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Index of area coverage of fuzzy image subsets and object extraction

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Abstract: Some new geometrical properties, e.g., length, breadth and index of area coverage (IOAC) of a fuzzy set along with their computational aspects are introdueed. An algorithm for providing both fuzzy and nonfuzzy segmentation based on these measures is also proposed. The proposed algorithm is found to be successful even for the input images containing multiple objects or an elongated object, where the existing fuzzy eompactness based algorithm (which is valid for extraeting a single compaet object) fails. This is illustrated on various images.

Key words: Image segmentation, fuzzy geometry, index of area coverage, compactness.

1. Introduction

When the regions in an image are ill-defined (i.e. fuzzy), it is natural and also appropriate to avoid committing ourselves to a specific segmentation/thresholding or skeletonization or to a specific (hard) decision by allowing the regions to be fuzzy subsets of the image. Fuzzy geometric properties (which are the generalization of those for ordinary regions) as defined by Rosenfeld [1] seem to provide a helpful tool for such analysis [2, 3].

Let us consider the work of Pal and Rosenfeld [2] describing an algorithm for both fuzzy and nonfuzzy image segmentation using fuzzy 'compactness' measure. For crisp sets, the measure is largest ($=1/4\pi$) for a disc. For a fuzzy disc, the measure is greater than or equal to $1/4\pi$. The algorithm in [2], therefore, assumes a single compact object for its extraction (segmentation) from an input image so that minimization of compactness of its fuzzy 'bright image plane' results in its optimum (least compact) segmented version. When the input image contains a number of isolated objects or an elongated object, the above algorithm fails.

The present work introduces some new geometrical measures, namely, length, breadth and index of area coverage (IOAC) of a fuzzy set. IOAC of an image gives a measure of the fraction of the maximum area (that can be covered by length and breadth) actually covered by the image. It takes spatial fuzziness of an image into account and is, therefore, minimized for gray level thresholding.

Besides these, the way of computing the various geometrical measures in terms of the cooccurrence matrix or row/column histogram is provided here. This, in turn, makes their conceptual realization easier and computationally faster. It is to be mentioned here that these aspects did not get attention while developing the concepts by Rosenfeld [1] or even while formulating a thresholding algorithm by Pal and Rosenfeld [2].

The superiority of the proposed algorithm based on IOAC measure over that of the compactness measure [2] is demonstrated on various input images having a wide range of histograms.

2. Fuzzy geometrical properties

A fuzzy subset of a set S is a mapping μ from S

into [0, 1]. For any $p \in S$, $\mu(p)$ is called the degree of membership of p in μ. The support of μ is an ordinary set and is defined as

$$S(\mu) = \{p \mid \mu(p) > 0\}.$$

p is called a cross-over point of μ if $\mu(p) = 0.5$. A crisp (ordinary, nonfuzzy) subset of S can be regarded as a special case of a fuzzy subset in which the mapping μ is into $\{0, 1\}$.

An image X of size $M \times N$ and L levels can be expressed [2, 3] as

$$X = \{\mu_X(x, y) = \mu(x, y)/(x, y):$$
$$x = 1, 2, \ldots, M,\ y = 1, 2, \ldots, N\},$$

where $\mu_X(x, y) = \mu(x, y)/(x, y)$ denotes the grade of possessing some property by the (x, y)th pixel. In defining the geometrical properties, we replace sometimes $\mu(x, y)$ simply by μ.

Some existing properties

Some geometrical properties of μ as defined by Rosenfeld are given below [1, 2, 3].

A. *Area.* The area of a fuzzy subset μ is defined as

$$a(\mu) = \int \mu \tag{1a}$$

where the integration is taken over a region outside which $\mu = 0$.

For μ being piecewise constant (in case of a digital image X of dimension $M \times N$) the area is

$$a(\mu) = \sum \mu = \sum_x \sum_y \mu(x, y) \tag{1b}$$

with $x = 1, 2, \ldots, M$, $y = 1, 2, \ldots, N$.

