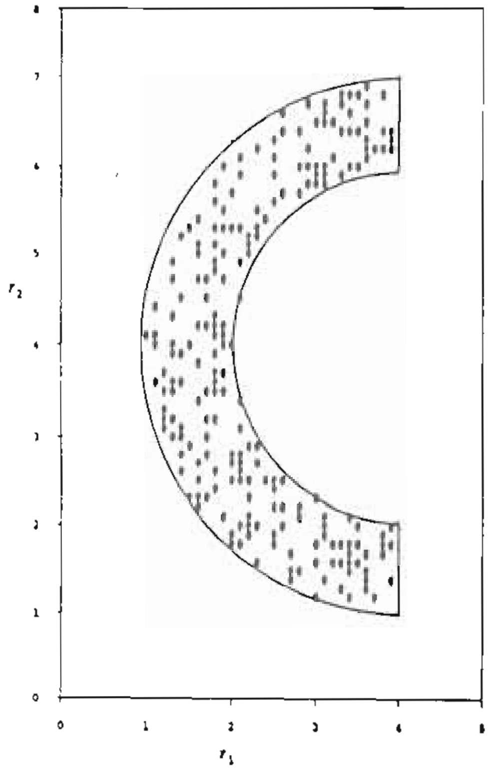
Various types of pattern classes are considered in this section to find the utility of the proposed procedure. The obtained shapes are found to be quite satisfactory in all the cases. Figures 6(a)–(d) show four typical pattern classes with 500 samples each. Note that the pattern class of Fig. 6(c) is originally (nearly) rectangular in shape, while others are not rectangular at all. The class shown in Fig. 6(d) has a hole whereas others do not have any hole. Training samples of size 50 are chosen randomly from each of the above four classes. The multivalued shapes of the above classes are shown in figures 7(a)–(d) with $\delta_N = 0.15$.

Initially in all the cases, the feature spaces are decomposed into small squares (which are called as pixels) and the squares are labelled in terms of their possibility values (θ) to be in the obtained pattern class. Recall that $0 \le \theta \le 1$. For the sake of presentation, the pixels are grouped into four categories as $0.5 \le \theta \le 1.0$, $0.25 \le \theta < 0.5$, $0 < \theta < 0.25$ and $\theta = 0$ and they are represented by the characters '@', '+', '?' and the blank character ' ' respectively in the figures 7(a)–(d). Here the pixels with blank character denote that the pixels are completely outside the pattern class. To obtain the *crisp* version of an output pattern class, the pixels with characters '+' and '?' along with the pixels with blank character may be considered to be outside the class. Observe that the algorithm has not decomposed the training sample set of Fig. 6(c) because it is originally nearly rectangular in shape.

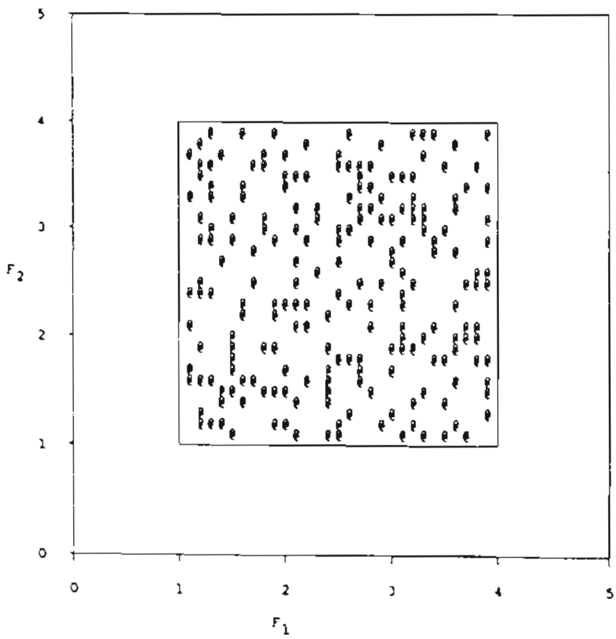
Note that the class shown in Fig. 6(d) has a hole. The algorithm initially decomposed the training sample set into two groups to detect the hole and finally the sample



(a)

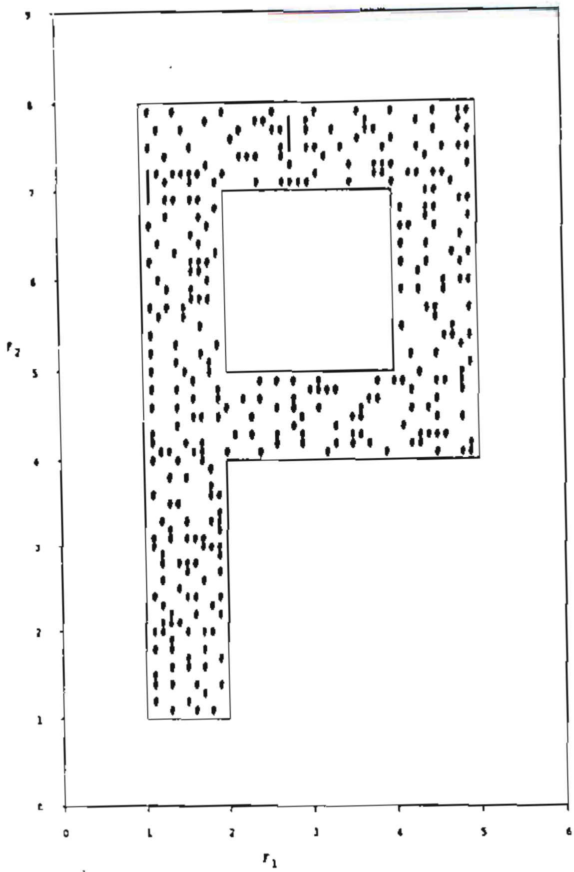


(b)



(c)

Figure 6(a)–(d) Four typical pattern classes.



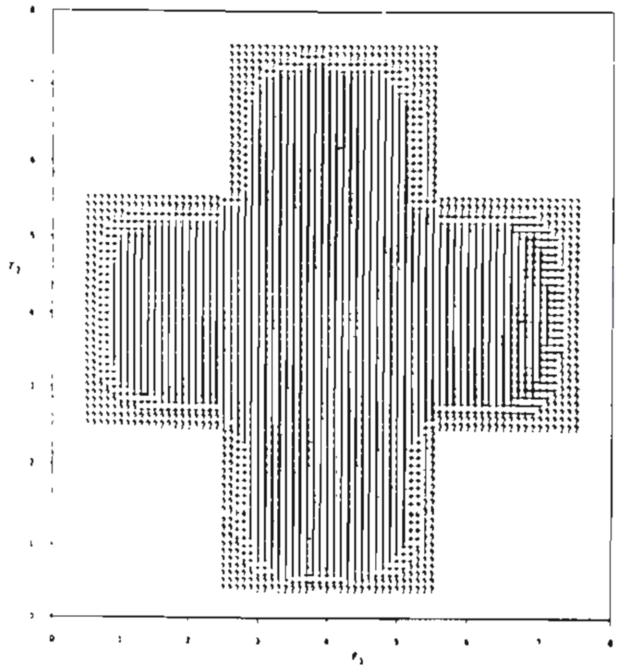
(d)

