Secure Computation with Constant Communication Overhead using Multiplication Embeddings

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Multiplication Embedding Problem

$\mathbf{a} \in \mathbb{F}^m$ $\mathbf{b} \in \mathbb{F}^m$

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$$\mathbf{a} \in \mathbb{F}^m \xrightarrow{\mathsf{Enc}} A \in \mathbb{K} \qquad B \in \mathbb{K} \xleftarrow{\mathsf{Enc}} \mathbf{b} \in \mathbb{F}^m$$

$\mathbb{K}$ is the degree-n extension field of some finite field $\mathbb{F}$

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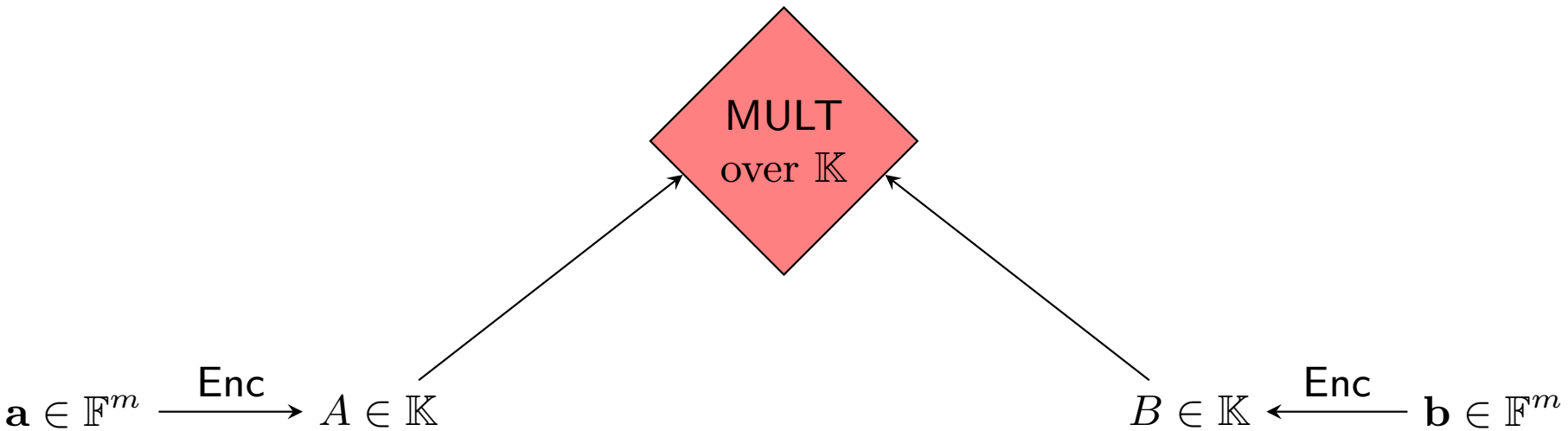

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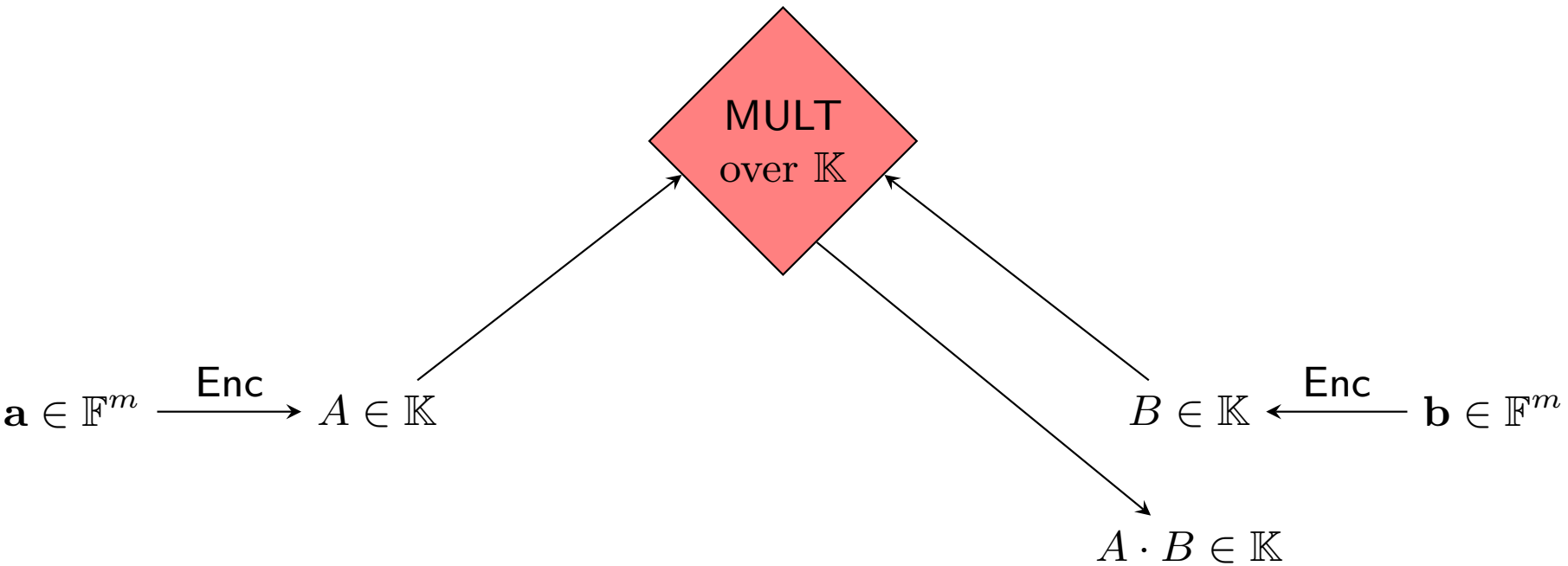

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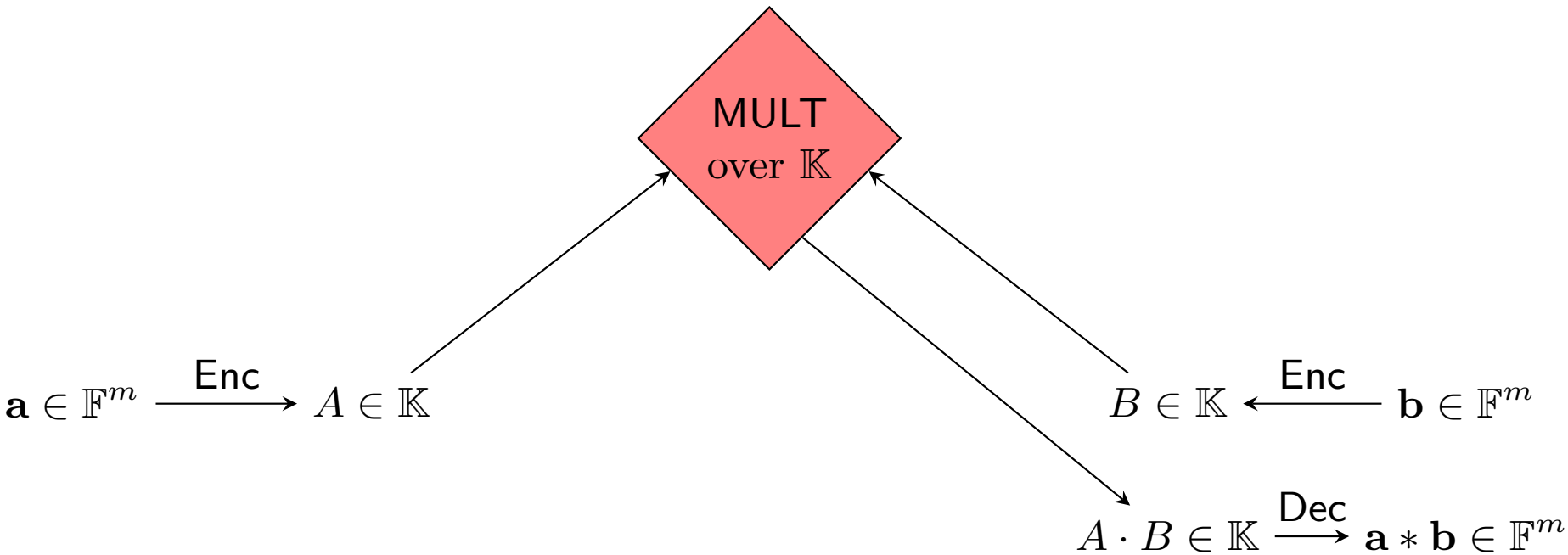