B. *Perimeter.* If μ is piecewise constant, the perimeter of μ is defined as

$$p(\mu) = \sum_{i,j,k} |\mu(i) - \mu(j)| \cdot |A(i, j, k)|. \tag{2a}$$

This is just the weighted sum of the lengths of the arcs $A(i, j, k)$ along which the regions having μ values $\mu(i)$ and $\mu(j)$ meet, weighted by the absolute difference of these values. In case of an image, if we consider the pixels as the piecewise constant regions, and the common arc length for adjacent pixels as unity, then the perimeter of an image is defined by

$$p(\mu) = \sum_{i,j} |\mu(i) - \mu(j)| \tag{2b}$$

where $\mu(i)$ and $\mu(j)$ are the membership values of two adjacent pixels.

C. *Compactness.* The compactness of a fuzzy set μ having an area of $a(\mu)$ and a perimeter of $p(\mu)$ is defined as

$$\text{comp}(\mu) = \frac{a(\mu)}{\{p(\mu)\}^2}. \tag{3}$$

Physically, compactness means the fraction of maximum area (that can be encircled by the perimeter) actually occupied by the object. In the nonfuzzy case, the value of compactness is maximum for a circle and this value is $1/4\pi$. In case of a fuzzy disc, where the membership value is only dependent on its distance from the center, this compactness value is $\geqslant 1/4\pi$ [2]. Of all possible fuzzy discs compactness is, therefore, minimum for its crisp version.

D. *Height and width.* The height of a fuzzy set μ is defined as

$$h(\mu) = \int \max_x \{\mu(x, y)\}\, dy \tag{4a}$$

where the integration is taken over a region outside which $\mu(x, y) = 0$.

Similarly, the width of a fuzzy set is defined by

$$w(\mu) = \int \max_y \{\mu(x, y)\}\, dx \tag{4b}$$

with the same condition over integration as above.

For digital pictures, x and y can take only discrete values, and since $\mu = 0$ outside the bounded region, the maxes are over a finite set. In this case the expressions take the form

$$h(\mu) = \sum_y \max_x \{\mu(x, y)\} \tag{5a}$$

and

$$w(\mu) = \sum_x \max_y \{\mu(x, y)\}. \tag{5b}$$

So physically, in case of a digital picture, height is the sum of the maximum membership values of each row. Similarly, by width we mean the sum of the maximum membership values of each column.

Example 1. Let μ be of the form

$$\begin{matrix} 0.2 & 0.4 & 0.3 \\ 0.2 & 0.7 & 0.6 \\ 0.6 & 0.5 & 0.6 \end{matrix}$$

The above mentioned properties are calculated:

$$\begin{aligned} a(\mu) &= 0.2+0.4+0.3+0.2+0.7 \\ &\quad +0.6+0.6+0.5+0.6=4.1, \\ p(\mu) &= |0.2-0.4|+|0.2-0.2|+|0.4-0.3| \\ &\quad +|0.4-0.7|+|0.3-0.6|+|0.2-0.6| \\ &\quad +|0.2-0.7|+|0.7-0.6|+|0.7-0.5| \\ &\quad +|0.6-0.6|+|0.6-0.5|+|0.5-0.6| \\ &= 2.3, \end{aligned}$$

$$\mathrm{comp}(\mu)=4.1/(2.3\times 2.3)=0.775,$$

$$h(\mu)=0.4+0.7+0.6=1.7,$$

$$w(\mu)=0.6+0.7+0.6=1.9.$$

New properties

Some new geometrical properties are introduced here along with their illustrations.

E. *Length.* The leugth of a fuzzy set μ is defined as

$$l(\mu)=\max_x\left\{\int \mu(x,y)\,\mathrm{d}y\right\} \tag{6}$$

where the integration is taken over a region outside which $\mu(x,y)=0$.

In case of a digital picture where x and y can take only discrete values, the expression takes the form

$$l(\mu)=\max_x\left\{\sum_y \mu(x,y)\right\}. \tag{7}$$

Physically speaking, the length of an image fuzzy subset gives its longest expansion in the Y-direction. If μ is crisp, $\mu(x,y)=0$ or 1 say, for background and object pixels respectively; the length then denotes the maximum number of object pixels in the Y-direction.