Figure 6 Continued

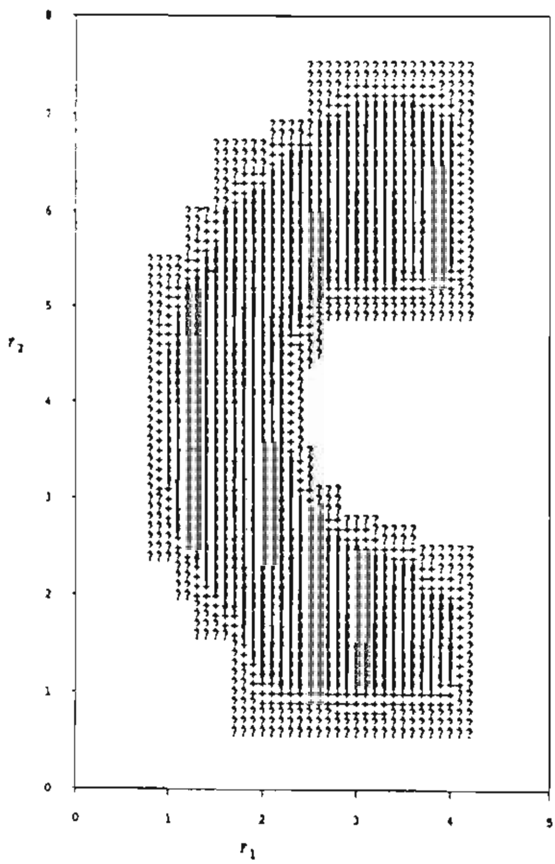
set has been decomposed into seven sample groups. The proposed methodology is explained below for a specific pattern class [Fig. 6(a)].

Fig. 7(a1) shows the training sample set with 50 samples corresponding to the class shown in Fig. 6(a). The algorithm first searches for the holes, but it could not find them. It then considered the boundaries in four perpendicular directions and found that the training sample set can be decomposed in all the four directions. The boundary variation is found maximum in the coded direction 1. Accordingly, the training sample set is decomposed into three groups with 12, 28 and 10 samples respectively [Fig. 7(a2)]. Here the sample points of different groups are shown by the characters *A*, *B* and *C*. Each of these groups are then analyzed and it could not find enough boundary variation in any of the considered four directions for which these groups can be decomposed again. As a result, it obtained three groups [Fig. 7(a2)] which are of nearly rectangular shape. The shapes of the three sub-classes are determined based on the sample points of these three groups. Finally these shapes are combined to obtain the multivalued shape [Fig. 7(a)] corresponding to the pattern class of Fig. 6(a).

To examine the practical applicability, the algorithm has been implemented on a set of Indian Telugu Vowel sounds in a consonant-vowel-consonant context uttered by three speakers in the age group 30 to 35 years. Fig. 8(a) shows the typical feature space of six vowel classes δ, a, i, u, e and o with 72, 89, 172, 151, 207 and 180 sample points respectively corresponding to the features F_1 and F_2. Here F_1 and F_2

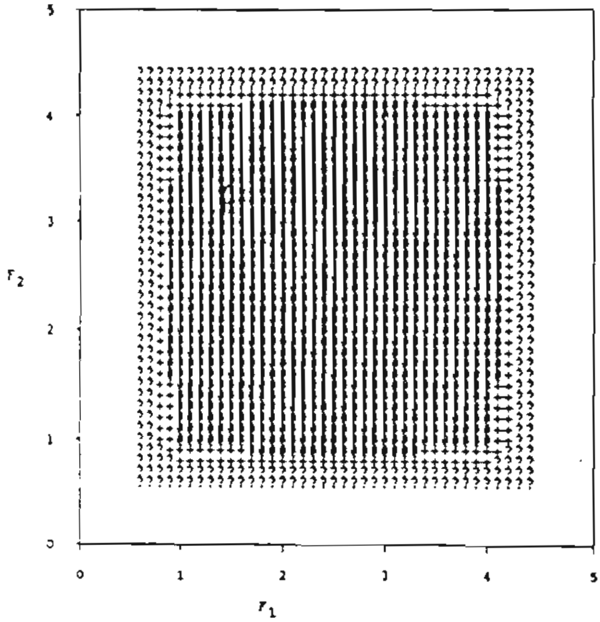


(a)

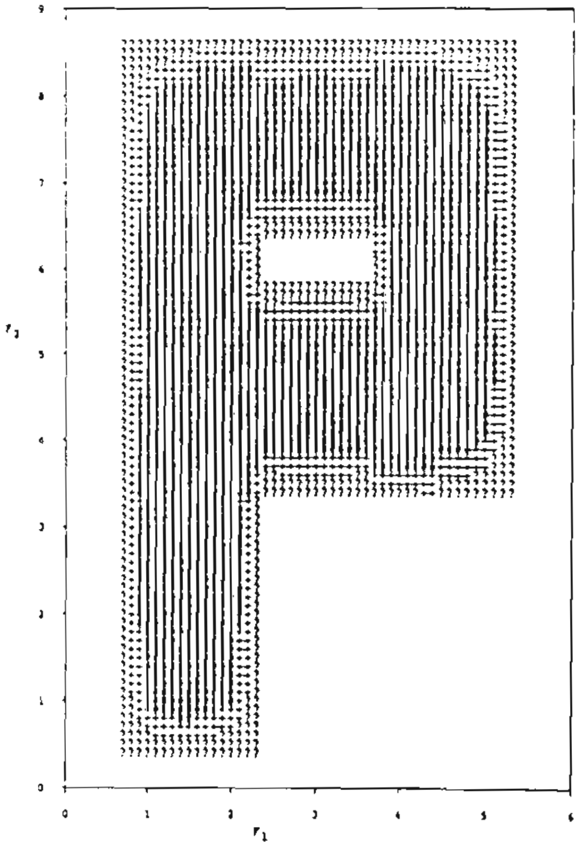


(b)

Figure 7(a)–(d) Estimated classes corresponding to the pattern classes of figures 6(a)–(d).



(c)



(d)

Figure 7 Continued

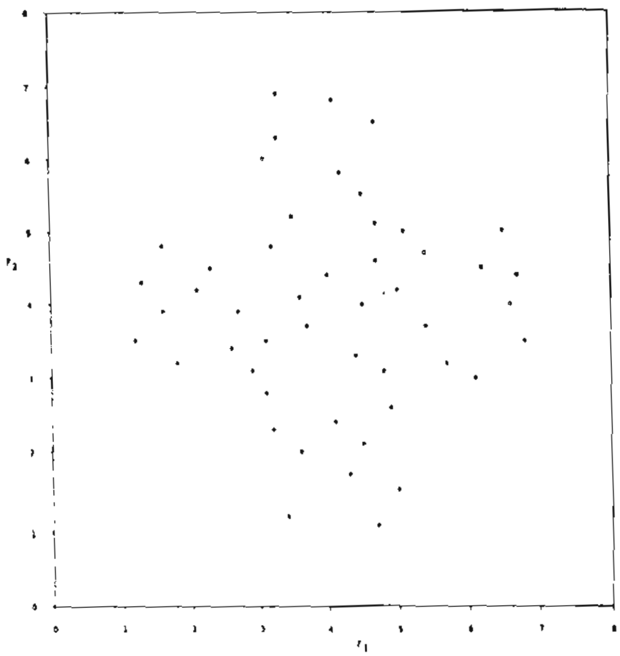


Figure 7(a1) A training set corresponding to the class of Fig. 6(a).

denote the first and second formant frequencies which are obtained through spectrum analysis of the speech data. The classes are seen to be overlapping and their boundaries are ill-defined (fuzzy).