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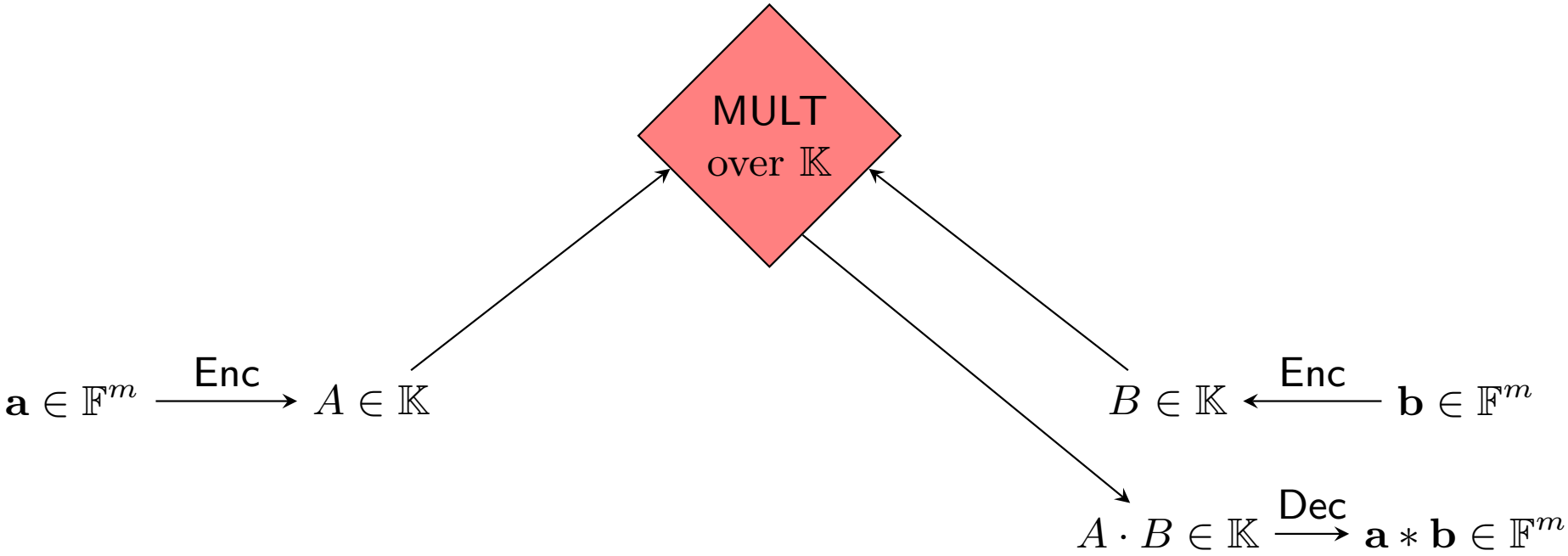

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Question

Given n and a field $\mathbb{F}$, find the largest m such that there exist two mappings $\mathsf{Enc} : \mathbb{F}^m \to \mathbb{K}$ and $\mathsf{Dec} : \mathbb{K} \to \mathbb{F}^m$ and

$$\forall\, \mathbf{a}, \mathbf{b} \in \mathbb{F}^m \text{ we have } \mathsf{Dec}(\mathsf{Enc}(\mathbf{a}) \cdot \mathsf{Enc}(\mathbf{b})) = \mathbf{a} * \mathbf{b}$$

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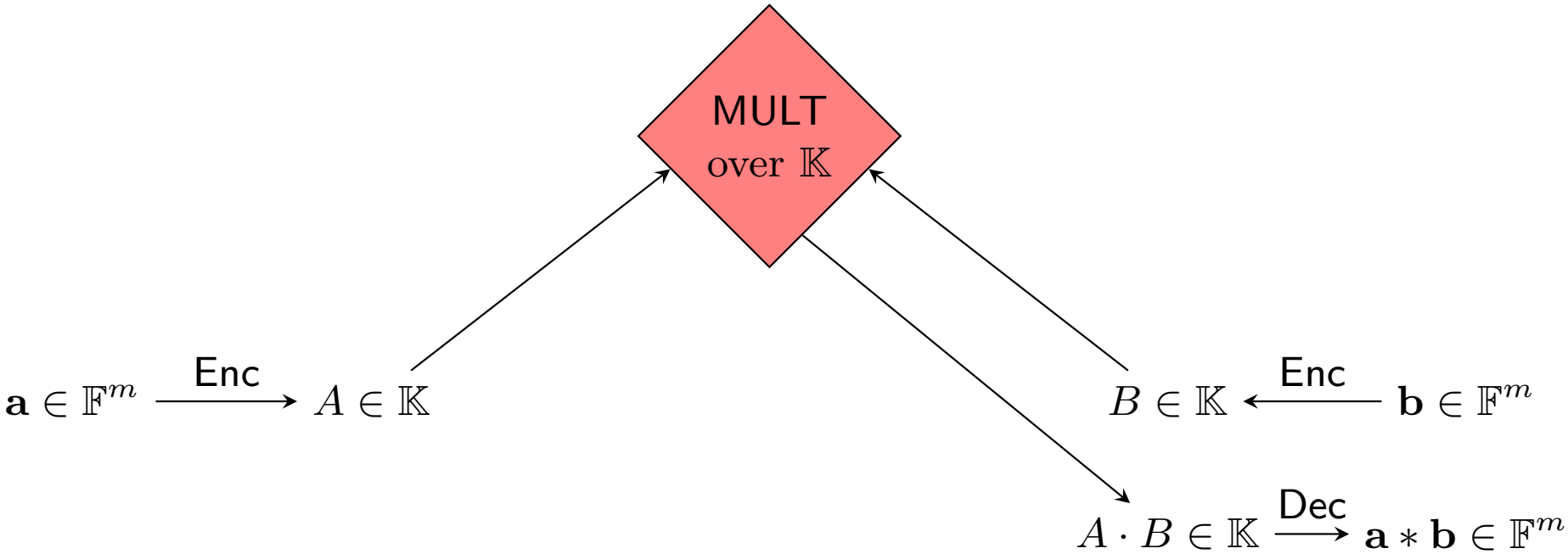

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Note

- Intuitively, we perform m multiplications over the base field $\mathbb{F}$ using one multiplication over the degree-n extension field $\mathbb{K}$
- For intuition, consider $\mathbb{K} = \mathbb{GF}[2^n]$ and $\mathbb{F} = \mathbb{GF}[2]$. However, our results hold for any degree-n extension field $\mathbb{K}$ of any base field $\mathbb{F}$.

Multiplication Embedding Problem: An Example

Suppose $\mathbb{K} = \mathbb{GF}[2^3]$ and $\mathbb{F} = \mathbb{GF}[2]$.
We interpret $\mathbb{K}$ as degree $\leqslant 2$ polynomials in X over $\mathbb{F}$.
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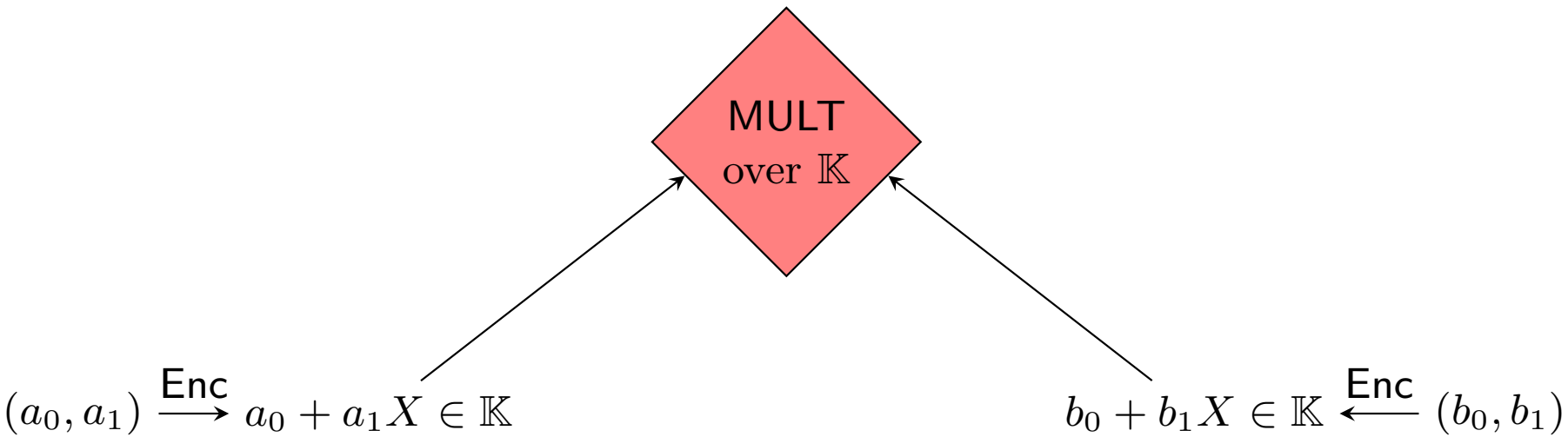