Comparing equation (7) with (5a) it is noted that the length is different from the height in the sense that the former takes the summation of the entries in a column first and then maximizes over different rows whereas the latter maximizes the entries in a column and then sums over different rows. It is also to be noted that

$$l(\mu)/h(\mu)\leqslant 1.$$

F. *Breadth.* The breadth of a fuzzy set μ is defined as

$$b(\mu)=\max_y\left\{\int \mu(x,y)\,\mathrm{d}x\right\} \tag{8}$$

where the integration is taken over a region outside which $\mu(x,y)=0$.

In case of a digital picture, where x and y can take only discrete values the expression takes the form

$$b(\mu)=\max_y\left\{\sum_x \mu(x,y)\right\}. \tag{9}$$

The breadth of an image fuzzy subset gives its longest expansion in the X-direction. If μ is crisp, $\mu(x,y)=0$ or 1; the breadth denotes the maximum number of object pixels in the X-direction.

From equations (9) and (5b) it is noted that the difference between breadth and width is the same as that between length and height. Here also,

$$b(\mu)/w(\mu)\leqslant 1.$$

G. *Index of area coverage (IOAC).* The index of area coverage of a fuzzy set may be defined as

$$\mathrm{IOAC}(\mu)=\frac{\mathrm{area}(\mu)}{l(\mu)\cdot b(\mu)}. \tag{10}$$

In the nonfuzzy case, IOAC has a maximum value of 1 for rectangles placed along the axes of measurement. For a circle this value is $\pi r^2/(2r\cdot 2r)=\pi/4$. Physically, by IOAC of a fuzzy image we mean the fraction (which may be improper also) of the maximum area (that can be covered by the length and breadth of the image) actually covered by the image. Note the difference between the IOAC and compactness measures.

For the fuzzy subset μ in Example 1, the new properties are

$$l(\mu) = 0.4 + 0.7 + 0.5 = 1.6,$$

$$b(\mu) = 0.6 + 0.5 + 0.6 = 1.7,$$

$$\mathrm{IOAC}(\mu) = 4.1/(1.6 \times 1.7) = 1.51.$$

3. Relation between compactness and IOAC

Let us, for example, consider an upright $M \times N$ rectangle. Let $\mu = \alpha$ inside the rectangle and $\mu = 0$ outside it. Then

$$a(\mu) = MN\alpha, \qquad p(\mu) = 2(M+N)\alpha,$$

$$l(\mu) = M\alpha, \qquad b(\mu) = N\alpha$$

where a, p, l, b represent the area, perimeter, length, and breadth of an image respectively. Hence

$$\mathrm{comp}(\mu) = \frac{MN\alpha}{(2(M+N)\alpha)^2} = \frac{MN}{4(M+N)^2} \times \frac{1}{\alpha},$$

$$\mathrm{IOAC}(\mu) = \frac{MN\alpha}{M\alpha\, N\alpha} = \frac{1}{\alpha}.$$

So for rectangles,

$$\mathrm{IOAC} = \frac{4(M+N)^2}{MN} \times \text{compactness}.$$

When $M = N$ (for a square),

$$\mathrm{IOAC} = 16 \times \text{compactness}.$$

If we consider a fuzzy disc (with the similar condition on α)

$$\mathrm{comp}(\mu) = \frac{\pi r^2 \alpha}{(2\pi r\alpha)^2} = \frac{1}{4\pi} \times \frac{1}{\alpha},$$

$$\mathrm{IOAC}(\mu) = \frac{\pi r^2 \alpha}{(2r\alpha)(2r\alpha)} = \frac{\pi}{4} \times \frac{1}{\alpha}.$$

So for a circle

$$\mathrm{IOAC} = \pi^2 \times \text{compactness}.$$

4. Histogram thresholding and image segmentation by IOAC optimization

In this section we are going to describe an algorithm for gray level thresholding and object/background separation considering the IOAC measure as the objective criterion.