The proposed algorithm was applied on each of the six vowel classes separately where the total available data are assumed as the training samples. *That is*, for the classes δ, a, i, u, e and o, the number of training samples are 72, 89, 172, 151, 207 and 180 respectively and correspondingly the accuracy factors are considered as 0.12, 0.12, 0.10, 0.10, 0.08 and 0.10 respectively. Fig. 8(b) shows the obtained

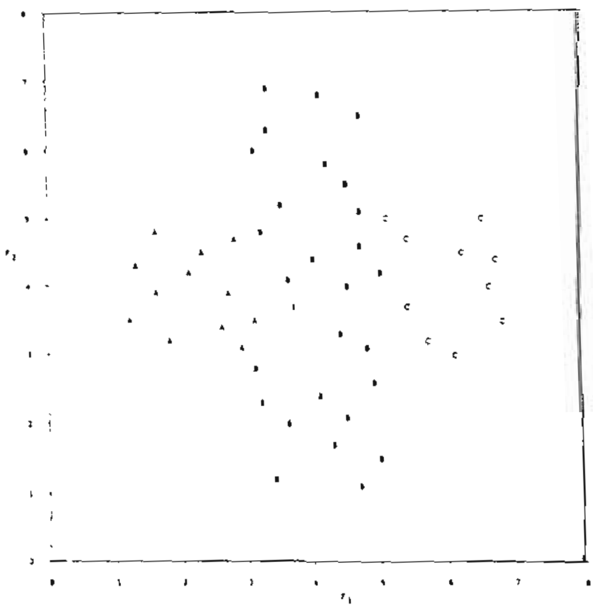


Figure 7(a2) Three decomposed sample groups of training sample set shown in Fig. 7(a1).

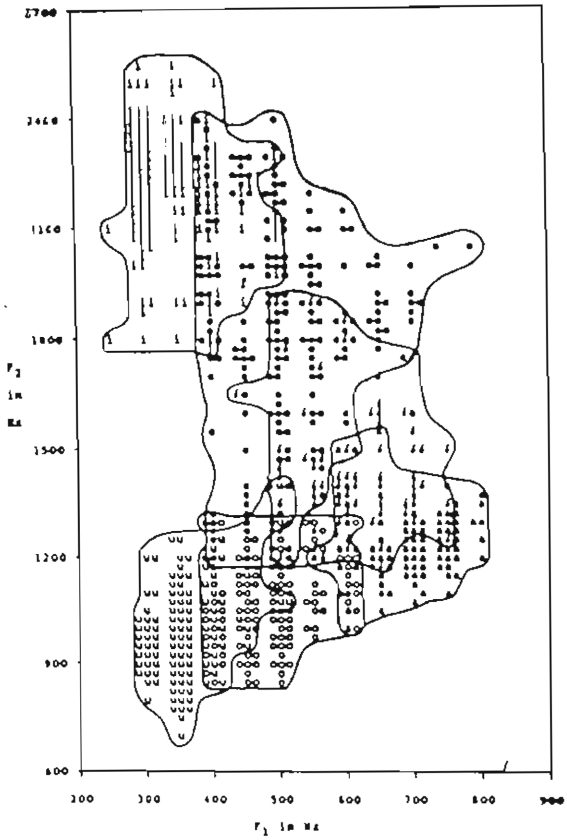


Figure 8(a) Six Telugu vowel sets.

multivalued shapes corresponding to the vowel classes *a* and *u*; Fig. 8(c) shows the shape corresponding to the class *e*; Fig. 8(d) shows the estimated shapes corresponding to the classes *i* and *o* and Fig. 8(e) shows the shape corresponding to the vowel class δ. In figures 8(b)–(e), the pixels with possibility values (θ) $\geq$ 0.5 are represented by the corresponding vowel characters; the pixels with θ satisfying $0 < \theta < 0.5$ are represented by '.' and the pixels with $\theta = 0$ are shown by blank character. It is to be observed that all the obtained vowel classes are good representations of their original classes.

VI. CONVERGENCE

For any shape determining approach based on training samples, the performance, in general, should improve with the increase in the size (number) of the training sample set. It will be shown in this section that the proposed class determining procedure also has this property. As the sample size (N) increases, the value of the accuracy factor decreases and hence accordingly the accuracy of the estimated shape increases.

An artificially generated data set [Fig. 9(a)] has been considered in this section to demonstrate the convergence property of the proposed algorithm. The pattern class is a disc of radius 2 with centre at (3, 3). Six different sets of data are chosen randomly from it with sizes 50, 100, 150, 200, 250 and 300. The algorithm was applied assuming these six data sets as sampled points with accuracy factors (δ_N) 0.16, 0.13, 0.10, 0.08, 0.07 and 0.06 respectively. The corresponding estimated

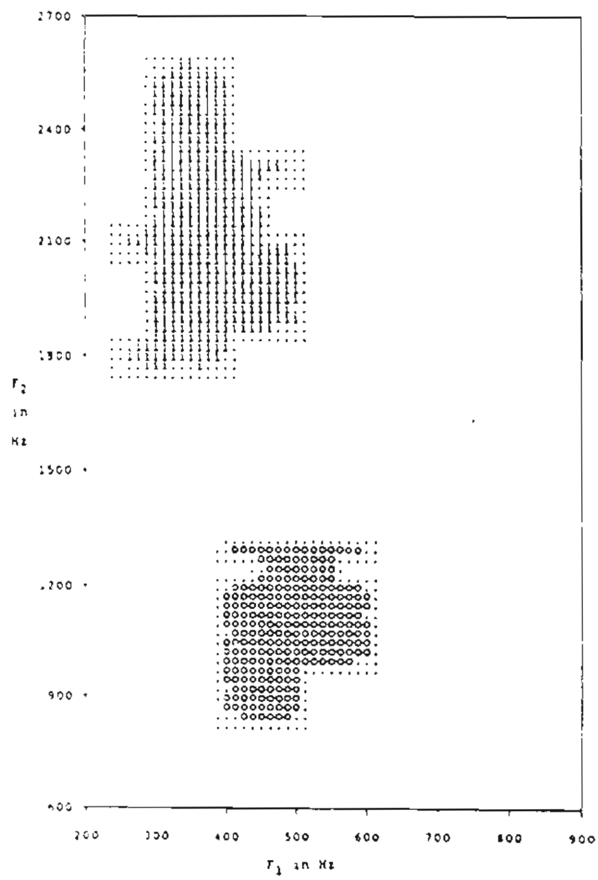


Figure 8(b) Estimated classes corresponding to the vowels a & u.

