

CORRELATION BETWEEN TWO FUZZY MEMBERSHIP FUNCTIONS

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The need for a measure of correlation between two membership functions is stated. The properties that the correlation measure may possess are examined. A measure of correlation is defined on the basis of those properties. The definition is extended to the cases when (a) the domain is finite and (b) the domain is a subset of R^n . The problems encountered while extending the definition are also discussed.

Keywords: Correlation, Random variable, Lebesgue measure, s - and π -type functions.

Introduction

One of the first motivations and one of the main aims of all research in fuzzy set theory, is to furnish mathematical models which are able to describe systems or classes of systems which escape traditional analysis. One way is to use known pieces of mathematics and try to build new calculi more apt than the existing ones for modelling the reality. I.e., abstractions are made from a conceptual point of view if not from a formal one, on the idea that in principle everything of which it is worth speaking, can be affirmed or denied with certainty [3].

In real life phenomena, we come across many characteristics and attributes which are similar in nature, e.g., tall and very tall, glamorous and beautiful. The distinguishing factor between the membership functions of 'tall' and 'very tall' is the degree of tallness. But glamorous and beautiful are too distinct characteristics. If the value of one membership is quite high, the other one cannot be very low. In this paper the authors try to explore this phenomenon. A measure of the relationship between two fuzzy membership functions, which is called 'correlation', is defined. This paper will not deal with the question of evaluating membership functions. It is assumed that membership functions are given.

Section 1

A few examples are given in this section where the need for defining a measure of correlation is stated.

Example 1.1. Let Ω be the set of all girls between the ages 20 and 30 in India. Let the two characteristics be glamorous and beautiful, represented by f_1 and f_2 respectively. In general those girls who are beautiful (the value of membership function is quite high, say around 0.8) are not glamourless (i.e. the value of f_1 is not as low as 0.01 or 0.05) and vice versa. For high values of f_1 , f_2 in general is also high. For low values of f_1 too, f_2 , in general, is not really high because our mind associates f_1 with f_2 upto certain extent. In fact we are trying to bring out this association in mathematical form.

Observe that Ω is not a subset of R^n .

Example 1.2. Let Ω_1 represent the set of heights measured in centimetres, say $\Omega_1 = [0, 300]$, and Ω_2 represent the set of weights measured in kilograms, say $\Omega_2 = [0, 200]$. Let us consider the membership functions for tall [f_1] and heavy [f_2].

It seems that the set of heights is sufficient in determining f_1 . But there are many examples where it is not true. If a person is very lean, though his height is not much, he looks tall. Similarly if a person is well built, he may not look tall. Now, is the membership function defined on the basis of the looks, or on the basis of the measurements? In reality, people form opinions on the basis of looks, but not on the basis of measurements. If the impressions of the people are taken as standard, then Ω_1 is not sufficient in determining f_1 though it gives a good idea. Some other factors are to be taken into consideration such as weight, chest measurements, etc. One can get several examples of adjectives like 'tall' where considering a set of lower dimension is good enough but the feelings of the people are not portrayed properly. In some cases it is impossible to state all factors which are responsible for that specific characteristic. Because of practical considerations sets of lower dimension are taken.

In this example, the membership functions are assumed to have been found on the basis of the measurements only and the other factors responsible are assumed to be the same for all the individuals. So $f_1: \Omega_1 \times \Omega_2 \rightarrow [0, 1]$ and $f_2: \Omega_1 \times \Omega_2 \rightarrow [0, 1]$. Observe that a person with a certain height has more weight than that individual whose height is less. And also a person with a certain height has a minimum weight. So if the value of f_1 is high, f_2 can't be really low. And it is clear that f_1 and f_2 are related and the relationship is similar to that of Example 1.1.

Example 1.3. Let $f_1: [0, \infty) \rightarrow [0, 1]$ and $f_2: [0, \infty) \rightarrow [0, 1]$ be membership functions for small and very small numbers respectively. Here also the relationship between f_1 and f_2 is similar to that of relationships expressed in Examples 1.1 and 1.2. Some authors [5] use f_1^2 for f_2 .

The difference between Examples 1.1, 1.2 and 1.3 is in connection with the domain. In Example 1.1, the domain is not a subset of R^n . In Example 1.2, the domain is a subset of R^n , but the value of n is not known because of ambiguities, whereas in Example 1.3, the domain is a subset of R^n and it is sufficient in determining the f_i 's.

Examples of membership functions f_1 and f_2 where for high values of f_1 , f_2 takes low values and vice versa can also be obtained. The above relationship between the membership functions is called 'correlation' between them.

Section 2

In this section, various properties which can be attributed to 'correlation' are discussed.

In statistics, the correlation r_{xy} between two variables x and y is defined on the basis of the ideas [2] expressed in Figures 1, 2 and 3.

Let $(x_i, y_i), i = 1, \dots, n$, be the given sample points, with mean \bar{x} and \bar{y} respectively.