A. Selection of membership function

From the discussions in Section 3 we notice that for a $M \times N$ upright rectangle represented by μ

$$\mathrm{comp}(\mu) = \frac{MN\alpha}{(2(M+N)\alpha)^2} = \frac{\text{const.}}{\alpha},$$

$$\mathrm{IOAC}(\mu) = \frac{MN\alpha}{M\alpha\, N\alpha} = \frac{1}{\alpha}.$$

Now, since α can take any value in $[0, 1]$, we can infer that concerned geometrical properties are dependent on the membership values. It is further to be noted that compactness and IOAC of a fuzzy region decrease as its μ value increases and they are smallest for a crisp one. Hence the choice of the membership function is an important criterion. In the case of a digital image we would like to have higher membership values for the pixels whose possibility of belonging to the object is high.

If the object pixels have higher gray levels we can select the standard S-function proposed by Zadeh as used in [2] to extract a fuzzy 'bright image plane'. The definition of the function is as follows:

$$\begin{aligned}
\mu(x) &= S(x; a, b, c) \\
&= 0 && \text{if } x \leqslant a, \\
&= 2\left(\frac{x-a}{c-a}\right)^2 && \text{if } a \leqslant x \leqslant b, \\
&= 1 - 2\left(\frac{x-c}{c-a}\right)^2 && \text{if } b \leqslant x \leqslant c, \\
&= 1 && \text{if } x \geqslant c
\end{aligned} \tag{11}$$

with cross-over point $b = (a+c)/2$ and window size $w = c - a$.

On the other hand, when the object pixels have lower gray levels compared to the background pixels, we would select a function complementary to the S-function (i.e. we would choose a $(1-S)$-function) which we will name 'Z'-function to represent a 'dark image plane'.

It is to be mentioned here that the criteria regarding the selection of membership functions along with the window size in image processing problems have recently been reported by Murthy and Pal [4]. The criteria involve symmetry in ambiguity around the cross-over point and bound functions based on the properties of correlation [5]. Zadeh's S-function (equation (11)) satisfies the aforesaid criteria.

B. Formulation of algorithm

From the definitions of area, length, breadth and index of area coverage (IOAC) (as given in equations (1), (7), (9) and (10) respectively), it has been noticed that for crisp sets the value of index of area coverage (IOAC) is maximum for a rectangle placed along the axes of measurement. Again, of all possible fuzzy rectangles IOAC is minimum for its crisp version. For this reason, we will use minimization of IOAC as a criterion for image segmentation.

C. Criteria for threshold selection

Suppose we use an S-function for obtaining the 'bright image' $\mu(X)$ of an image X. Then for a particular cross-over point selected at say, $b=s$, the pixels having gray levels $>s$ will have membership values >0.5 and those having gray levels $<s$ will have membership values <0.5. This implies allocation of the gray levels into two regions. The term $\mathrm{IOAC}(\mu)$ then reflects the amount of ambiguity in the geometry (i.e. in spatial domain) of X. Therefore, modification of the cross-over point will result in different $\mu(X)$ planes (and hence different segmented versions), with varying amount of IOAC denoting fuzziness in the spatial domain. The $\mu(X)$ plane having minimum IOAC value can be regarded as an optimum fuzzy segmented version of X. This is optimum in the sense that for any other selection of s, the value of IOAC will be greater.

Computational steps of the proposed algorithm are similar to those based on the compactness measure [2]; the steps are summarized here for the convenience of the readers.

Algorithm

Given an L level image X $(M\times N)$ with minimum and maximum gray level values $l_{\min}$ and $l_{\max}$ respectively.

Step 1. Construct the membership plane μ (using equation (11)) as

$$\mu(m,n)=\mu(l)=S(l;a,b,c) \qquad \text{(bright image plane)}$$

or

$$\mu(m,n)=\mu(l)=1-S(l;a,b,c) \qquad \text{(dark image plane)}$$

with cross-over point b and particular window size $w=c-a$.

Step 2. Compute the area, length, breadth and IOAC of X using equations (1), (7), (9) and (10) respectively.

Step 3. Vary b between $l_{\min}$ and $l_{\max}$ and select those $\mu(m,n)$ planes for which $\mathrm{IOAC}(X)$ has local minima. Among the local minima let the global one have a cross-over point s.

The $\mu(m,n)$ plane, corresponding to the cross-over point s, can then be viewed as a fuzzy segmented version of the image X. For the purpose of nonfuzzy segmentation, one can take s as the threshold or boundary for classifying/segmenting an image into object and backgrouud. For images having multiple regions, one would have a set of such optimum $\mu(X)$ planes.