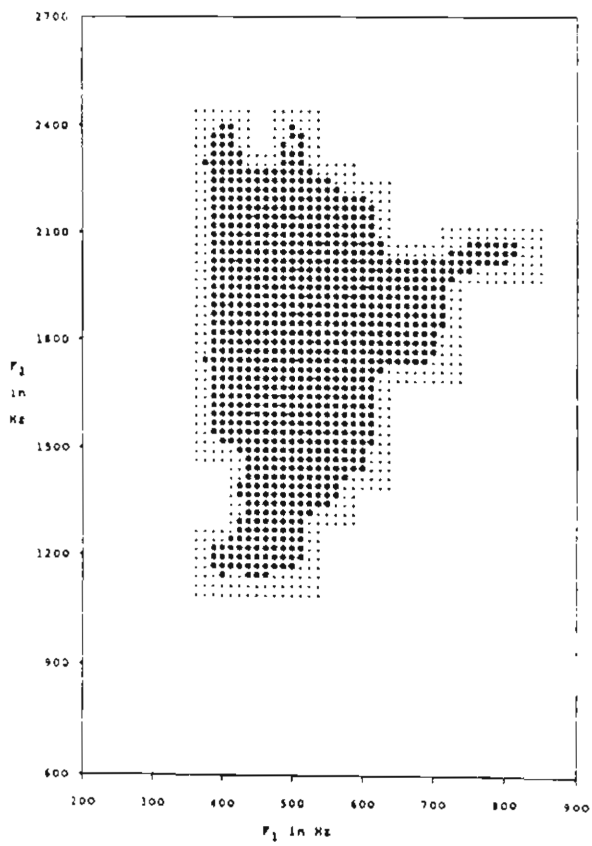


Figure 8(c) Estimated class corresponding to the vowel e.

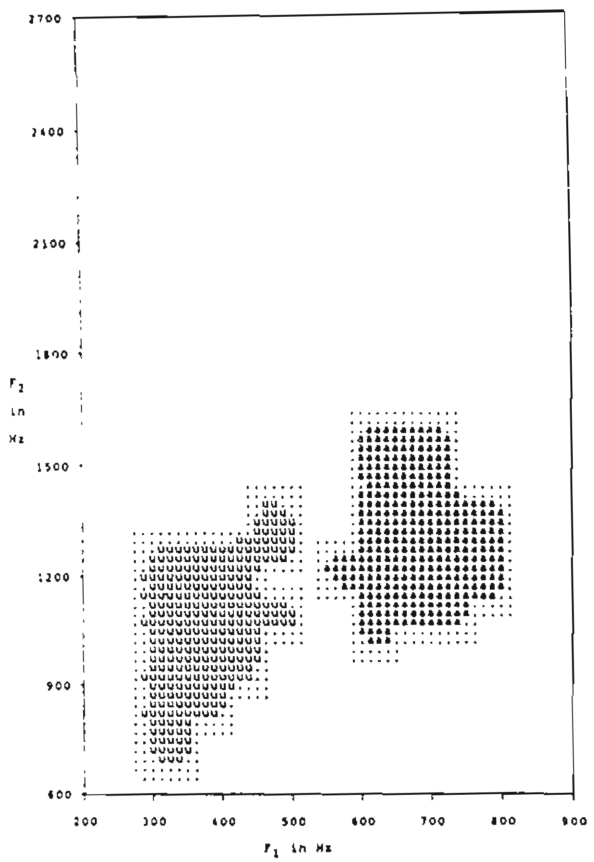


Figure 8(d) Estimated classes corresponding to the vowels i & o.

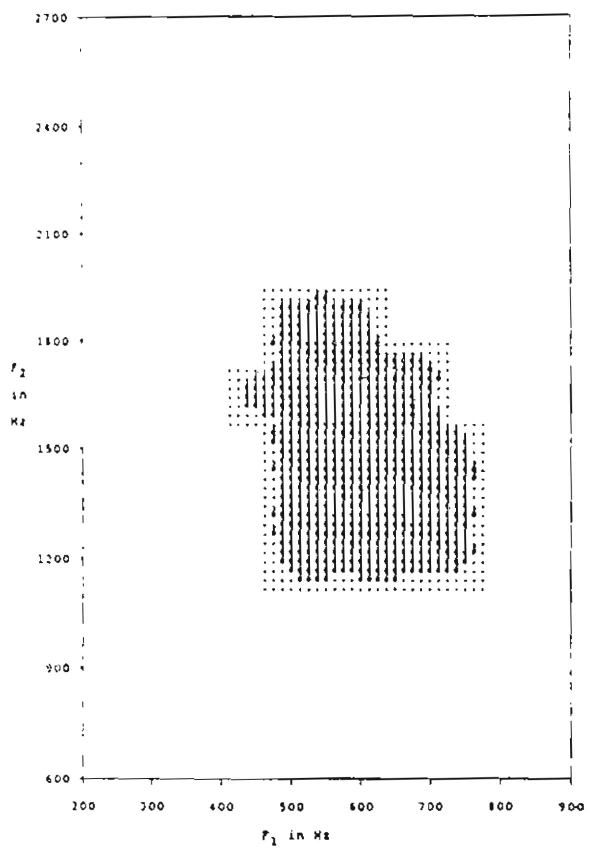


Figure 8(e) Estimated class corresponding to the vowel δ.

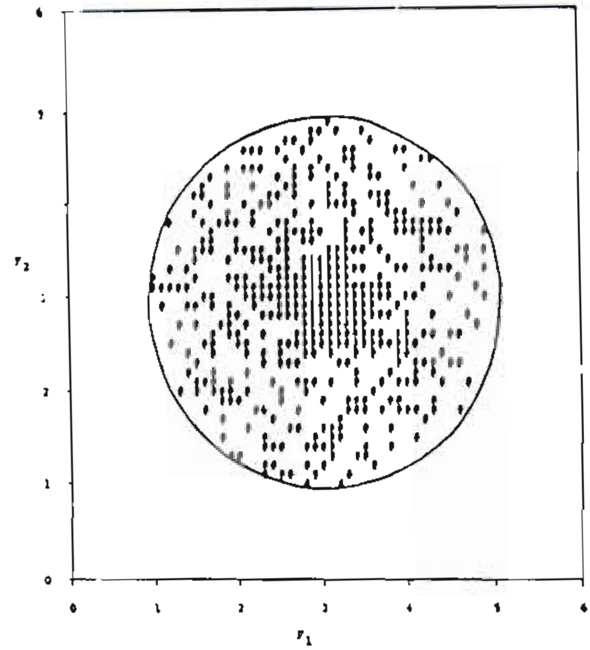
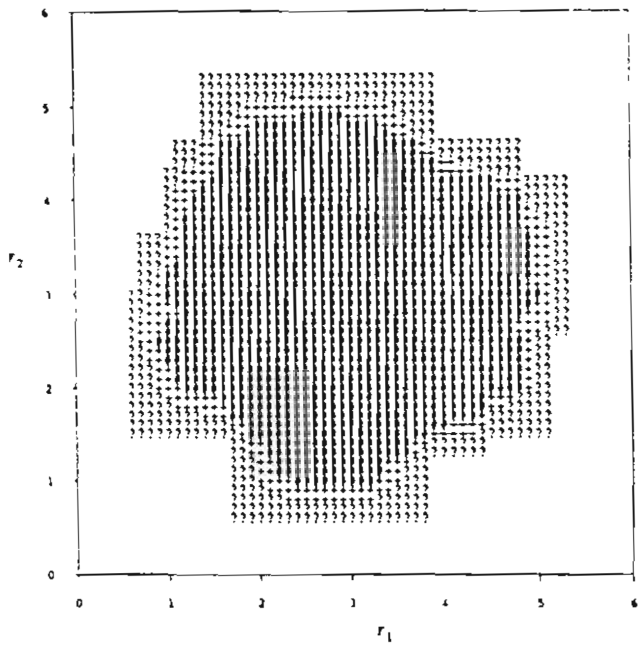