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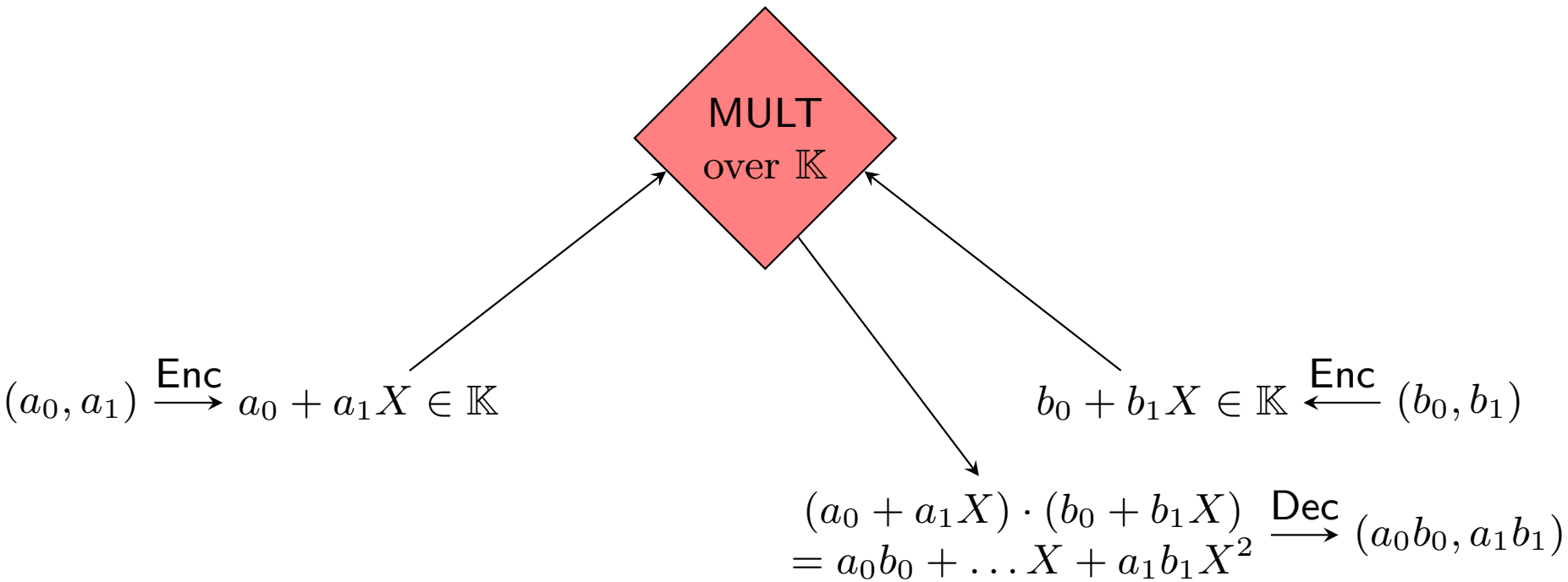

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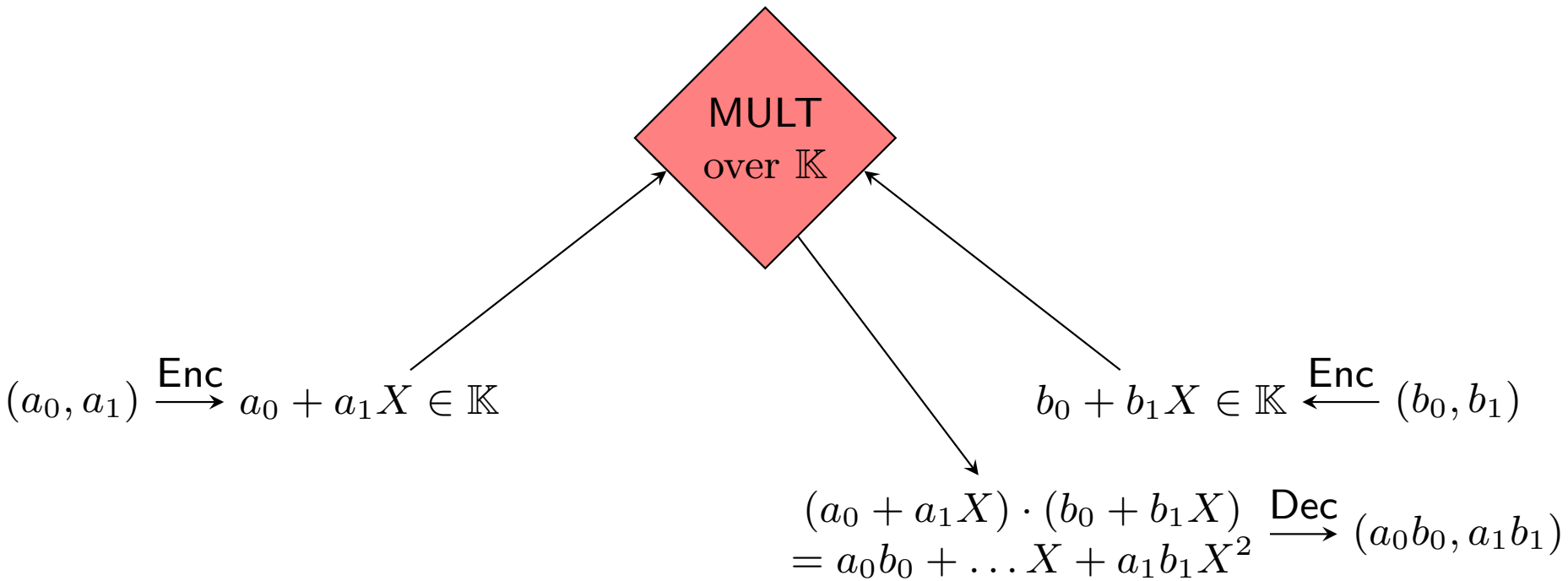

Note

Any element of the degree-n extension field $\mathbb{K}$ of a base field $\mathbb{F}$ can be represented as a polynomial of degree less than n with coefficients in $\mathbb{F}$.

Related Problem: Bi-linear Multiplication

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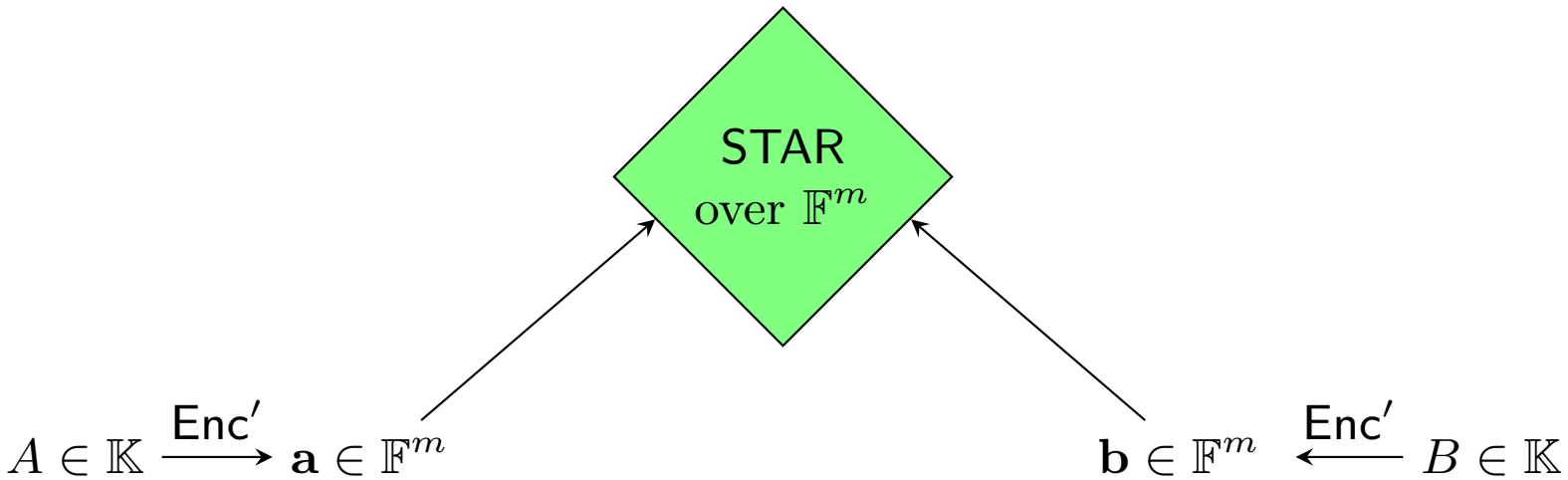