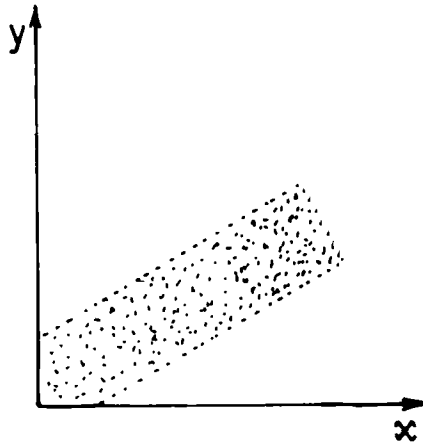


Fig. 1

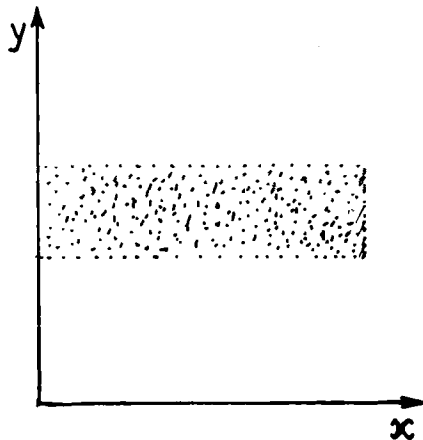


Fig. 2

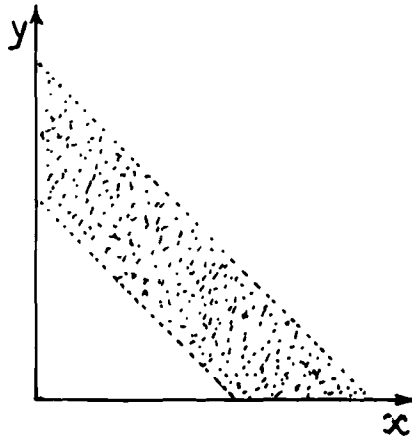


Fig. 3

Let

$$s_x^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2, \quad s_y^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2$$

and the covariance between x and y

$$\text{Cov}(x, y) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}).$$

In Figure 1, as x increases, y increases too whereas in Figure 2, irrespective of whether x increases or decreases the value of y remains more or less fixed. In Figure 3, as x increases, y decreases. $\text{Cov}(x, y)$ is positive in Figure 1, near to zero in Figure 2 and negative in Figure 3. The correlation coefficient is obtained by standardizing covariance by s_x and s_y :

$$r_{xy} = \frac{\text{Cov}(x, y)}{s_x s_y}, \quad r_{xy} = r_{yx} \quad \text{and} \quad |r_{xy}| \leq 1.$$

For continuous random variables X and Y , the correlation coefficient is defined as

$$\rho_{xy} = E(X - \mu_x)(Y - \mu_y) / \sigma_x \sigma_y$$

where μ_x , μ_y , σ_x^2 and σ_y^2 are means and variances of X and Y respectively. E denotes 'expectation'. It does not give good results when X and Y are non-linearly related. For example $\rho_{xy} = 0 \not\Rightarrow X$ and Y are independent. There are examples where X is a function of Y but $\rho_{xy} = 0$ [1].

The relationship between membership functions (as explained in Section 1) is called correlation because the ideas are quite similar to the ones expressed above. In all the three examples that are given in Section 1, it was observed that, for high values of one function, the other one does not take low values and vice versa. It was also stated that there are examples where, for high values of one function, the other one takes low values and vice versa. So a measure of relationship – like the correlation coefficient in statistics – may be defined in the context of fuzzy membership functions

also. Let the correlation between two fuzzy membership functions f_1 and f_2 , defined on the same domain Ω , be represented by c_{f_1, f_2} .

2.a. If Ω is a subset of R and if $x \uparrow$ then if both $f_1(x)$ and $f_2(x) \uparrow$ then $c_{f_1, f_2} > 0$, if $f_1(x) \uparrow$ and $f_2(x) \downarrow$ then $c_{f_1, f_2} < 0$ and lastly if $f_1(x) \uparrow$ and $f_2(x)$ is more or less the same then c_{f_1, f_2} must be very near to zero.

2.b. If Ω is always taken in such a way that $f_1(\Omega) = [0, 1] = f_2(\Omega)$, then the case that "increase in $f_1(x)$ does not change the value of f_2 throughout the length of Ω " does not arise. The correlation measure defined on such an Ω would give a good idea of the relationship between f_1 and f_2 .

Following is the list of properties which may be attributed to c_{f_1, f_2} . The detailed mathematical formulation of the domain Ω , membership functions f_i , and c_{f_1, f_2} will be dealt with in later sections. Here the properties are only discussed.

p₁. If for higher values of $f_1(x)$, $f_2(x)$ takes higher values and if the converse is also true then c_{f_1, f_2} must be very high (from Section 1).

p₂. If Ω is a subset of R and
 (a) $x \uparrow$ implies $f_1(x) \uparrow$ and $f_2(x) \uparrow$, then $c_{f_1, f_2} > 0$,
 (b) $x \uparrow$ implies $f_1(x) \uparrow$ and $f_2(x) \downarrow$, then $c_{f_1, f_2} < 0$ (2.a).

p₃. c_{f_1, f_1} is the supremum of all c_{f_1, f_2} . That is $c_{f_1, f_1} \geq c_{f_1, f_2} \forall f_1, f_2$ defined on Ω .

This property is the natural counterpart of correlation in statistics.

p₄. $c_{f_1, f_1} = 1 \forall f_1$.

Given that $c_{f_1, f_2} \leq c_{f_1, f_1} \forall f_1$, c_{f_1, f_1} may be set equal to either 1 or ∞ . If c_{f_1, f_1} is a positive number then the expression for c_{f_1, f_2} may be normalized to give it the value 1. In this paper, the theory is developed on the assumption that $c_{f_1, f_1} = 1 \forall f_1$. Similar theory may be developed assuming $c_{f_1, f_1} = \infty$.

p₅. $c_{f_1, 1-f_1} = -1 \forall f_1$.

