RMO–1995

1. In triangle ABC, K and L are points on the side BC (K being closer to B than L) such that $BC \cdot KL = BK \cdot CL$ and AL bisects $\angle KAC$. Show that AL is perpendicular to AB.
2. Call a positive integer n **good** if there are n integers, positive or negative, and not necessarily distinct, such that their sum and product are both equal to n (e.g. 8 is **good** since

$$8 = 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1 \cdot 1(-1)(-1) = 4 + 2 + 1 + 1 + 1 + 1 + (-1) + (-1)).$$

   Show that integers of the form $4k + 1$ ($k \geq 0$) and $4l$ ($l \geq 2$) are **good**.
3. Prove that among any 18 consecutive 3-digit numbers there is at least one number which is divisible by the sum of its digits.
4. Show that the quadratic expression

$$x^2 + 7x - 14(q^2 + 1) = 0,$$

   where q is an integer, has no integer root.
5. Show that for any triangle ABC, the following inequality is true :

$$a^2 + b^2 + c^2 > \sqrt{3} \max\{|a^2 - b^2|, |b^2 - c^2|, |c^2 - a^2|\},$$

   where a, b, c are, as usual, the sides of the triangle.
6. Let $A_1A_2A_3 \ldots A_{21}$ be a 21-sided regular polygon inscribed in a circle with center O. How many triangles $A_iA_jA_k$, $1 \leq i < j < k \leq 21$, contain the point O in their interior.
7. Show that for any real number x,

$$x^2 \sin x + x \cos x + x^2 + \frac{1}{2} > 0.$$