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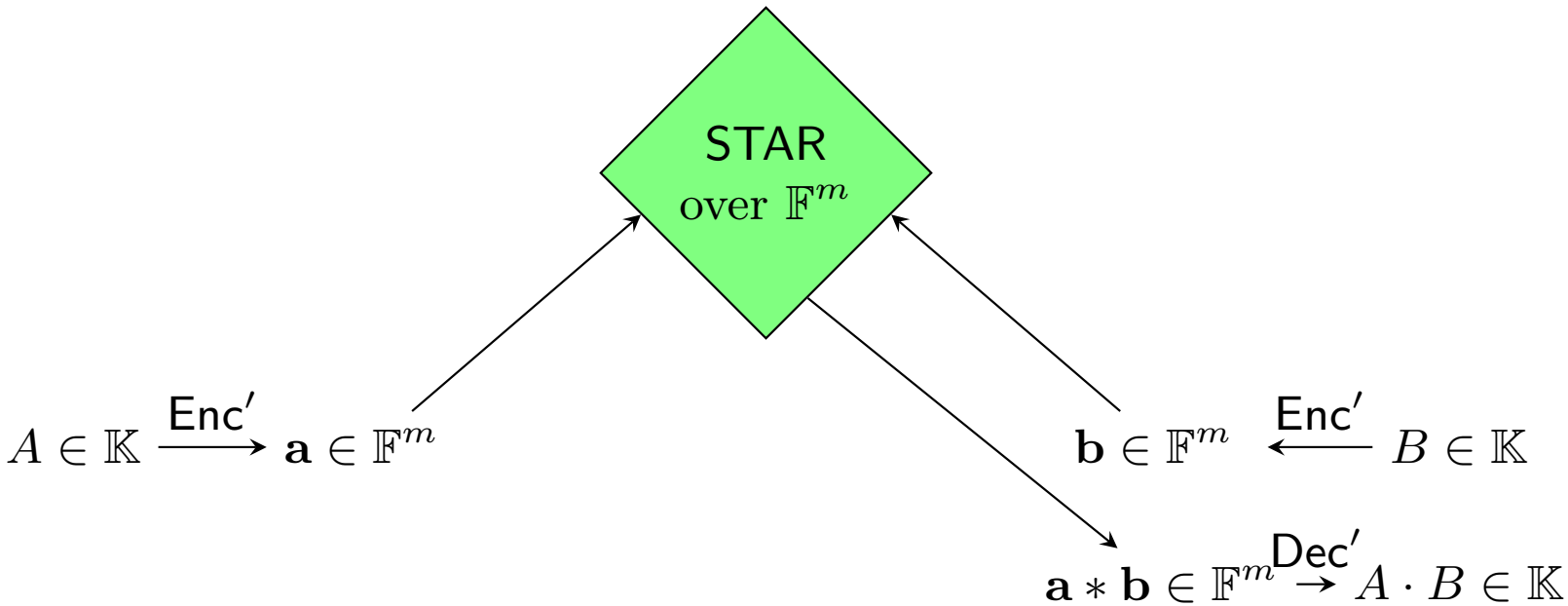

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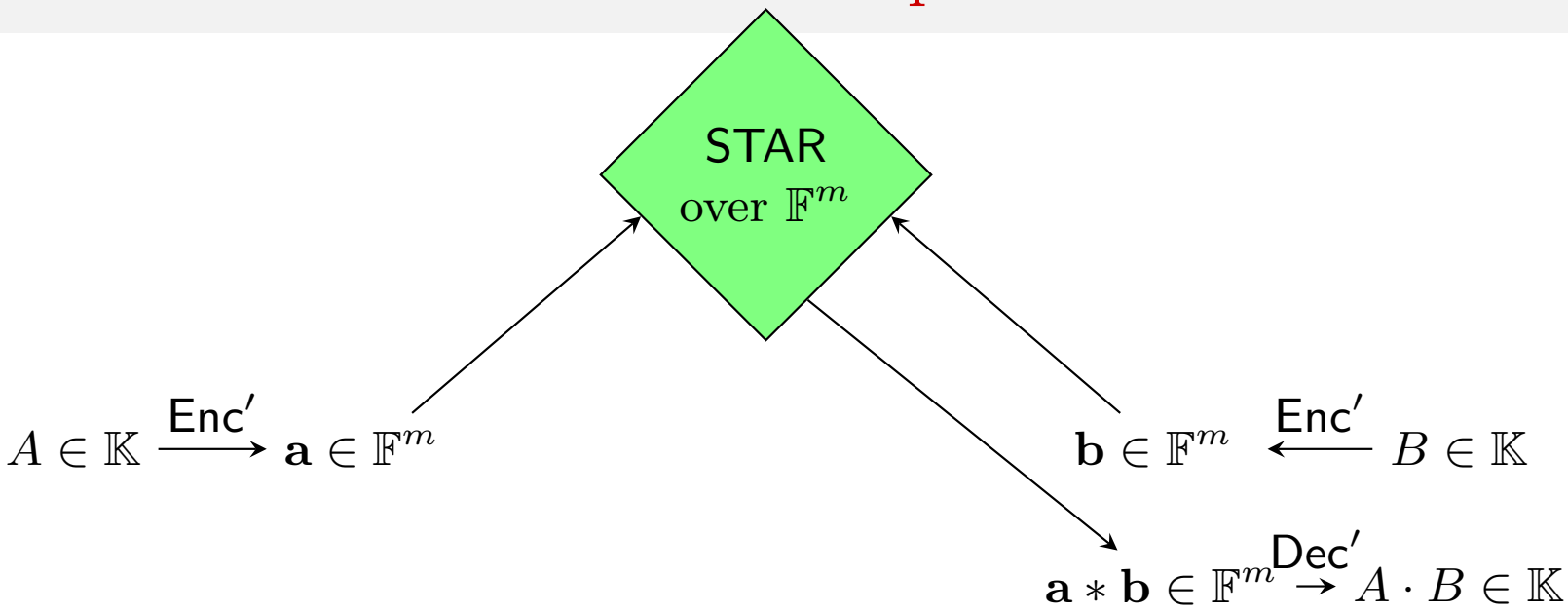