The natural complement of a fuzzy membership function f is $1-f$. If f increases (decreases) then $1-f$ decreases (increases) with exactly the same gradient but with different signs. And, this is true for every $x \in \Omega$ where the derivative exists.

p₆. $-1 \leq c_{f_1, f_2} \leq 1 \forall f_1, f_2$.

Example 2.1. Let Ω be a closed interval of R and $f_1: \Omega \rightarrow [0, 1]$ such that $f_1(\Omega) = [0, 1]$. In Figure 4, $\{(f_1(x), g(x)); x \in \Omega\}$ is plotted for various membership functions g .

As f_1 increases, f_4 increases, so also f_5 . Hence $c_{f_1, f_4} > 0$ and $c_{f_1, f_5} > 0$. But for

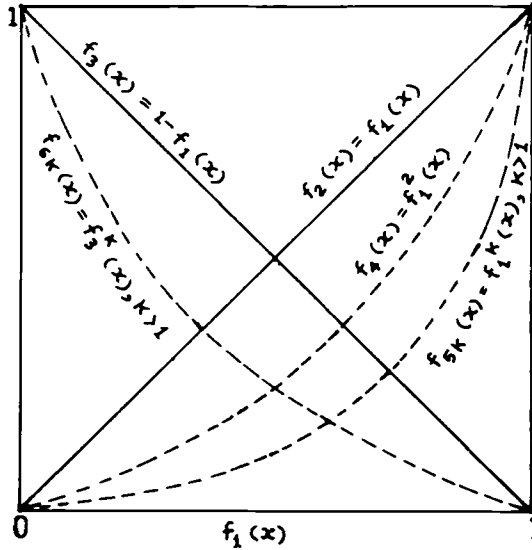


Fig. 4

higher values of f_1, f_5 may take low values [$f_{5k}(x) = f_1^k(x); k > 1$]. So $c_{f_1, f_5} < c_{f_1, f_4} < c_{f_1, f_2} = 1$. That is, as k increases, c_{f_1, f_1^k} must decrease. As the curve becomes flatter and comes closer to the x -axis, c_{f_1, f_1^k} must become smaller though always greater than zero.

Observe that f_4 and f_5 completely lie below the line $f_2(x) = f_1(x)$. If a membership function is considered which lies completely above the $f_2(x) = f_1(x)$ line, the behaviour of the correlation coefficient must be similar to that of c_{f_1, f_1^k} . As the membership function tends to be more parallel to the x -axis, the correlation coefficient must become smaller though always greater than zero.

2.c. c_{f_1, f_1^k} decreases both as $k \rightarrow \infty$ and as $k \rightarrow 0$. That is, if the differences between f_1 and f_1^k are considered to define the measure of correlation between f_1 and f_1^k , then the absolute differences are more important than the actual differences.

Consider $f_{6k}(x) = [f_3(x)]^k$. Then $c_{f_3, f_{6k}}$ must be greater than zero $\forall k > 0$ and must decrease both as $k \uparrow \infty$ and as $k \downarrow 0$ according to the previous arguments. $c_{f_1, f_{6k}}$ must be less than 0 $\forall k > 0$, because as f_1 increases, f_{6k} decreases and also for high values of f_1, f_{6k} takes low values. So $c_{f_1, f_{6k}}$ must be less than zero. Now, there are two possibilities for $\lim_{k \rightarrow \infty} c_{f_1, f_{6k}}$.

2.d. $c_{f_1, f_{6k}}$ increases as $k \rightarrow \infty$. That means, importance is given to the fact that there does not exist any other membership, other than f_3 whose derivative is the same as that of f_1 at every point but with a different sign. I.e., $c_{f_1, f_{6k}} \geq c_{f_1, f_3} \forall k$.

2.e. $c_{f_1, f_{6k}}$ decreases as $k \rightarrow \infty$.

Here importance is given to p_1 . Since for higher values of f_1 , f_{6k} takes much lower values than f_3 , $c_{f_1, f_{6k}}$ has to be less than c_{f_1, f_3} .

Similar things can be stated about $c_{f_1, 1-f_{6k}}$, $c_{f_3, f_{5k}}$ and $c_{f_3, 1-f_{5k}}$. So p_6 could as well have been $-\infty \leq c_{f_1, f_2} \leq 1$. Depending on the problem, p_6 is to be chosen. In this paper, we considered $-1 \leq c_{f_1, f_2} \leq 1$ as p_6 .

Observe that p_6 is not independent of p_4 and p_5 .

$$p_7. \quad c_{f_1, f_2} = c_{f_2, f_1} \quad \forall f_1, f_2.$$

p_7 essentially reflects the idea that the extent of influence of f_1 on f_2 is same as that of f_2 on f_1 . The correlation is 'between' f_1 and f_2 but not one 'on' the other.

$$p_8. \quad c_{f_1, f_2} = -c_{1-f_1, f_2} \quad \forall f_1, f_2.$$

If $f_2 = f_1$ then $c_{f_1, f_1} = -c_{f_1, 1-f_1}$ which is satisfied if $c_{f_1, f_1} = 1$ and $c_{f_1, 1-f_1} = -1$. But, if $c_{f_1, f_1} = \infty$ and $c_{f_1, 1-f_1} = -1$ then this is not satisfied. Under p_6 , if there exists a function f_2 such that $c_{f_1, f_2} = 0$ then naturally $c_{1-f_1, f_2} = 0$. Thus for a function f_2 if $c_{f_1, f_2} > 0$ then c_{1-f_1, f_2} must be less than zero. Inherently p_4 , p_5 and p_6 are being utilized in making this assumption of equality. I.e., p_8 is not independent of p_4 , p_5 and p_6 .

$$p_9. \quad c_{f_1, f_2} = c_{1-f_1, 1-f_2} \quad \forall f_1, f_2.$$

The correlation between two functions f_1 and f_2 is same as the correlation between their complements. This property is not independent of p_4 , p_5 and p_6 .