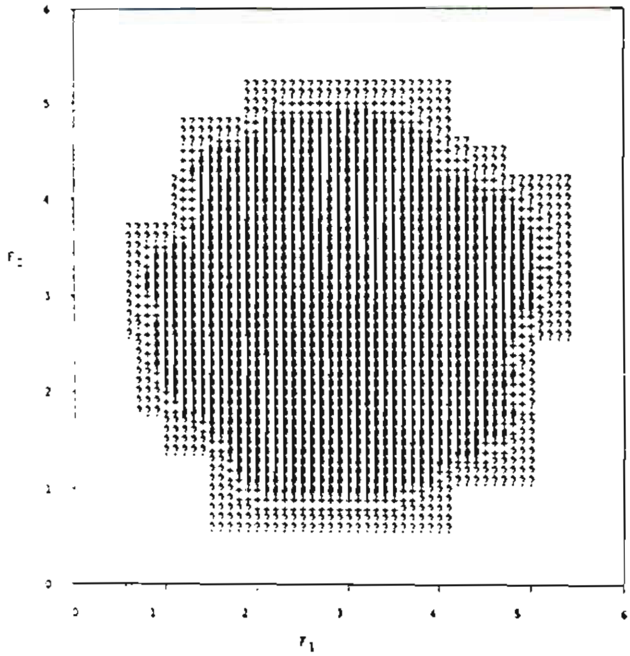
Figure 9(a) A circular pattern class.



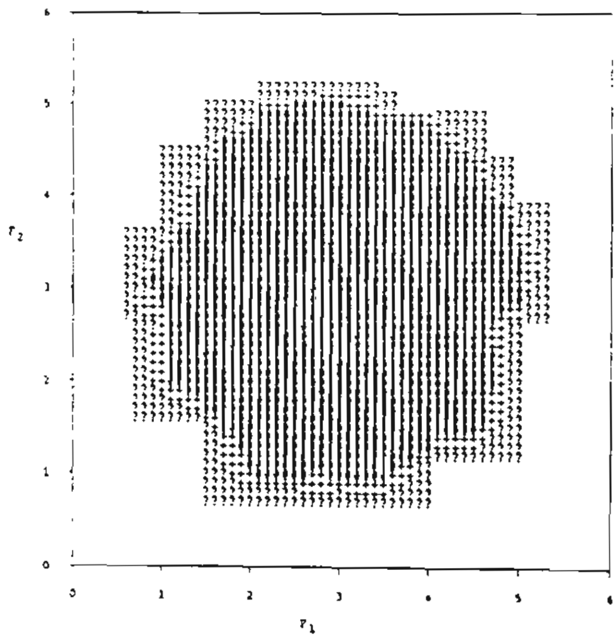
(b)

Figure 9(b)–(g) Estimated classes based on 50, 100, 150, 200, 250 and 300 training samples respectively corresponding to the pattern class of Fig. 9(a).

shapes are shown in figures 9(b)–(g). It can be seen from these results that the obtained shapes of the pattern classes are gradually converging to the shape of the original pattern class, as the sample size (N) increases. This shows the efficiency of the algorithm.

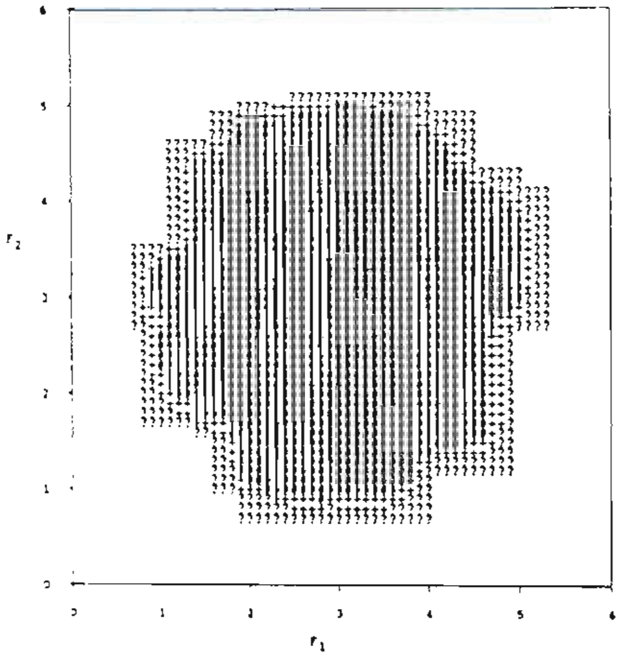


(c)

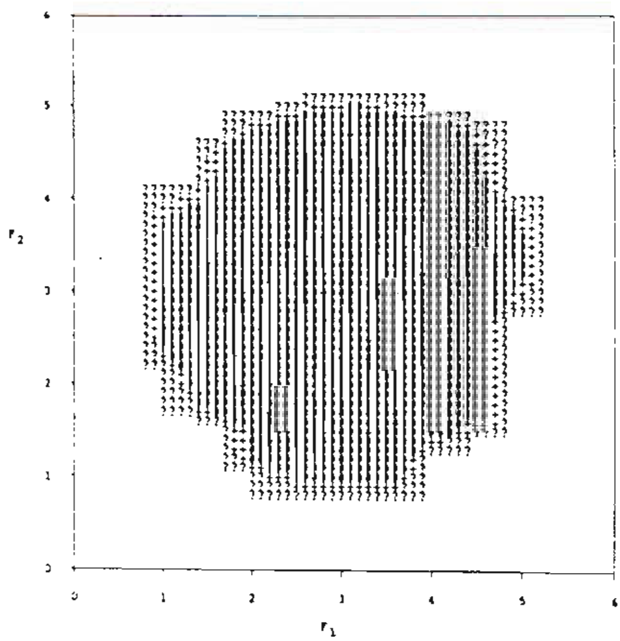


(d)

