A Las Vegas algorithm to solve Elliptic Curve Discrete Logarithm Problem

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Overview of the talk

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We made some progress in solving Problem L, we report on that.

We implemented our algorithm using NTL, we report on some experimental data.

The discrete logarithm problem

We assume that $\mathcal{E}(\mathbb{F}_q)$ be a group of prime order p. Let P and Q be two non-identity points, such that, $Q = mP$ for $1 \leq m < p$. The discrete logarithm problem is to find the m.

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- Other kind of attacks are the index-calculus kind of attacks on DLP.

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Non-generic attack

Our attack is of the second kind.

A theorem

Definition

An elliptic curve $\mathcal{E}$ over a field $\mathbb{F}_q$ is a non-singular plane curve of degree 3 together with a point $\mathcal{O}$.

Theorem

Let $\mathcal{E}$ be an elliptic curve over $\mathbb{F}_q$ and $P_1, P_2, \ldots, P_k$ be points on that curve, where $k = 3n'$ for some positive integer n'. Then $\sum_{i=1}^{k} P_i = \mathcal{O}$ if and only if there is a curve $\mathcal{C}$ of degree n' that passes through these points. Multiplicities are intersection multiplicities.

How to use the theorem

- Choose k such that $k = 3n'$ for some positive integer n'.
- Choose random non-identity points $P_1, P_2, \ldots, P_s$ and $Q_1, Q_2, \ldots, Q_t$ such that $s + t = k$.
- Check if there is a homogeneous curve of degree n' that passes through these points; where $P_i = n_i P$ and $Q_j = -n'_j Q$ for some integers n_i and n'_j.
- If there is a curve, the discrete logarithm problem is solved.
- Otherwise repeat the process.
- To choose these points P_i and Q_j, we choose a random point n_i, n'_j and compute $n_i P$ and $-n'_j Q$.
- We would choose n_i and n'_j to be distinct from the ones chosen before to give rise to distinct points P_i and Q_j on $\mathcal{E}$.

How do we know if there is a curve

Let $C = \sum_{i+j+k=n'} a_{ijk} x^i y^j z^k$ be a complete homogeneous curve of degree n'. An ordering of i, j, k is fixed and C is presented according to that ordering. By complete we mean that the curve has all the possible monomials of degree n'. We need to check if P_i, $i = 1, 2, \ldots, s$ and Q_j for $j = 1, 2, \ldots, t$ satisfy the curve C. Note that, there is no need to compute the values of a_{ijk}, just mere existence will solve the discrete logarithm problem.

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Let P be a point on $\mathcal{E}$. We denote by $\overline{P}$ the value of C when the values of x, y, z in P is substituted in C. Similarly for Qs. We now form a matrix $\mathcal{M}$ where the rows of $\mathcal{M}$ are $\overline{P_i}$ for $i = 1, 2, \ldots, s$ and $\overline{Q_j}$ for $j = 1, 2, \ldots, t$. If this matrix has a non-zero left-kernel, we have solved the discrete logarithm problem. By *left-kernel* we mean the kernel of $\mathcal{M}^{\mathsf{T}}$, the transpose of $\mathcal{M}$.

Why look at left-kernel

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(B) $\mathcal{K}'$ *only contain curves that are a multiple of* $\mathcal{E}$.

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Take the number of points $k = 3n' + l$, then

Theorem

If $l \geq 1$, *the dimension of the left kernel of* $\mathcal{M}$ *is* l.

What really works

Corollary

Assume that $\mathcal{M}$ has $3n' + l$ rows, computed from the same number of points of the elliptic curve $\mathcal{E}$. If there is a curve $\mathcal{C}$ intersecting $\mathcal{E}$ non-trivially in $3n'$ points among $3n' + l$ points, then there is a vector v in $\mathcal{K}$ with at least l zeros. Conversely, if there is a vector v in $\mathcal{K}$ with at least l zeros, then there is a curve $\mathcal{C}$ passing through those $3n'$ points that correspond to the non-zero entries of v in $\mathcal{M}$.

What is the advantage

In the exhaustive search we would have picked a random set of $3n'$ points and then checked to see if the sum of those points is Q. In the above algorithm we are taking a set of $3n' + l$ points and then checking all possible $3n'$ subsets of this set simultaneously. There are $\binom{3n'+l}{l}$ such subsets. This is one of the main advantage of our algorithm.

The algorithm

A sketch of the algorithm

Choose $k = 3n' + l$.

- Choose points from the elliptic curve.
- Construct the matrix $\mathcal{M}$.
- Compute the left-kernel $\mathcal{K}$ of $\mathcal{M}$.
- If the left kernel contains a vector with l zeros we are done.
- If not, repeat.

Problem L

Problem

Let W be a l-dimensional subspace of a n-dimensional vector space V. The vectors in the vectors space are presented as linear sum of some fixed basis of V. The problem is to determine, if W contains a vector with l zeros. If there is one such vector, find that vector.