D. Implementation and results

The algorithm has been implemented on the images of Biplane and Lincoln (Figures 1a, 2a) having black object and white background. Here $l_{\min}=1$ and $l_{\max}=32$. It has also been tested on an image of handwritten characters 'Shu' (Figure 3a) having white object and black background. The corresponding histograms are shown in Figures 1b, 2b and 3b. For a black object (white background) a $(1-S)$-, i.e., Z-function and for a white object (black background) a standard S-function is used for extracting the membership planes.

Figure 1. Biplane image: (a) input, (b) histogram.

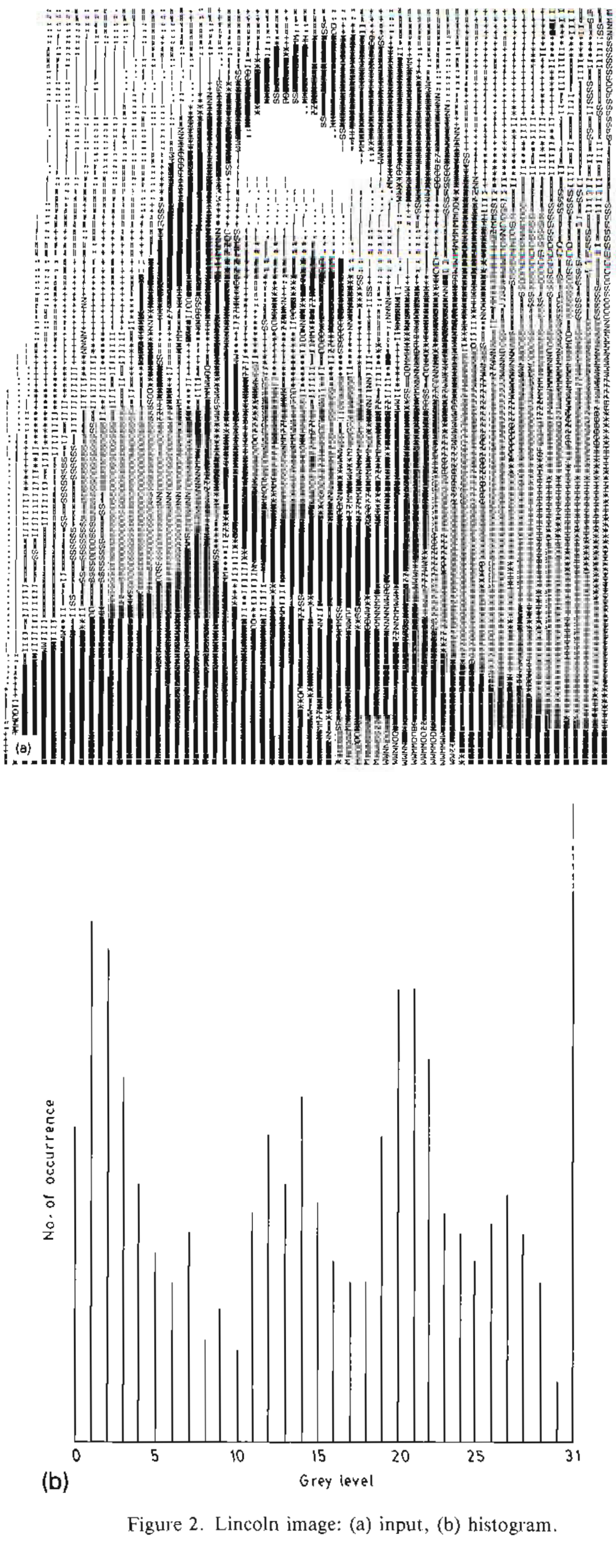

Figure 2. Lincoln image: (a) input, (b) histogram.

The different minima obtained by the proposed algorithm for different window sizes are shown in Tables 1 and 2. The thresholds obtained by compactness minimization [2] are also included here for comparison. As a typical illustration, the crisp segmented versions of the images corresponding to the thresholds obtained by both the measures (IOAC and compactness) for a fixed window size (12) are given in Figure 4 (gray values below threshold are put zero except Figure 4f which is two-tone).