Figure 9 Continued

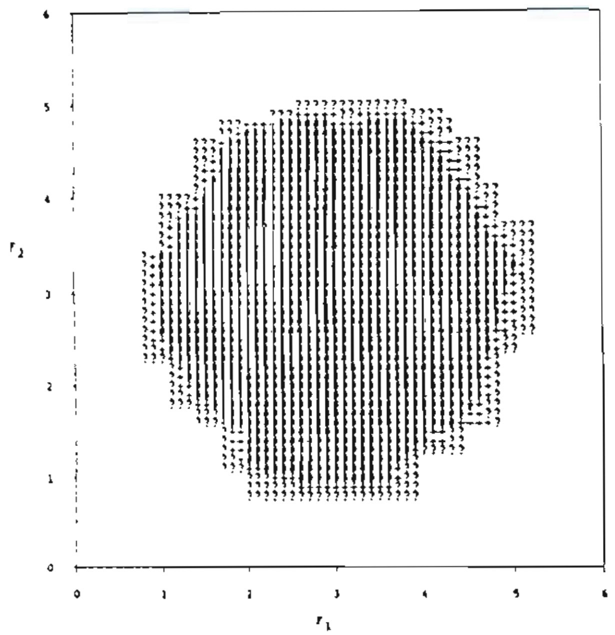


(e)



(f)

Figure 9 Continued



(g)

Figure 9 Continued

Criteria for Goodness of Fit

The convergence of the suggested algorithm has been visually demonstrated above. In order to show the same analytically, a metric between two sets is to be defined and the value of the metric should tend towards zero as $N \to \infty$. In this paper two metrics are used for showing the convergence property. One of them is a standard metric between two sets namely the Hausdorff metric. The other one is a new metric that has been defined. It has been shown experimentally that the values of both tend towards zero as $N \to \infty$.

A. Hausdorff Metric[10]

Normally, to find the similarity or dissimilarity, a distance measure is often used. Hausdorff metric has been used here for this purpose and its definition is given below.

DEFINITION 3[10] Let $\mathbb{A}$ and $\mathbb{B}$ be two closed sets in $\mathbb{R}^2$. Then the distance between $\mathbb{A}$ and $\mathbb{B}$, denoted by $Dist(\mathbb{A}, \mathbb{B})$, is defined as

$$Dist(\mathbb{A}, \mathbb{B}) = max\{\sup_{x \in \mathbb{A}} \delta(x, \mathbb{B}), \sup_{y \in \mathbb{B}} \delta(y, \mathbb{A})\} \tag{13}$$

where

$$\delta(x, \mathbb{B}) = \inf_{y \in \mathbb{B}} \|x - y\|,$$

$$\delta(y, \mathbb{A}) = \inf_{x \in \mathbb{A}} \|x - y\| \quad \text{and}$$

the sets $\mathbb{A}$ and $\mathbb{B}$ are non-empty. Here *max*, *sup* and *inf* denote maximum, supremum and infimum respectively. ■

If the original sets are assumed to be finite, the *sup* and *inf* can be replaced by *max* and *min* (minimum) respectively *i.e.*,

$$Dist(\mathbb{A}, \mathbb{B}) = max\{\max_{x \in \mathbb{A}} \delta(x, \mathbb{B}), \max_{y \in \mathbb{B}} \delta(y, \mathbb{A})\} \quad (14)$$

where

$$\delta(x, \mathbb{B}) = \min_{y \in \mathbb{B}} \|x - y\| \quad \text{and}$$

$$\delta(y, \mathbb{A}) = \min_{x \in \mathbb{A}} \|x - y\|.$$

This distance measure $Dist(\mathbb{A}, \mathbb{B})$ is considered here as one of the criteria for goodness of fit, where $\mathbb{A}$ is considered as the boundary of the estimated set or class and $\mathbb{B}$ is considered as the boundary of the original class. This distance measure is applied on the six estimated sets or classes [figures 9(b)–(g)] with the original set [Fig. 9(a)] in the following way.

The boundary of the disc [Fig. 9(a)] is approximated by 180 equally spaced points. This set of 180 points is considered here as the set $\mathbb{B}$. Note that the estimated classes are multivalued. Hence, in order to apply this measure, three levels of estimated boundaries based on the possibility values (θ), namely $\theta \geq 0.5$, $\theta \geq 0.25$ and $\theta > 0$, are considered. The values of the *Dist* measure are shown in Fig. 10(a). These results confirm the convergence property of the proposed algorithm.

B. A New Similarity Measure

Note that the Hausdorff metric reflects the overall similarity between two closed sets. In order to incorporate the similarity between each element of one set with the other set, a new measure has been defined below.

DEFINITION 4 Let $\mathbb{A}$ and $\mathbb{B}$ be 2 finite sets with N_A and N_B elements respectively. Then a similarity measure between $\mathbb{A}$ and $\mathbb{B}$, denoted by $Sim(\mathbb{A}, \mathbb{B})$, is defined as

$$Sim(\mathbb{A}, \mathbb{B}) = \frac{1}{N_A}\sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B}\sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) + Dist(\mathbb{A}, \mathbb{B}) \quad (15)$$

Here the first term denotes the average similarity of the elements of $\mathbb{A}$ to $\mathbb{B}$, the second term denotes the average similarity of the elements of $\mathbb{B}$ to $\mathbb{A}$ and the last term $Dist(\mathbb{A}, \mathbb{B})$ denotes the overall similarity between $\mathbb{A}$ and $\mathbb{B}$. This is a metric and the proof is provided in the appendix. ■

The metric *Sim* has also been applied between each of the six estimated sets (with three levels of boundaries as $\theta \geq 0.5$, $\theta \geq 0.25$ and $\theta > 0$) and the original set. The values of the *Sim* measure are shown in Fig. 10(b). These results again confirm the convergence property mathematically of the proposed algorithm.

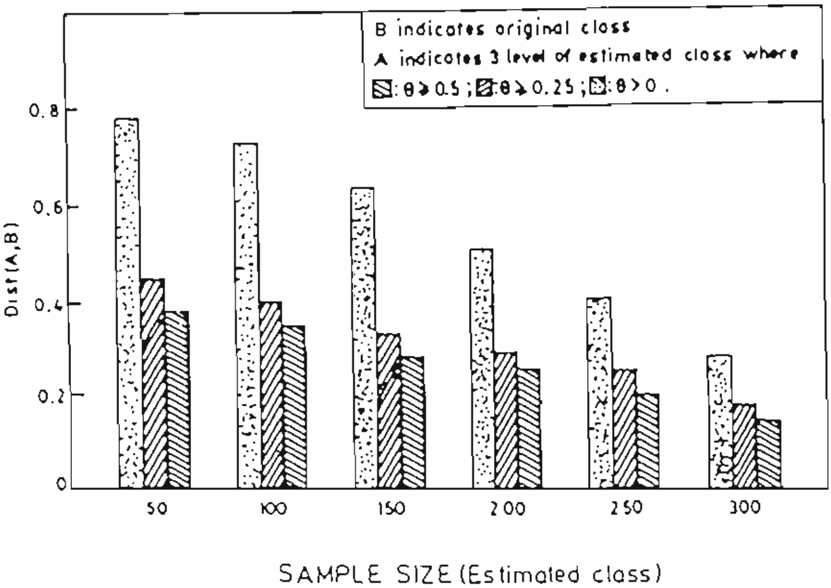


Figure 10(a) The values of *Dist* measure between the estimated classes of figures 9(b)–(g) and the pattern class of Fig. 9(a).