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Question

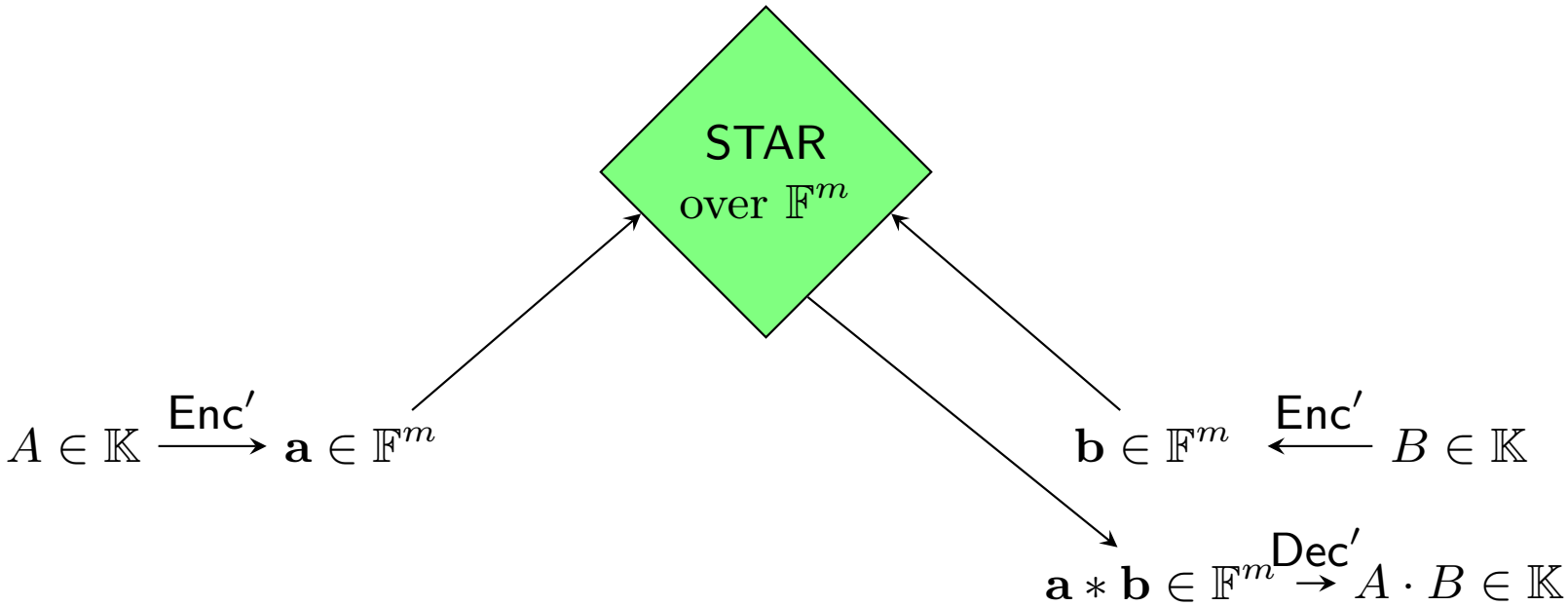
Given n and a field $\mathbb{F}$, find smallest m such that there exist Enc' and Dec' satisfying

$$\mathsf{Dec}'(\mathsf{Enc}'(A) * \mathsf{Enc}'(B)) = A \cdot B \text{ for all } A, B \in \mathbb{F}^m$$

Intuition

We can perform one multiplication over the degree-n extension field $\mathbb{K}$ using m multiplications over the base field $\mathbb{F}$.

Related Problem: Bi-linear Multiplication



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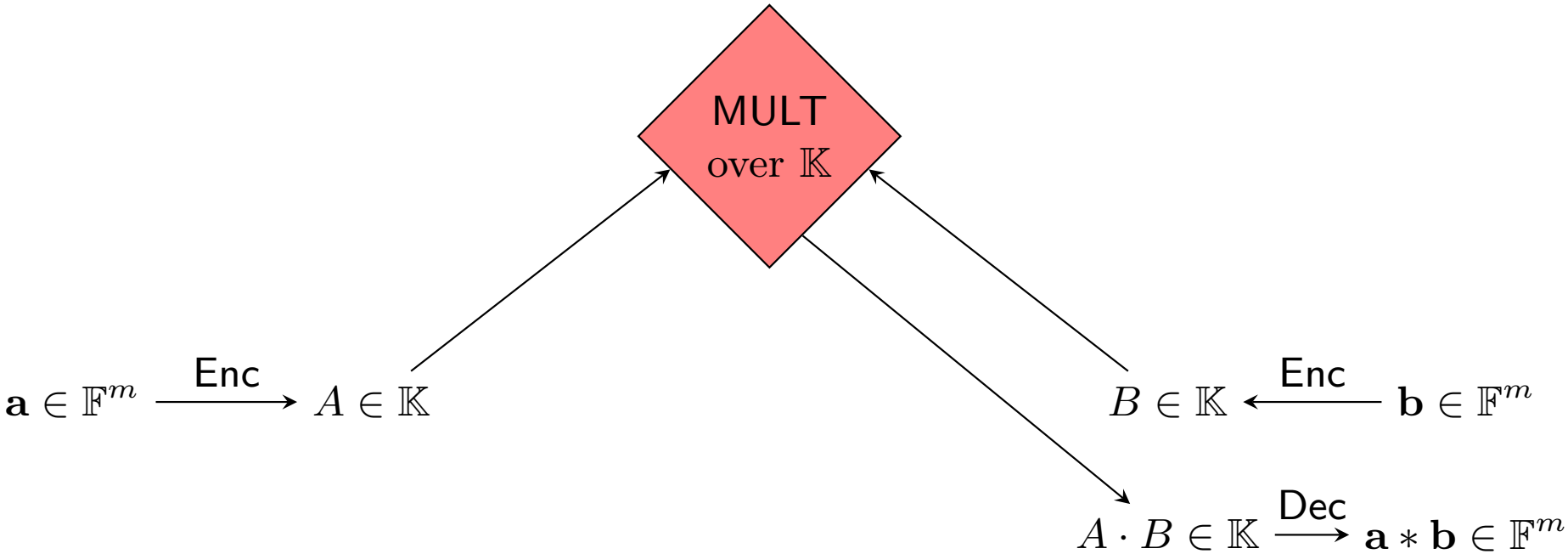
Theorem (Chudnovsky-Chudnovsky [CC87])

There exist mappings $\mathsf{Enc}' : \mathbb{K} \to \mathbb{F}^m$ *and* $\mathsf{Dec}' : \mathbb{F}^m \to \mathbb{K}$ *such that*

- m *is linear in* n
- $\mathsf{Dec}'(\mathsf{Enc}'(A) * \mathsf{Enc}'(B)) = A \cdot B$ *for all* $A, B \in \mathbb{F}^m$

Our Main Result



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Theorem (Multiplication Embedding)

There exist two linear maps $E : \mathbb{F}^m \to \mathbb{K}$ and $D : \mathbb{K} \to \mathbb{F}^m$ such that

- *m is linear in n*
- *$D(E(\mathbf{a}) \cdot E(\mathbf{b})) = \mathbf{a} * \mathbf{b}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{F}^m$*

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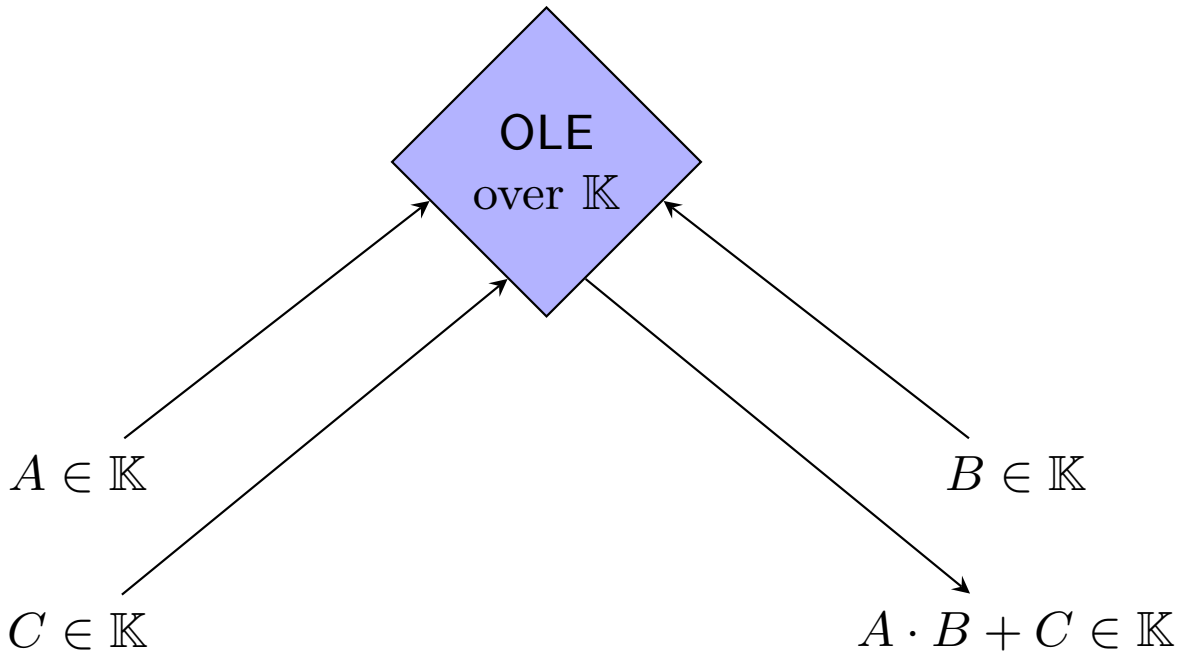
Recent Independent Work

Cascudo et al. [CCXY18] (CRYPTO–2018) studied this embedding problem (referred to as the "Reverse Multiplication-Friendly Embeddings" (RMFE))

- Provide a linear-rate construction.
- Use this result to achieve new amortization results in MPC.