The results show that the global thresholds (for any window size) obtained by the compactness measure, as expected, are very much worse for the 'Shu' image than those of the IOAC measure in extracting the object. It is also to be noted that the

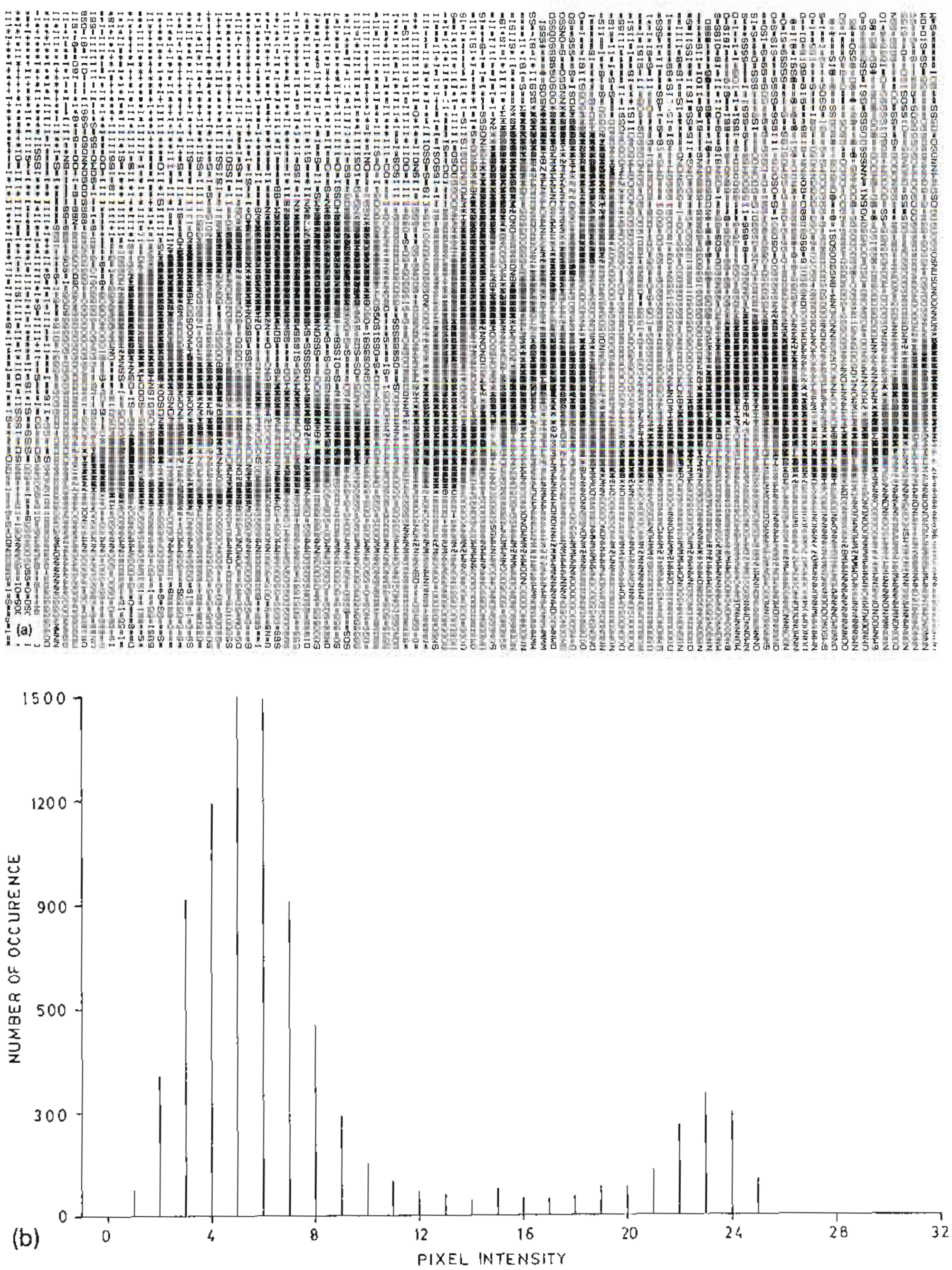

Figure 3. Shu image: (a) input, (b) histogram.