VII. CONCLUSION

In this paper we have proposed an algorithm to determine the shape of a pattern class in the plane from its training samples. A few basic concepts like accuracy factor, extension factors *etc.* are introduced for the said purpose. We have also discussed about some of its applications along with the convergence of the algorithm. The multivalued pattern classes can very easily be converted to the usual *crisp* pattern classes by assuming that the points with $\theta \geq 0.5$ are within the classes and the points with $\theta < 0.5$ are outside the pattern classes. The convergence of the estimated set

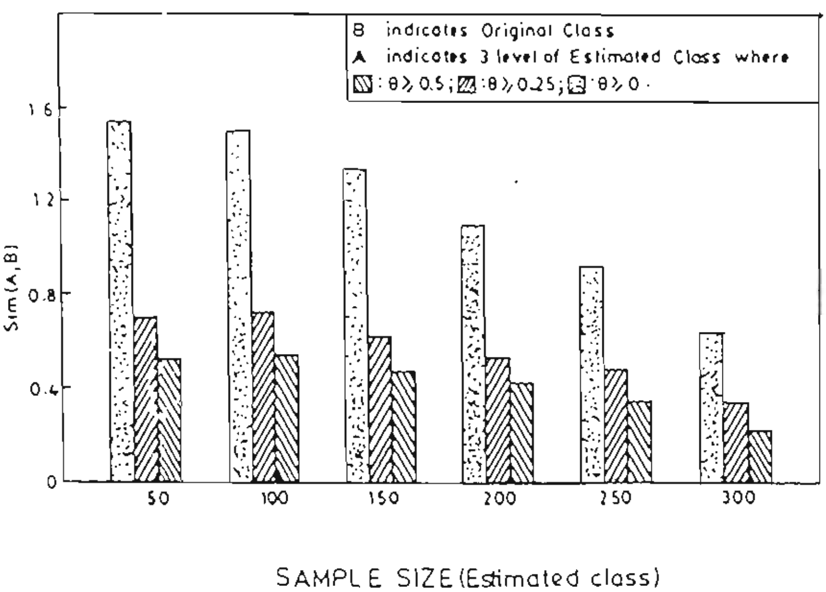


Figure 10(b) The values of *Sim* measure between the estimated classes of figures 9(b)–(g) and the pattern class of Fig. 9(a).

to the original set has been verified successfully using Hausdorff metric as well as the new similarity metric defined between sets. This new metric may be used to find the similarity or dissimilarity between any two finite sets.

One of the major advantages of the procedure is that any pattern class can be considered as the union of nearly rectangular shaped sub-classes. This result was found to be very useful in our proposed recognition system.[7] Also this can reduce the memory space to store a pattern class or set. Another advantage of the method is that every heuristic value, except δ_N, is determined automatically from the training sample set. δ_N also is not chosen arbitrarily. The guideline for the selection of δ_N is given in Eq. (1).

Finally, the algorithm can be extended to the pattern classes in three or more dimensions. In such cases, the boundaries are to be considered in six or more directions.

ACKNOWLEDGEMENT

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APPENDIX

A. Accuracy Factor

Selection of accuracy factor (δ_N) has been dealt with here. An existing definition and a theorem have been stated below in this regard.

DEFINITION A1[11] Let $X_1, X_2, \ldots, X_N$ be independent and identically distributed (i.i.d.) random vectors from a distribution $\mathbb{P}_\alpha$ which supports a set α. Let α_N^* be an estimated set on the basis of $X_1, X_2, \ldots, X_N$. Then α_N^* is said to be consistent estimate of α if

$$E_\alpha[\mu(\alpha_N^* \Delta \alpha)] \rightarrow 0 \quad \text{as} \quad N \rightarrow \infty$$

where E represents expectation, μ is a σ-finite measure and Δ represents symmetric difference. ■

THEOREM A1[11] Let $\xi_N \rightarrow 0$ and $N\xi_N^2 \rightarrow \infty$ and $\mathbb{P}_\alpha$ be the uniform distribution over α where $\alpha \in \mathcal{A}$. Let $\alpha_N^* = \cup_{i=1}^N \{x : \|x_i - x\| \leq \xi_N\}$. Then $E_\alpha[\lambda(\alpha_N^* \Delta \alpha)] \rightarrow 0$ as $N \rightarrow \infty$ where λ is Lebesgue measure in $\mathbb{R}^2$. ■

COROLLARY A1 Let $X_i = (X_{i1}, X_{i2})'$. Let $\alpha_N^* = \cup_{i=1}^N \{[X_{i1} - \xi_N, X_{i1} + \xi_N] \times [X_{i2} - \xi_N, X_{i2} + \xi_N]\}$ where $\xi_N \rightarrow 0$ and $N\xi_N^2 \rightarrow \infty$. Then α_N^* is a consistent estimator of α. ■

COROLLARY A2 Let $\alpha_N^* = \cup_{i=1}^N \{[X_{i1} - \xi_{1N}, X_{i1} + \xi_{1N}] \times [X_{i2} - \xi_{2N}, X_{i2} + \xi_{2N}]\}$ where $\xi_{iN} \rightarrow 0$ and $N\xi_{iN}^2 \rightarrow \infty$ for $i = 1, 2$. Then also α_N^* is consistent to α. ■

COROLLARY A3 Let $a_N \to a$ and $b_N \to b$, where $a, b > 0$. Let $\xi_{1N} = a_N\delta_N$ and $\xi_{2N} = b_N\delta_N$ where $\delta_N \to 0$ and $N\delta_N^2 \to \infty$. Then α_N^*, as defined in corollary A2, is consistent to α. ■

Notes:

1. The above theorem and corollaries basically take the union of certain neighbourhoods for every point as an estimate of the original set α.
2. In theorem A1, the distribution was assumed to be uniform. But it can be shown that for any continuous distribution on a compact path connected α, similar result holds.[1] Observe that a pattern class can always be assumed to be a path connected set. Hence corollary A3 can be used for any continuous distribution.
3. Corollary A3 has been used in the present paper where a_N's and b_N's are taken to be the ranges of the individual features and δ_N satisfies the inequality (1).

B. *Criteria for Goodness of Fit*

The new similarity measure $Sim(\mathbb{A}, \mathbb{B})$, as introduced in section VI, has been considered as one of the criteria for goodness of fit of our proposed procedure. It is shown here that Sim is a metric.