Note. The properties p_1 to p_9 are not exhaustive. Similar properties can be attributed to c_{f_1, f_2} assuming that $-\infty \leq c_{f_1, f_2} \leq 1$.

An example is stated below where the validity of the property $c_{f_1, 1-f_1} = -1 \quad \forall f$ is in doubt.

Example 2.2. Let $x \in [0, 1]$ and $I_k = [1/2^k, 1 - 1/2^k]$, $k \in [1, \infty)$. Define

$$\begin{aligned} f_k(x) &= \frac{1}{2} && \text{if } x \in I_k, \\ &= x2^{k-1} && \text{if } x \leq 1/2^k, \\ &= [x2^k - 2^k + 2]/2 && \text{if } x \geq 1 - 1/2^k. \end{aligned}$$

As $k \rightarrow \infty$, the difference between f_k and $1 - f_k$ decreases. In fact

$$\begin{aligned} \lim_{k \rightarrow \infty} f_k(x) &= 0 \quad \text{if } x = 0, \\ &= 1 \quad \text{if } x = 1, \\ &= \frac{1}{2} \quad \text{otherwise,} \end{aligned}$$

and

$$\begin{aligned}\lim_{k \rightarrow \infty} [1 - f_k(x)] &= 1 \quad \text{if } x = 0, \\ &= 0 \quad \text{if } x = 1, \\ &= \frac{1}{2} \quad \text{otherwise.}\end{aligned}$$

Except at $x = 0$ and $x = 1$,

$$\lim_{k \rightarrow \infty} f_k(x) = \lim_{k \rightarrow \infty} [1 - f_k(x)].$$

But $c_{f_k, 1-f_k} = -1 \quad \forall k$.

This example is discussed in detail in Section 4.

Section 3

In this section a measure of correlation is defined between two fuzzy membership functions.

In order to get an expression for c_{f_1, f_2} , the principle applied in statistics cannot be taken here. The concept of 'mean' of a fuzzy membership function does not exist. Hence 'variance' of a fuzzy membership function is non-existent. In statistics, random variables are defined on a probability space and they induce a probability measure [1]. In fuzzy sets, membership functions are defined from a domain to $[0, 1]$ and they don't induce any membership function. Above all even with the above drawbacks, c_{f_1, f_2} may be defined as

$$\begin{aligned}c_{f_1, f_2} &= \left[\int_{\Omega} (f_1(x) - f_1(\bar{x}_1))(f_2(x) - f_2(\bar{x}_2)) dx \right] \\ &\quad / \sqrt{\int_{\Omega} [f_1(x) - f_1(\bar{x}_1)]^2 dx} \sqrt{\int_{\Omega} [f_2(x) - f_2(\bar{x}_2)]^2 dx}\end{aligned}$$

where \bar{x}_1 is the mean under f_1 , i.e., $\bar{x}_1 = \int_{\Omega} x f_1(x) dx$ and \bar{x}_2 is the mean under f_2 . c_{f_1, f_2} may be defined in the above fashion given that $f_1(\bar{x}_1)$ and $f_2(\bar{x}_2)$ are defined. If Ω is finite then \bar{x}_1 and \bar{x}_2 cannot be defined in many cases. Multiplication of x with $f_1(x)$ may become meaningless. So the above approach is ruled out.

Definition 1. Let (a) the domain Ω be a closed interval in R . Let the membership functions f_1 and f_2 be such that

(b) $f_1: \Omega \rightarrow [0, 1]$ and $f_2: \Omega \rightarrow [0, 1]$ are continuous,

(c) $f_1(\Omega) = f_2(\Omega) = [0, 1]$, and

(d) $\forall x \in \Omega^c, f_i(x) = 0$ or 1 or undefined $\forall i = 1, 2$.

It is clear from the above assumptions that given f_1 and f_2 another domain S may be obtained which satisfies the above assumptions. It can be done by extending Ω or by reducing the length of the interval. In order to avoid this confusion, let

$$\mathcal{S}_{f_1, f_2} = \{S: S \text{ is a domain for } f_1 \text{ and } f_2 \text{ satisfying a, b, c, d above}\}$$

and

$$\Omega = \bigcap_{S \in \mathcal{S}_{f_1, f_2}} S.$$

Let

$$X_1 = \int_{\Omega} [f_1 - (1 - f_1)]^2 dx = \int_{\Omega} (2f_1 - 1)^2 dx,$$

$$X_2 = \int_{\Omega} [f_2 - (1 - f_2)]^2 dx = \int_{\Omega} (2f_2 - 1)^2 dx.$$

Now define

$$c_{f_1, f_2} = 1 - \frac{4}{X_1 + X_2} \int_{\Omega} (f_1 - f_2)^2 dx.$$

Note 1. In Example 2.1 it was mentioned that the absolute differences between f_1 and f_2 are important in defining c_{f_1, f_2} (2.c). Here instead of taking absolute differences, the squares are taken and they are summed up.