Table 1
Thresholds for the images of Lincoln and Biplane

	Lincoln		Biplane	
w	Comp	IOAC	Comp	IOAC
6	10*	11* 23	5* 14 27	11 19 24*
8	10*	11* 23	5* 26	11 19 24*
10	10*	11* 23	5* 26	12 24*
12	10*	11* 23	5* 24	13 24*
14	9*	11* 23	6* 23	13 24*
16	9*	11*	6* 21	14 24*

The thresholds with superscript * denote the global minima.

Table 2
Thresholds for Shu image

w	Comp	IOAC
8	8* 24	8 12*
10	8* 24	9 12*
12	8* 23	9 13*
14	9* 23	13*
16	10* 22	14*

The thresholds with superscript * denote the global minima.

appropriate thresholds (12 to 16) did not come out even as local minima by the compactness measure. This is because of the fact that the former measure attempts to make circular approximation of the object for its extraction. As a result, some of the background portions get treated as object; thus failing to remove background noise. For the Biplane image, neither compactness nor IOAC has been able to provide a global threshold appropriate for its extraction. However, it is interesting to note that the IOAC measure has been able to detect the appropriate thresholds (11 to 14) of the Biplane image as local minima for any window size whereas the compactness measure failed to do so. For the image of Lincoln the thresholds are more or less the same for both measures.

Furthermore, for a wide range of window sizes, the variation in global thresholds is seen to be insignificant. This corroborates the theoretical findings of Murthy and Pal [4] and establishes further the flexibility of the fuzzy set theoretic approach.

5. Computational aspects

From the proposed algorithm in Section 4 and the algorithm of Pal and Rosenfeld [2] it appears that one needs to scan an L level image L times (corresponding to L cross-over points of the membership function) for computing the parameters for detecting its threshold. The time of computation can be reduced significantly by scanning it only once for computing its cooccurrence matrix, row histogram and column histogram, and by computing $\mu(l)$, $l=1,2,\ldots,L$ every time with the membership function of a particular cross-over point.

Let $h(i)$, $i=1,2,\ldots,L$, be the number of occurrences of the level i, let $C[i,j]$, $i=1,2,\ldots,L$, $j=1,2,\ldots,L$, be the cooccurrence matrix and let $\mu(i)$, $i=1,2,\ldots,L$, be the membership vector for a fixed cross-over point of an L level image X. Then determine the area and perimeter as

$$a(X)=\sum_{i=1}^{L} h(i)\,\mu(i), \tag{12}$$

$$p(X)=\sum_{i=1}^{L}\sum_{j=1}^{L} C[i,j]\,|\mu(i)-\mu(j)|. \tag{13}$$

Let the row-histogram $R[m,l]$, $m=1,\ldots,M$, $l=1,\ldots,L$, represent the number of occurrences of the gray level l in the mth row and let the column-histogram $C[n,l]$, $n=1,\ldots,N$, $l=1,\ldots,L$, represent the number of occurrences of the gray level l in the nth column of the image.

Then calculate length and breadth as

$$l(X)=\max_{n}\sum_{l=1}^{L} C[n,l]\,\mu(l), \tag{14}$$

$$b(X)=\max_{m}\sum_{l=1}^{L} R[m,l]\,\mu(l). \tag{15}$$

It is to be mentioned here that the computational aspects of the perimeter did not get attention earlier in [1, 2].

6. Discussion and conclusions

An attempt has been made here to introduce some new measures, e.g., length, breadth and index of area coverage (IOAC) on fuzzy geometrical properties. The IOAC measure removes the drawbacks of the existing 'compactness' measure in object/background classification problems.