Assuming Linear-rate Solution for the Embedding Problem: Consequences for MPC

Oblivious Linear-function Evaluation over a field $\mathbb{K}$:



Note

1-out-of-2-OT $\equiv$ OLE($\mathbb{GF}[2]$) since $x_b = (x_1 + x_0)b + x_0$.

Assuming Linear-rate Solution for the Embedding Problem: Consequences for MPC

Theorem

There exists a 2-party semi-honest secure protocol that

- *performs only one call to* $\mathsf{OLE}(\mathbb{K})$ *(no additional communication)*
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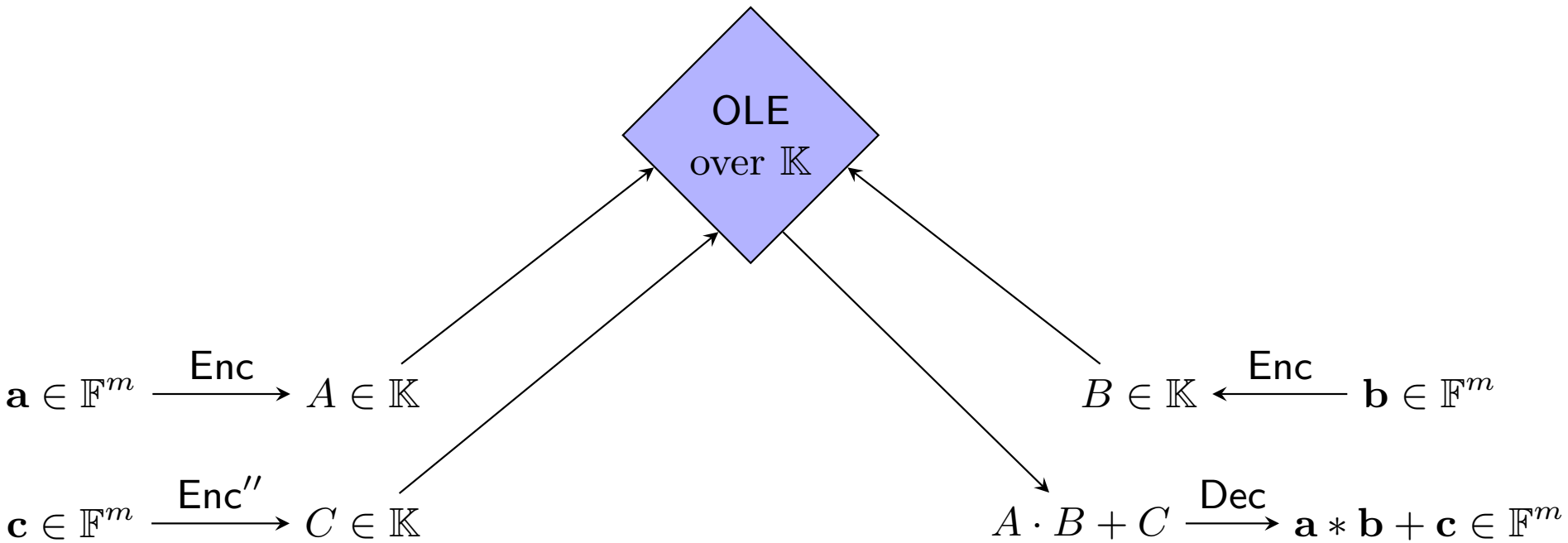

Assuming Linear-rate Solution for the Embedding Problem: Consequences for MPC (Corollary 1)

Corollary

There exists a computationally secure protocol that is

- *Based on any computational hardness assumption that helps construct one* OLE *over an extension field, and*
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Example (Computational Hardness Assumption)

- Ishai-Prabhakaran-Sahai [IPS09] and Naor-Pinkas [NP06]: Pseudorandomness of noisy random RS codewords $\Rightarrow \mathsf{OLE}(\mathbb{K})$.
- Applebaum et al. [ADI+17]: Arithmetic analogues of well-studied cryptographic assumptions $\Rightarrow \mathsf{OLE}(\mathbb{K})$.

Assuming Linear-rate Solution for the Embedding Problem: Consequences for MPC (Corollary 2)

Corollary

For all $\varepsilon \in [0, 1/2)$, there exists an n-bit correlated private randomness such that

- *Despite $t = (1/2 - \varepsilon)n$ bits of leakage*
- *We can securely compute a linear number of independent and secure OTs from the leaky correlated private randomness*

Prior Best Solution to the Embedding Problem: BMN Embedding

Theorem (Block-Maji-Nguyen [BMN17])

There exist two mappings $\mathsf{Enc} : \mathbb{F}^m \to \mathbb{K}$ *and* $\mathsf{Dec} : \mathbb{K} \to \mathbb{F}^m$ *satisfying*

- $m = n^{1-o(1)}$
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Note

Reduces the construction to the famous "Tri-colored Sum-free Set" problem in combinatorics (Kleinberg [Kle16], Blasiak et.al. [BCC+17], Kleinberg-Speyer-Sawin [KSS18]).

Let Us Try to Solve the Embedding Multiplication Problem

Principal Observation: Two equivalent ways to multiply polynomials

- Direct technique: compute the convolution of the two polynomials

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Implications

There are also 2 ways to perform coordinate-wise product.

First Attempt: Reed-Solomon Codes + BMN Embeddings

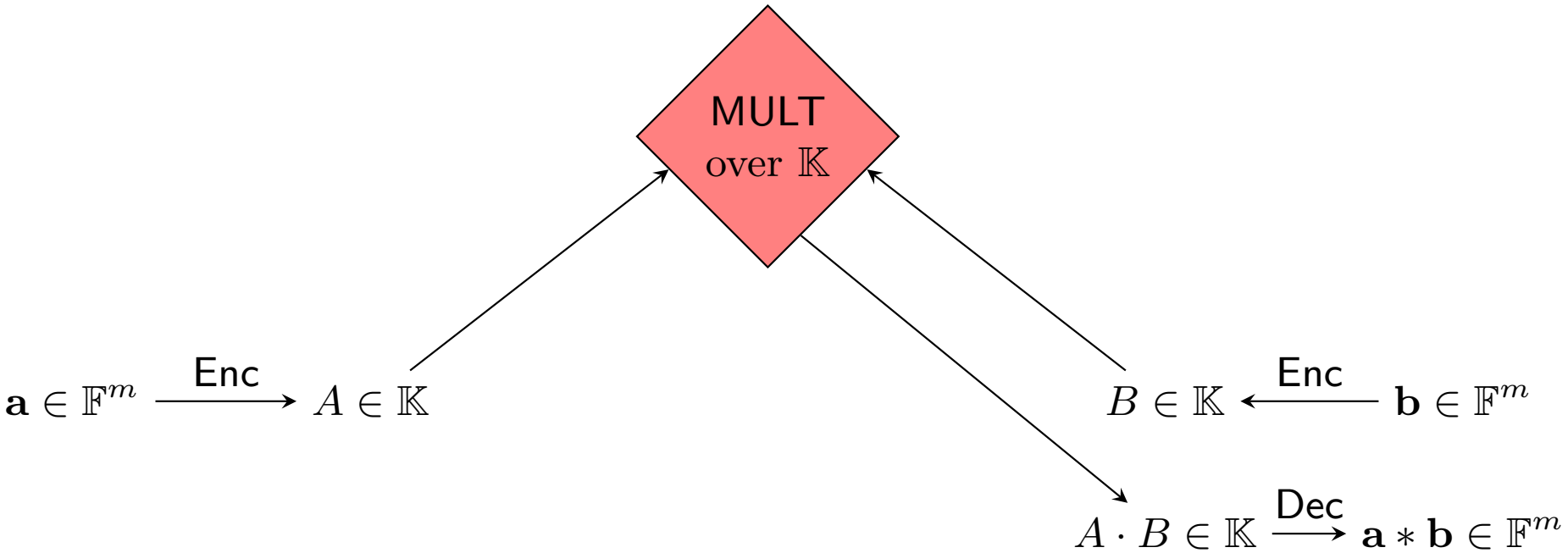
Reed-Solomon Codes

- For a field $\mathbb{F}$ of size $|\mathbb{F}| \geqslant \eta$, and $S = \{\alpha_1, \alpha_2, \ldots, \alpha_\eta\} \subseteq \mathbb{F}$, the Reed-Solomon code with block length η and dimension k is defined as

$$\mathsf{RS}_{\mathbb{F},S}(\eta, k) = \{(p(\alpha_1), \ldots, p(\alpha_\eta)) \in \mathbb{F}^\eta \mid \deg p(X) \text{ is less than } k\}.$$

First Attempt: Reed-Solomon Codes + BMN Embeddings

Recall that we want to do:



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Think about $\mathbf{a}, \mathbf{b}$ as codewords in $\mathsf{RS}_{\mathbb{F},S}(m, m)$.