Note 2. Assumption (d) states that for $x \in \Omega^c$, $f_i(x) = 0$ or 1 or undefined, $i = 1, 2$. If f_1 represents 'tall' and Ω represents height in cms, say, $[60, 300]$ then for $x < 0$, $f_1(x)$ may be taken to be either 0 or undefined.

Proposition 1. $c_{f_1, f_2} \leq 1 \forall f_1, f_2$.

Proof. It suffices to show that $2 \int_{\Omega} (f_1 - f_2)^2 dx \leq X_1 + X_2$. After some calculations it follows

$$X_1 + X_2 - 2 \int_{\Omega} (f_1 - f_2)^2 dx = 2 \int_{\Omega} (f_1 + f_2 - 1)^2 dx \geq 0.$$

Proposition 2. $c_{f_1, f_2} = -c_{f_1, 1-f_2} \forall f_1, f_2$.

Proof. From

$$c_{f_1, f_2} = 1 - \frac{4}{X_1 + X_2} \int_{\Omega} (f_1 - f_2)^2 dx$$

and

$$c_{f_1, 1-f_2} = 1 - \frac{4}{X_1 + X_2} \int_{\Omega} (f_1 + f_2 - 1)^2 dx$$

it follows

$$\begin{aligned} c_{f_1, f_2} + c_{f_1, 1-f_2} &= 2 - \frac{4}{X_1 + X_2} \left[\int_{\Omega} (f_1 - f_2)^2 dx + \int_{\Omega} (f_1 + f_2 - 1)^2 dx \right] \\ &= 2 - \frac{2}{X_1 + X_2} [X_1 + X_2] \quad (\text{from Proposition 1}) \\ &= 0, \end{aligned}$$

which proves the proposition.

Proposition 3.

$$c_{f_1, f_1} = 1 \quad \text{and} \quad c_{f_1, 1-f_1} = -1 \quad \forall f_1,$$

$$c_{f_1, f_2} = c_{f_2, f_1} \quad \text{and} \quad c_{1-f_1, 1-f_2} = c_{f_1, f_2} \quad \forall f_1, f_2.$$

Proof. Trivial.

Note. c_{f_1, f_2} is a measure of closeness of f_1 and f_2 but not a distance measure. For given f_1, f_2, f_3 and f_4 , $c_{f_1, f_2} = c_{f_3, f_4} \Rightarrow$ the difference between f_1 and f_2 compared with the sum of the differences between f_1 and $1-f_1$ and f_2 and $1-f_2$ is same as the difference between f_3 and f_4 compared with the sum of the differences between f_3 and $1-f_3$ and f_4 and $1-f_4$. It does not mean that $d(f_1, f_2) = d(f_3, f_4)$ where d is a metric on membership functions.

The main properties which c_{f_1, f_2} has to satisfy are p_1 and p_2 as stated in Section 2. An example is provided below when the correlation coefficient is calculated for various functions defined on the same domain. In general the functions dealt with in fuzzy set theory are s - and π -type functions [4]. Simple examples of s - and π -type functions are given.

Example 3.1. Let $\Omega = [0, 1]$, and

$$f_1(x) = x, \quad f_2(x) = 2 \min(x, 1-x),$$

$$f_3(x) = 1 - f_1(x) = 1 - x,$$

$$f_{4k}(x) = x^k, \quad f_{5k}(x) = (1-x)^k \quad \forall k \in (0, \infty),$$

$$\bar{f}_{4k}(x) = 1 - f_{4k}(x), \quad \bar{f}_{5k}(x) = 1 - f_{5k}(x).$$

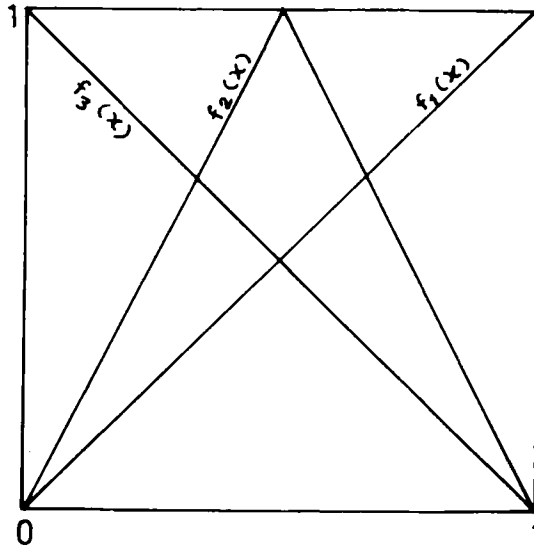


Fig. 5

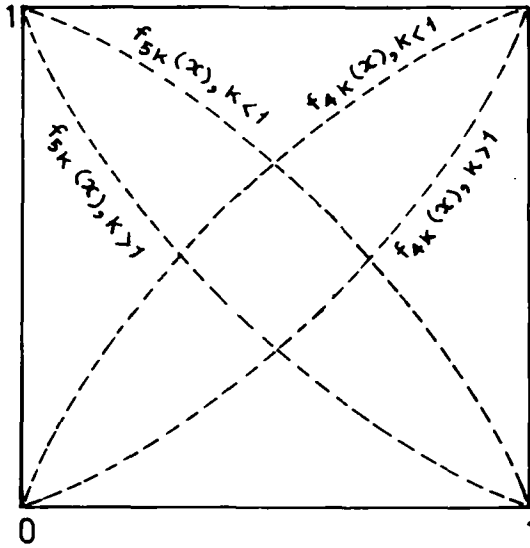


Fig. 6

$$1. X_{f_1} = X_{f_3} = \int_{\Omega} (2x-1)^2 dx = \frac{1}{3}.$$

$$2. X_{f_2} = \int_{\Omega} (2f_2-1)^2 dx = \frac{1}{3}.$$

$$3. X_{f_{4k}} = \int_{\Omega} (2f_{4k}-1)^2 dx = 1 - \frac{4}{k+1} + \frac{4}{2k+1} \quad \forall k \in (0, \infty).$$