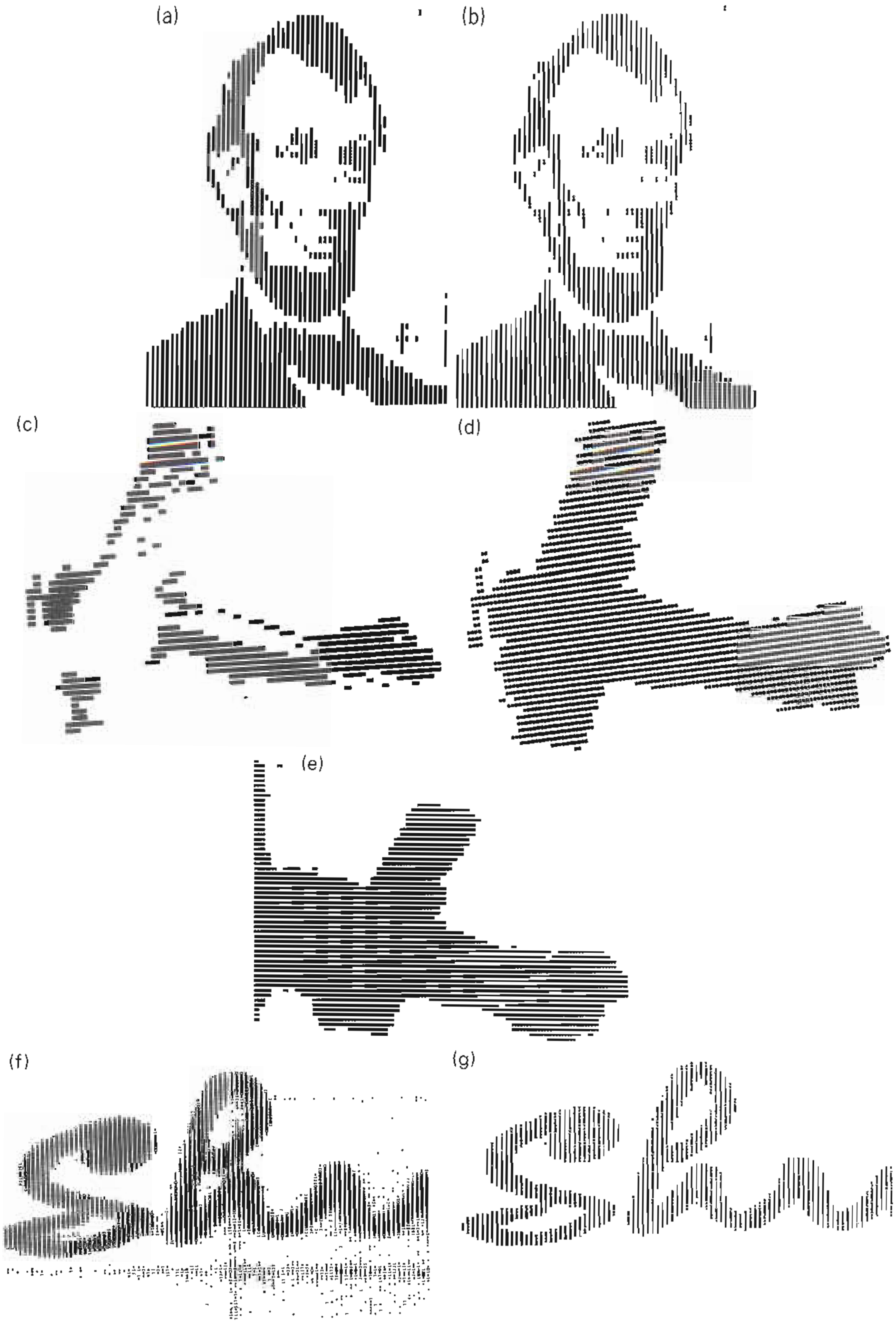

Figure 4. The various segmented versions of the images for the window size 12. Thresholded at (a) 10, (b) 11, (c) 5, (d) 13, (e) 24, (f) 8, (g) 13.

When the input image contains multiple objects or an elongated object, the segmentation algorithm [2] based on the compactness measure is seen to fail to extract the appropriate boundary even as one of its local minima. On the other hand, the proposed algorithm based on the IOAC measure has been found to be successful in this regard.

Computational formulae of various geometrical measures in order to make the algorithm fast have also been provided. One may also use the new measures for finding the skeleton of a fuzzy segmented image as done with the compactness measure in [3].

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