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If $|\mathbb{F}| \geqslant m = n/2$, let $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ be distinct elements of $\mathbb{F}$.

$$\mathbf{a} \in \mathbb{F}^m \xrightarrow[\text{L.I.}]{\mathsf{Enc}} \begin{array}{l} A(X) \in \mathbb{K} \\ \deg A < m \\ A(\alpha_i) = a_i \end{array}$$

$$\begin{array}{l} B(X) \in \mathbb{K} \\ \deg B < m \\ B(\alpha_i) = b_i \end{array} \xleftarrow[\text{L.I.}]{\mathsf{Enc}} \mathbf{b} \in \mathbb{F}^m$$

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- Enc is Largrange Interpolation (LI).

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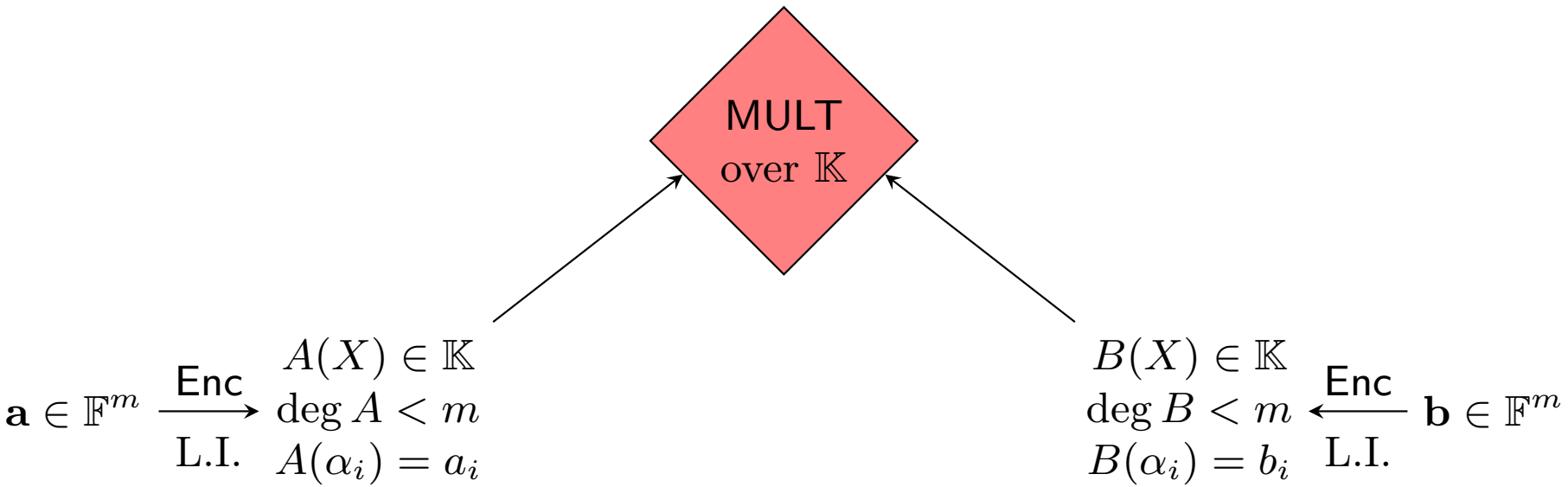

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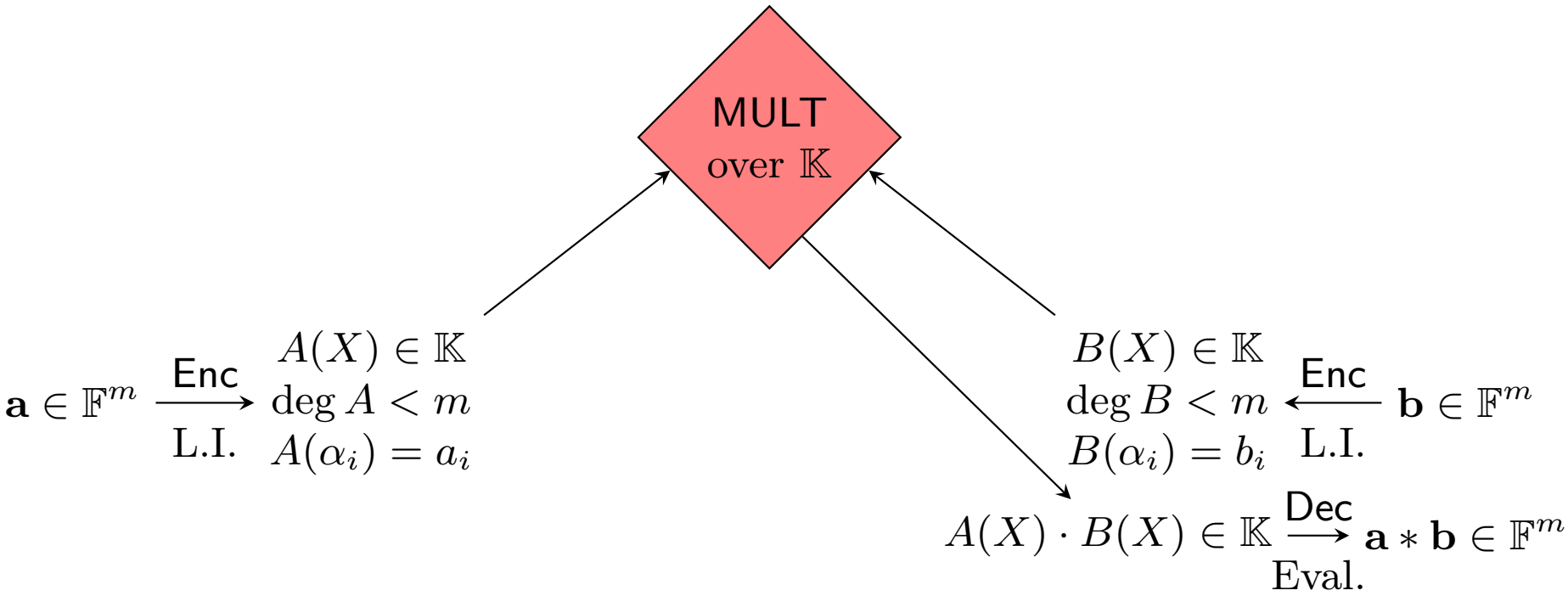

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- Dec is an evaluation (Eval.) map such that $A(\alpha_i) \cdot B(\alpha_i) = a_i \cdot b_i$.

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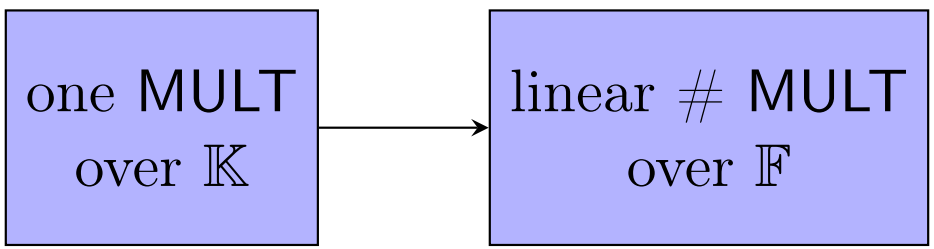
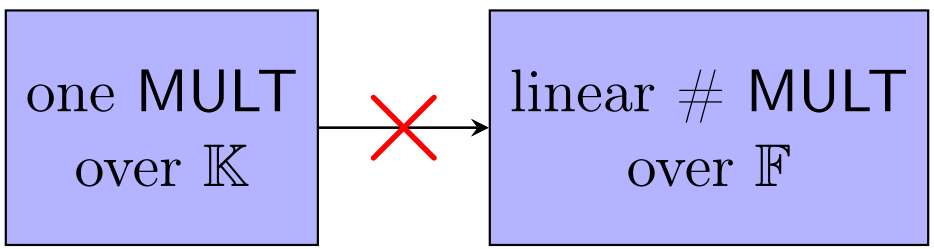
If $\mathbb{F}$ has sufficiently many places ($|\mathbb{F}|$ is linear in n)



First Attempt: Reed-Solomon Codes + BMN Embeddings

What if $\mathbb{F}$ does not have enough places ($|\mathbb{F}|$ is constant)?