$$4. X_{f_{5k}} = X_{f_{4k}} \quad \forall k \in (0, \infty).$$

$$5. \int_{\Omega} (f_1 - f_2)^2 dx = \frac{1}{6}.$$

$$6. \int_{\Omega} (f_1 - f_{4k})^2 dx = \frac{1}{3} + \frac{1}{2k+1} - \frac{2}{k+2}$$

$$= \int_{\Omega} (f_3 - f_{5k})^2 dx = \int_{\Omega} (f_1 - \bar{f}_{5k})^2 dx \quad \forall k > 0.$$

$$7. \int_{\Omega} (f_1 - f_{5k})^2 dx = \frac{1}{3} + \frac{1}{2k+1} - \frac{2}{(k+1)(k+2)}$$

$$= \int_{\Omega} (f_3 - f_{4k})^2 dx = \int_{\Omega} (f_1 - \bar{f}_{4k})^2 dx \quad \forall k > 0.$$

$$8. \int_{\Omega} (f_2 - f_3)^2 dx = \frac{1}{6}.$$

9.
$$\int_{\Omega} (f_2 - f_{4k})^2 dx = \frac{1}{2k+1} + \frac{1}{3} - \frac{4}{[k+1][k+2]} + \frac{2}{(k+1)(k+2)2^k}$$

$$= \int_{\Omega} (f_2 - f_{5k})^2 dx \quad \forall k > 0.$$
10.
$$\int_{\Omega} (f_2 - \bar{f}_{4k})^2 dx = \frac{1}{3} + \frac{1}{2k+1} - \frac{2k}{(k+1)(k+2)} - \frac{2}{2^k(k+1)(k+2)}$$

$$= \int_{\Omega} (f_2 - \bar{f}_{5k})^2 dx \quad \forall k > 0.$$
11.
$$c_{f_1, f_2} = 1 - \frac{4}{\frac{1}{3} + \frac{1}{3}} \cdot \frac{1}{6} = 0.$$
12.
$$c_{f_1, f_{4k}} = 1 - \frac{(k+1)(2k^2 - 4k + 2)}{(k+2)(2k^2 + 1)} = c_{f_3, f_{5k}} = c_{f_1, \bar{f}_{5k}}.$$

$$c_{f_1, f_{4k}} > 0 \quad \forall k > 0.$$

$$c_{f_1, f_{4k}} \rightarrow 0 \quad \text{as } k \rightarrow 0 \text{ and as } k \rightarrow \infty.$$
13.
$$c_{f_1, f_{5k}} = 1 - \frac{2k^3 + 10k^2 + 4k + 2}{(k+2)(2k^2 + 1)} = c_{f_3, f_{4k}} = c_{f_1, \bar{f}_{4k}}.$$

$$c_{f_1, f_{5k}} < 0 \quad \forall k > 0.$$

$$c_{f_1, f_{5k}} \rightarrow 0 \quad \text{as } k \rightarrow 0 \text{ and as } k \rightarrow \infty.$$
14.
$$c_{f_2, f_{4k}} = 1 - \frac{2k^3 + 10k^2 - 8k - 4 + [12k + 6]/2^k}{[k+2][2k^2 + 1]} = c_{f_2, \bar{f}_{5k}}.$$

$$c_{f_2, f_{4k}} \geq 0 \quad k \leq 1,$$

$$\leq 0 \quad k \geq 1.$$

$$c_{f_2, f_{4k}} \rightarrow 0 \quad \text{as } k \rightarrow \infty \text{ and as } k \rightarrow 0.$$
15.
$$c_{f_2, \bar{f}_{4k}} = 1 - \frac{2k^3 - 2k^2 + 10k + 8 - 6[2k + 1]/2^k}{[k+2][2k^2 + 1]} = c_{f_2, f_{5k}}.$$

$$c_{f_2, \bar{f}_{4k}} \leq 0 \quad k \leq 1,$$

$$\geq 0 \quad k \geq 1.$$

$$c_{f_2, f_{4k}} \rightarrow 0 \quad \text{as } k \rightarrow 0 \text{ and as } k \rightarrow \infty.$$

The results indicate that c_{f_1, f_2} follows the properties stated in Section 2.

Note. (1) In the definition of the correlation coefficient between f_1 and f_2 an assumption was made inherently. If f_1 and f_2 are defined on the same domain Ω then correlation exists between them. If there does not exist any correlation, then c_{f_1, f_2} has to be equal to zero.