DEFINITION A2[10] Let $\mathbb{A}$ & $\mathbb{B} \subseteq \mathbb{R}^2$. Then

$$\delta(\mathbb{A}, \mathbb{B}) = \inf_{x \in \mathbb{A}, y \in \mathbb{B}} \|x - y\| \tag{A1}$$

This is referred as the lower distance between $\mathbb{A}$ and $\mathbb{B}$ and the following relations hold.

$$1)\ \delta(x, y) = \|x - y\| \quad \text{where } x, y \in \mathbb{R}^2 \tag{A2}$$

$$2)\ \delta(\bar{\mathbb{A}}, \bar{\mathbb{B}}) = \delta(\mathbb{A}, \mathbb{B}) \tag{A3}$$

$$3)\ \{\delta(x, \mathbb{A}) = 0\} \Rightarrow \{x \in \bar{\mathbb{A}}\} \tag{A4}$$

$$4)\ \delta(\mathbb{A}, \mathbb{C}) \leq \delta(\mathbb{A}, \mathbb{B}) + \delta(\mathbb{B}, \mathbb{C}) + \rho(\mathbb{B}) \tag{A5}$$

where $\rho(\mathbb{B})$ denotes the diameter of the set $\mathbb{B}$ and it is defined as the least upper bound of the distances of the elements of $\mathbb{B}$. ■

DEFINITION A3[10] Let $\mathbb{A}$ and $\mathbb{B}$ be two closed sets in $\mathbb{R}^2$. Then the distance between $\mathbb{A}$ and $\mathbb{B}$, denoted by $Dist(\mathbb{A}, \mathbb{B})$, is defined as

$$Dist(\mathbb{A}, \mathbb{B}) = max\{\sup_{x \in \mathbb{A}} \delta(x, \mathbb{B}), \sup_{y \in \mathbb{B}} \delta(y, \mathbb{A})\} \tag{A6}$$

■

If the original sets are assumed to be finite, the *sup* and *inf* of equations (A6) and (A1) can be replaced by *max* and *min* respectively *i.e.*,

$$Dist(\mathbb{A}, \mathbb{B}) = max\{\max_{x \in \mathbb{A}} \delta(x, \mathbb{B}), \max_{y \in \mathbb{B}} \delta(y, \mathbb{A})\} \tag{A7}$$

$$\text{where} \quad \delta(\mathbb{A}, \mathbb{B}) = \min_{x \in \mathbb{A}, y \in \mathbb{B}} \|x - y\| \tag{A8}$$

DEFINITION A4 Based on Definitions A2 and A3, a new similarity measure $Sim(\mathbb{A}, \mathbb{B})$ between two finite sets $\mathbb{A}$ and $\mathbb{B}$ in $\mathbb{R}^2$ was defined in section VI as

$$Sim(\mathbb{A}, \mathbb{B}) = \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) + Dist(\mathbb{A}, \mathbb{B}) \tag{A9}$$

where N_A and N_B denote the number of elements in the sets $\mathbb{A}$ and $\mathbb{B}$ respectively. ■

Proposition $Sim(\mathbb{A}, \mathbb{B})$ is a metric.

Proof Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ be three finite sets in $\mathbb{R}^2$. Clearly $Sim(\mathbb{A}, \mathbb{B}) = 0$ iff $\mathbb{A} = \mathbb{B}$. Again it is obvious from the definition that $Sim(\mathbb{A}, \mathbb{B}) = Sim(\mathbb{B}, \mathbb{A})$. It only remains to prove the triangular inequality *i.e.*,

$$Sim(\mathbb{A}, \mathbb{C}) \le Sim(\mathbb{A}, \mathbb{B}) + Sim(\mathbb{B}, \mathbb{C}) \tag{A10}$$

Without loss of generality, we can assume

$$Dist(\mathbb{A}, \mathbb{C}) = max\{\max_{x \in \mathbb{A}} \delta(x, \mathbb{C}), \max_{z \in \mathbb{C}} \delta(z, \mathbb{A})\} = \max_{x \in \mathbb{A}} \delta(x, \mathbb{C}) \tag{A11}$$

$$\text{So,} \quad Sim(\mathbb{A}, \mathbb{B}) = \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) + \max_{x \in \mathbb{A}} \delta(x, \mathbb{C}) \tag{A12}$$

If $x \in \mathbb{A}$ and $y \in \mathbb{B}$, then by (A5),

$$\delta(x, \mathbb{C}) \le \|x - y\| + \delta(\mathbb{B}, \mathbb{C}) \le \|x - y\| + Dist(\mathbb{B}, \mathbb{C})$$

whence

$$\begin{aligned} \delta(x, \mathbb{C}) &\le \inf_{y \in \mathbb{B}} \|x - y\| + Dist(\mathbb{B}, \mathbb{C}) = \delta(x, \mathbb{B}) + Dist(\mathbb{B}, \mathbb{C}) \\ \Rightarrow \max_{x \in \mathbb{A}} \delta(x, \mathbb{C}) &\le \max_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + Dist(\mathbb{B}, \mathbb{C}) \\ &\le Dist(\mathbb{A}, \mathbb{B}) + Dist(\mathbb{B}, \mathbb{C}) \end{aligned} \tag{A13}$$

By (A5) and assuming $\mathbb{A} = \{x\}$,

$$\delta(x, \mathbb{C}) \leq \delta(x, \mathbb{B}) + \delta(\mathbb{B}, \mathbb{C})$$

$$\Rightarrow \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{C}) \leq \frac{1}{N_A} \sum_{x \in \mathbb{B}} \delta(x, \mathbb{B}) + \delta(\mathbb{B}, \mathbb{C}) \tag{A14}$$

Again,

$$\delta(\mathbb{B}, \mathbb{C}) = \inf_{y \in \mathbb{B}} \delta(y, \mathbb{C}) \leq \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{C}) \tag{A15}$$

Using (A15) in (A14), we get

$$\frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{C}) \leq \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{C}) \tag{A16}$$

Similarly, one can get

$$\frac{1}{N_C} \sum_{z \in \mathbb{C}} \delta(z, \mathbb{A}) \leq \frac{1}{N_C} \sum_{z \in \mathbb{C}} \delta(z, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) \tag{A17}$$

Using (A16), (A17) and (A13) in (A12),

$$\begin{aligned}
Sim(\mathbb{A}, \mathbb{C}) \leq{} & \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{C}) && \text{[by (A16)]} \\
& + \frac{1}{N_C} \sum_{z \in \mathbb{C}} \delta(z, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) && \text{[by (A17)]} \\
& + Dist(\mathbb{A}, \mathbb{B}) + Dist(\mathbb{B}, \mathbb{C}) && \text{[by (A13)]} \\
={} & \frac{1}{N_A} \sum_{x \in \mathbb{A}} \delta(x, \mathbb{B}) + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{A}) + Dist(\mathbb{A}, \mathbb{B}) \\
& + \frac{1}{N_B} \sum_{y \in \mathbb{B}} \delta(y, \mathbb{C}) + \frac{1}{N_C} \sum_{z \in \mathbb{C}} \delta(z, \mathbb{C}) + Dist(\mathbb{B}, \mathbb{C}) \\
& \Rightarrow Sim(\mathbb{A}, \mathbb{C}) \leq Sim(\mathbb{A}, \mathbb{B}) + Sim(\mathbb{B}, \mathbb{C}).
\end{aligned}$$

Hence the defined measure *Sim* is a *metric*. ■

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