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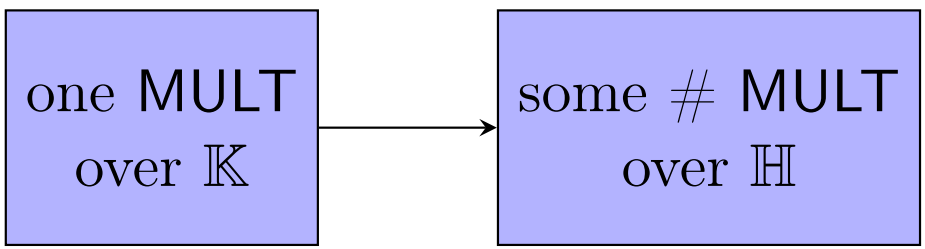
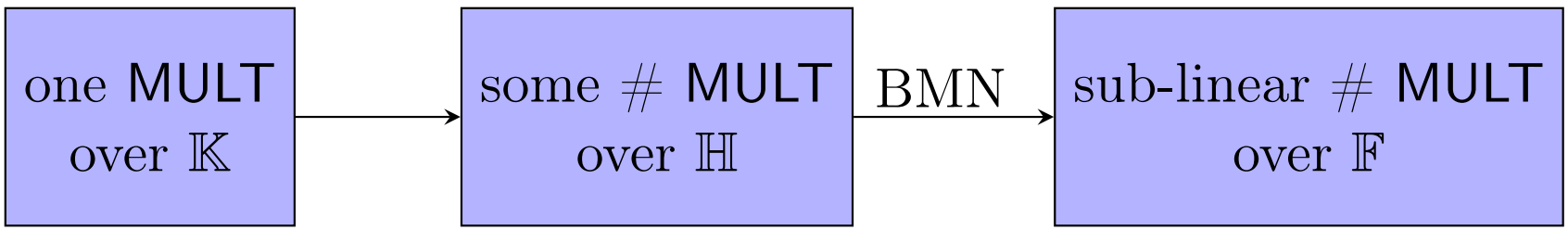


Note

- $\mathbb{K} = \mathbb{GF}[2^n], \mathbb{H} = \mathbb{GF}[2^s], \mathbb{F} = \mathbb{GF}[2]$, where $s \approx \lg n$
- RS encoding: n/s copies of $\mathsf{OLE}(\mathbb{H})$

First Attempt: Reed-Solomon Codes + BMN Embeddings

What if $\mathbb{F}$ does not have enough places ($|\mathbb{F}|$ is constant)?



Note

- $\mathbb{K} = \mathbb{GF}[2^n], \mathbb{H} = \mathbb{GF}[2^s], \mathbb{F} = \mathbb{GF}[2]$, where $s \approx \lg n$
- RS encoding: n/s copies of $\mathsf{OLE}(\mathbb{H})$
- BMN embedding: each $\mathsf{OLE}(\mathbb{H})$ gives $s^{1-o(1)}$ copies of OTs
- $\#\mathsf{OT} = n \cdot \dfrac{1}{(\lg n)^{o(1)}}$

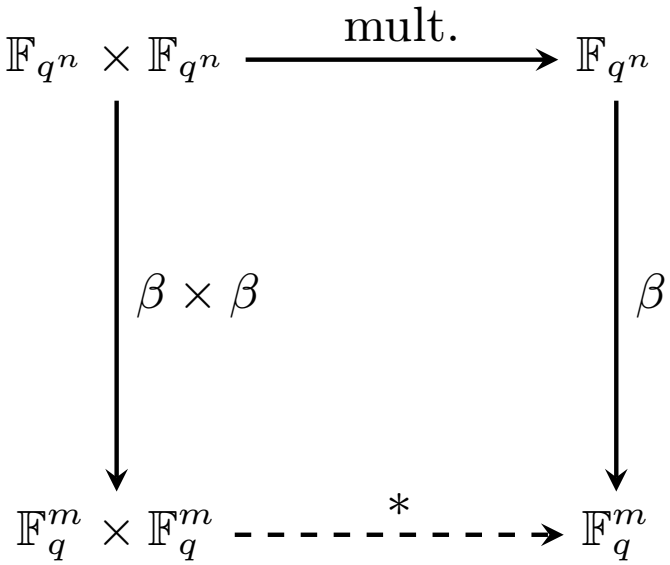
Getting to Multiplication Embeddings with a Linear Rate

Note

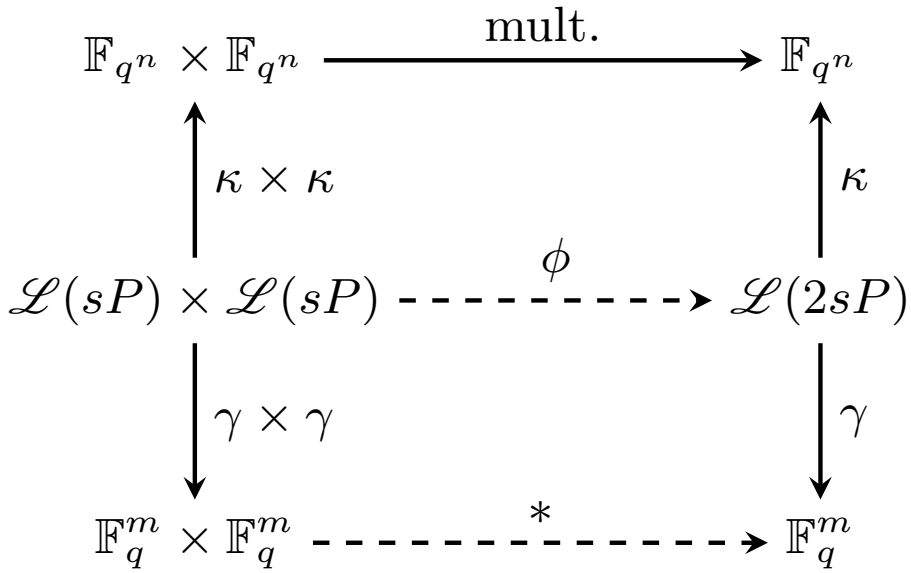
- Reed-Solomon codes: Maximum Distance Separable (MDS) over large-size fields.
- Algebraic Geometry codes (Goppa [Gop81]): near MDS with multiplicative property just like RS but over constant-size fields.

Getting to Multiplication Embeddings with a Linear Rate

Commutative diagrams for multiplication embedding:



Using RS codes



Using AG codes

Note

- $\gamma : z \mapsto (z(P_1), \ldots, z(P_m))$, where P_i's is prime divisors of degree one.
- $\kappa : z \mapsto z(Q)$, where Q is a prime divisor of degree n.
- κ must be surjective to make our embedding work.

Getting to Multiplication Embeddings with a Linear Rate

Note

We use algebraic function fields (Burgisser-Clausen-Shokrollahi [BCS97], Garcia-Stichtenoth [GS96]) to construct the multiplication embeddings for every base field $\mathbb{F}$.

Theorem

There exist two mappings $\mathsf{Enc} : \mathbb{F}^m \to \mathbb{K}$ *and* $\mathsf{Dec} : \mathbb{K} \to \mathbb{F}^m$ *satisfying*

- m *is* *linear* *in* n
- $\mathsf{Dec}(\mathsf{Enc}(\mathbf{a}) \cdot \mathsf{Enc}(\mathbf{b})) = \mathbf{a} * \mathbf{b}$ *for all* $\mathbf{a}, \mathbf{b} \in \mathbb{F}^m$.

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Thank You!