(2) The definition of c_{f_1, f_2} given in this section can also be used when Ω is finite. In that case let

$$X_1 = \sum_{x \in \Omega} [2f_1(x) - 1]^2, \quad X_2 = \sum_{x \in \Omega} [2f_2(x) - 1]^2,$$

and

$$c_{f_1, f_2} = 1 - \frac{4}{X_1 + X_2} \sum_{x \in \Omega} [f_1(x) - f_2(x)]^2,$$

$$= 1 \quad \text{if } X_1 + X_2 = 0.$$

But it is to be remembered that the correlation coefficient when Ω is infinite gives a better idea than the correlation when Ω is finite (2.b) because there is no loss of information when Ω is infinite. When Ω is finite, the greater the number of elements in Ω , the better the correlation coefficient would be.

The generalization of c_{f_1, f_2} to R^2 will be dealt with in Section 5. In the next section an example is discussed which is similar to Example 2.2.

Section 4

Example 4.1. Let $\Omega = [0, 1]$, and

$$f_k(x) = \begin{cases} \frac{x}{2k} & \text{if } 0 \leq x \leq k, \\ \frac{1}{2} & \text{if } k \leq x \leq 1-k, \quad \forall k \in (0, \frac{1}{2}], \\ \frac{x-1+2k}{2k} & \text{if } 1-k \leq x \leq 1, \end{cases}$$

Then $c_{f_k, 1-f_k} = -1 \quad \forall k \in (0, \frac{1}{2}]$ according to the previous definition. But

$$\lim_{k \rightarrow 0} f_k(x) = \lim_{k \rightarrow 0} (1 - f_k(x)) \quad \forall x \in (0, 1)$$

The definition of correlation which will be given here is based on the following principle. In the region where $f_1 = f_2$, the correlation must be equal to 1 and in the complementary region one has to look for whether $f_1(x)$ takes high values when

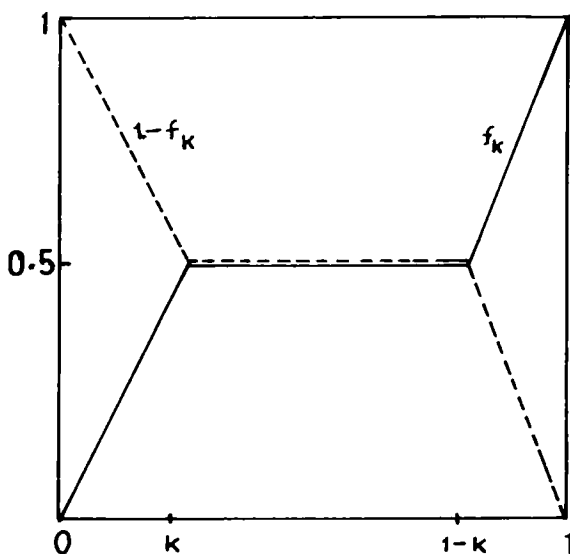


Fig. 7

$f_2(x)$ takes high values and so on. If c_{f_1, f_2} is defined in that way then $c_{f_1, f_2} + c_{f_1, 1-f_2}$ need not be equal to zero, because f_1 need not be equal to $1-f_2$ in the region $f_1 = f_2$. The information which is provided by f_1 and f_2 is being condensed to give a single value. So, naturally one has to look for the best way of condensing the information within the given limitations. The properties which are to be preserved depend on the problem at hand. The definition which is given below is just another definition, taking some other properties into consideration.

Let f_1, f_2 and Ω follow the same properties as defined in Section 3.

Definition 2. Let

$$D = \{x : f_1(x) = f_2(x), x \in \Omega\},$$

$$X_1 = \int_{\Omega-D} [2f_1 - 1]^2 dx, \quad X_2 = \int_{\Omega-D} (2f_2 - 1)^2 dx.$$

Define

$$c_{f_1, f_2} = \frac{\lambda_1(D)}{\lambda_1(\Omega)} + \frac{\lambda_1(\Omega-D)}{\lambda_1(\Omega)} \left[1 - \frac{4}{X_1 + X_2} \int_{\Omega-D} (f_1 - f_2)^2 dx \right],$$

$$= 1 \quad \text{if } X_1 + X_2 = 0,$$

where λ_1 represents the Lebesgue measure on R [length of a set].

Using the above definition, correlations are calculated for functions defined in Example 4.1. The calculations are shown below.

Let

$$D_k = \{x : f_k(x) = 1 - f_k(x), x \in [0, 1]\} \quad \forall k \in (0, \frac{1}{2}).$$

$$= [k, 1 - k],$$

$$\lambda_1(\Omega - D_k) = 1 - (1 - 2k) = 2k.$$

$$c_{f_k, 1-f_k} = (1 - 2k) + 2k \left[1 - \frac{4}{X_1 + X_1} X_1 \right] = 1 - 4k \quad \forall k \in (0, \frac{1}{2}).$$

Observe that for $k = \frac{1}{2}$, $c_{f_k, 1-f_k} = -1$ and $c_{f_k, 1-f_k} \uparrow$ as $k \downarrow$.

Note. A similar definition, when Ω is finite, is given below.

Let

$$D = \{x : f_1(x) = f_2(x), x \in \Omega\},$$

$$X_1 = \sum_{x \in \Omega-D} [2f_1 - 1]^2, \quad X_2 = \sum_{x \in \Omega-D} [2f_2 - 1]^2,$$

$$c_{f_1, f_2} = \frac{\#(D)}{\#(\Omega)} + \frac{\#(\Omega-D)}{\#(\Omega)} \left[1 - \frac{4}{X_1 + X_2} \sum_{x \in \Omega-D} [f_1(x) - f_2(x)]^2 \right],$$

$$= 1 \quad \text{if } X_1 + X_2 = 0,$$

where '#' denotes the number of elements in a set.