Theorem

When p tends to infinity, the probability of success of the above algorithm is approximately $1 - \frac{1}{e} \approx 0.6321$. The number of rows of the matrix $\mathcal{M}$ required to reach this probability is $O(\log p)$. This makes our algorithm polynomial in both time and space complexity.

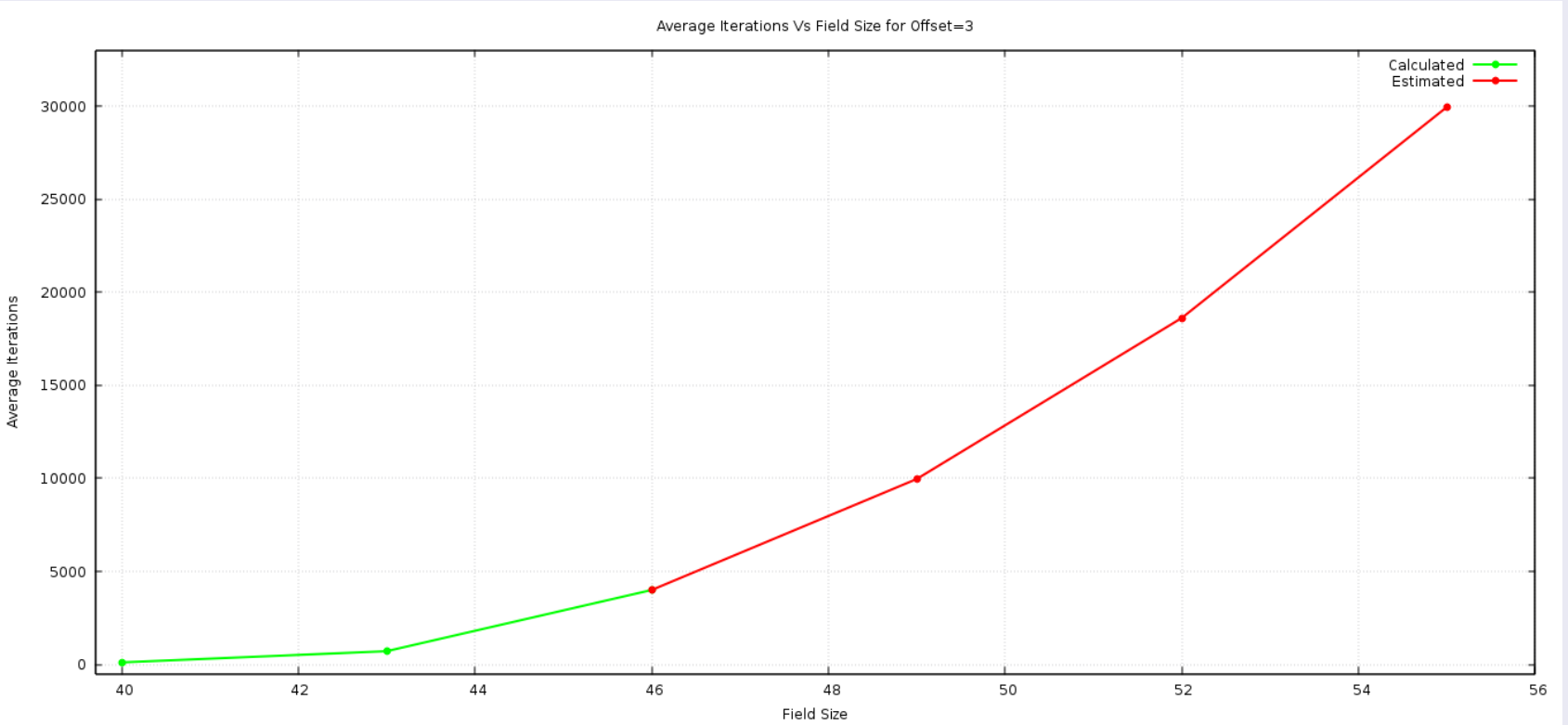
Solving Problem L

Take $3n' = l$. Then $\mathcal{K}$ is two $l \times l$ matrix stacked sideways.

$$\mathcal{K} = \left[\begin{array}{cccc|cccc} * & * & \dots & \cdots & 0 & \cdots & 0 & 1 \\ * & * & \dots & \cdots & 0 & \cdots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & \dots & \cdots & 1 & 0 & \cdots & 0 \end{array}\right]$$

- The matrix $\mathcal{K}$ Is the basis for the left-kernel $\mathcal{K}$. The size of the matrix is $l \times 2l$. We look for a square sub-matrix with determinant zero from the left part.
- In particular, for the purpose of this talk we look at the 2×2 submatrix.

Some experimental results



- Number of iterations 40. The average number of kernels computed is the y-axis. The x-axis is the size of the field.
- Number of cores used 66.