As of now, membership functions are assumed to be defined on a subset of R . In the next section it is generalized for R^2 .

Section 5

Let Ω_1 and Ω_2 be bounded closed intervals of R and $\Omega = \Omega_1 \times \Omega_2$. Let $f_1: \Omega_1 \times \Omega_2 \rightarrow [0, 1]$ and $f_2: \Omega \rightarrow [0, 1]$ be such that

- (a) f_1 and f_2 are continuous,
- (b) $f_1(\Omega) = f_2(\Omega) = [0, 1]$,
- (c) $x \in \Omega^c$, $f_i(x) = 0$ or 1 or undefined $\forall i = 1, 2$, and
- (d) Ω is the minimal set satisfying the above properties.

Before c_{f_1, f_2} is defined, a few adjustments are made in the following way.

Note. Let $\Omega_1 = [a_1, b_1]$, $\Omega_2 = [a_2, b_2]$, $a_1 < b_1$ and $a_2 < b_2$ and $g_i: [0, 1] \times [0, 1] \rightarrow [0, 1]$ be such that

$$g_i(x, y) = f_i(a_1 + x[b_1 - a_1], a_2 + y[b_2 - a_2]) \quad \forall i = 1, 2,$$

i.e., Ω_1 and Ω_2 are normalized and correspondingly two membership functions g_1 and g_2 are defined on $[0, 1] \times [0, 1]$. Then define

$$c_{f_1, f_2} = c_{g_1, g_2}.$$

I.e., it suffices to get an expression for c_{g_1, g_2} to get c_{f_1, f_2} .

Thus, from now on let $\Omega_1 = \Omega_2 = [0, 1]$ and f_1 and f_2 be membership functions defined on $\Omega = \Omega_1 \times \Omega_2$ satisfying the properties stated at the beginning of this section.

Example 5.1. Let $f_1: \Omega \rightarrow [0, 1]$, $f_2: \Omega \rightarrow [0, 1]$ be such that

$$\begin{aligned} f_1(x, y_1) &= f_1(x, y_2) \quad \forall y_1, y_2 \in [0, 1], \forall x \in [0, 1], \\ f_2(x_1, y) &= f_2(x_2, y) \quad \forall x_1, x_2 \in [0, 1], \forall y \in [0, 1]. \end{aligned}$$

Then c_{f_1, f_2} must be equal to zero since f_1 is not dependent on f_2 .

On the basis of the above example and Section 4, c_{f_1, f_2} is defined.

Definition 3. Let

$$\begin{aligned} A_1 &= \{x: x \in \Omega_1, f_1(x, y_1) = f_1(x, y_2) \quad \forall y_1, y_2 \in \Omega_2\}, \\ A_2 &= \{x: x \in \Omega_1, f_2(x, y_1) = f_2(x, y_2) \quad \forall y_1, y_2 \in \Omega_2\}, \\ B_1 &= \{y: y \in \Omega_2, f_1(x_1, y) = f_1(x_2, y) \quad \forall x_1, x_2 \in \Omega_1\}, \\ B_2 &= \{y: y \in \Omega_2, f_2(x_1, y) = f_2(x_2, y) \quad \forall x_1, x_2 \in \Omega_1\}, \\ D &= \{z: z \in \Omega, f_1(z) = f_2(z)\}, \\ \Omega_3 &= (A_1 \times B_2) \cup (A_2 \times B_1), \quad \Omega_4 = D - \Omega_3, \quad \Omega_5 = \Omega_3 \cup D, \\ X_1 &= \int_{\Omega - \Omega_5} [2f_1(z) - 1]^2 dz, \quad X_2 = \int_{\Omega - \Omega_5} [2f_2(z) - 1]^2 dz. \end{aligned}$$

Define

$$c_{f_1, f_2} = \lambda_2(\Omega_4) + \lambda_2(\Omega - \Omega_5) \left[1 - \frac{4}{X_1 + X_2} \int_{\Omega - \Omega_5} (f_1(z) - f_2(z))^2 dz \right],$$

$$= \lambda_2(\Omega_4) + \lambda_2(\Omega - \Omega_3) \quad \text{if } X_1 + X_2 = 0,$$

where λ_2 is the Lebesgue measure on R^2 (area).

Explanation for Definition 3: In the region on which f_1 is independent of f_2 , the correlation must be equal to zero. The region D is to be considered separately as mentioned in Section 4. Thus the expressions are integrated over $\Omega - \Omega_5$. Ω_4 is taken in the definition of c_{f_1, f_2} because D may have a non-empty intersection with Ω_3 .

Under this definition, $c_{f, f} = \lambda_2(\Omega - \Omega_5) + \lambda_2(\Omega_4)$ may be less than one since there is a possibility that $\lambda_2(\Omega_3) > 0$. Also $c_{f_1, f_2} + c_{f_1, 1-f_2}$ need not be equal to zero.

For Example 5.1, $A_1 = \Omega_1$, $B_2 = \Omega_2$, $\Omega_3 = \Omega$, $\Omega_4 = \emptyset$, $\Omega - \Omega_5 = \emptyset$ and so $c_{f_1, f_2} = 0$.

A similar definition can be given when Ω is a two-dimensional, finite set as given in Section 4. The definition can be extended when the domain Ω is a subset of R^n [finite cross product of bounded closed intervals] and also when Ω is a finite-dimensional finite set.

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