A note on the security of CSIDH

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Shor's algorithm

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All the currently deployed public key infrastructure will need to be replaced.

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In November 30th 2017, NIST initiated a standardization process for PQ primitives.

NIST
National Institute of Standards and Technology
U.S. Department of Commerce

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- (Ideal) lattice-based crypto
- Code-based crypto
- Multivariate crypto
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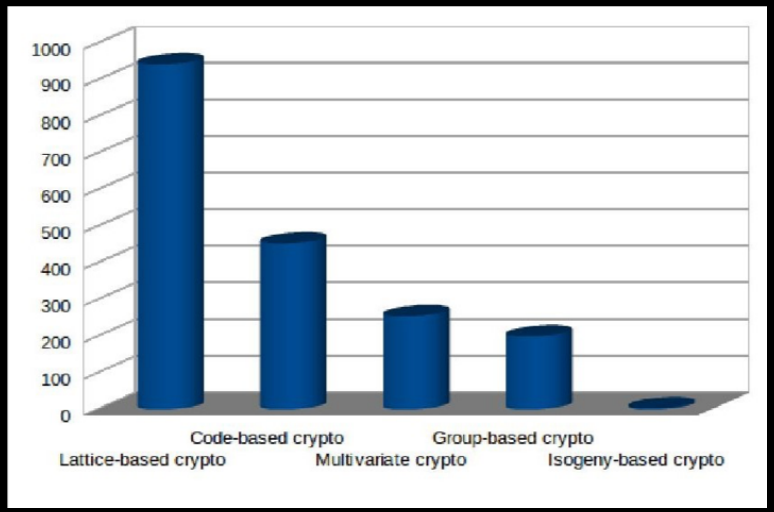
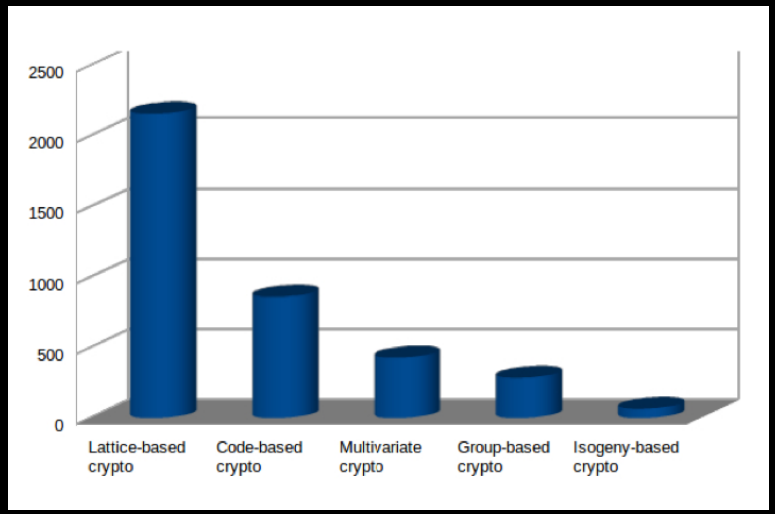


YYZ: 02/01/2015



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Elliptic curves and isogenies

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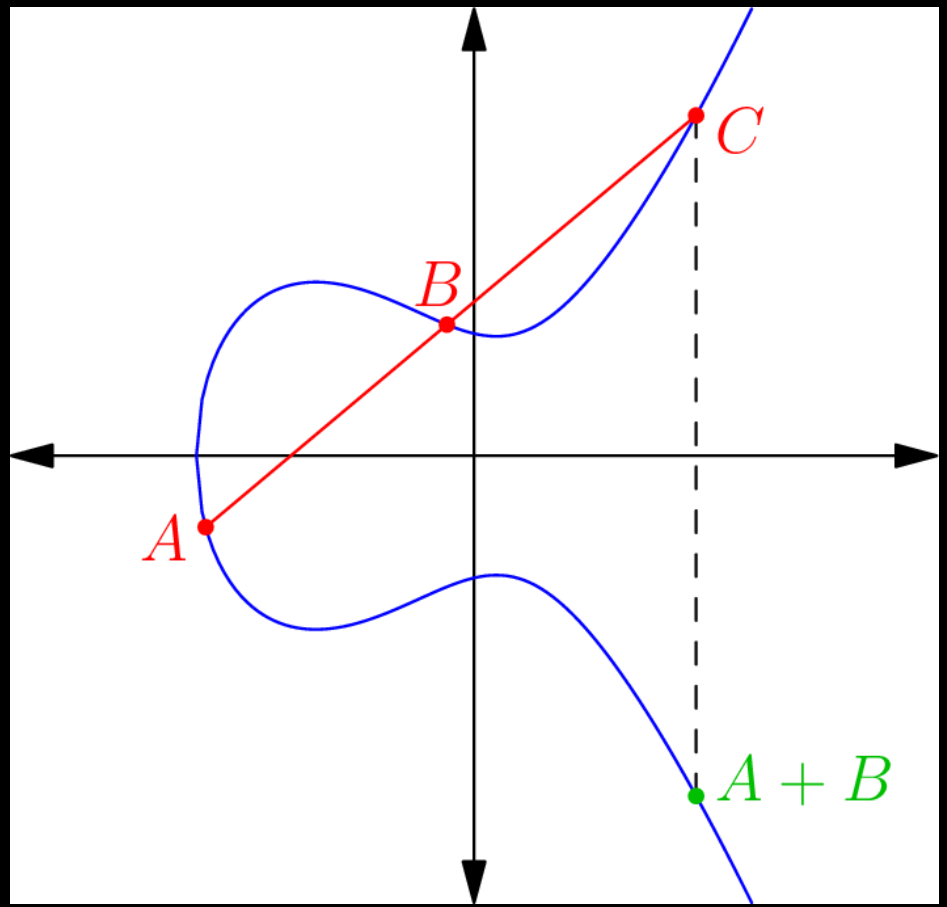

Group law

Elliptic curves and isogenies

$$\begin{array}{rccc} \varphi : & E & \longrightarrow & E' \\ & (x,y) & \mapsto & \left(\frac{f_1(x,y)}{g_1(x,y)}, \frac{f_2(x,y)}{g_2(x,y)}\right) \end{array}$$

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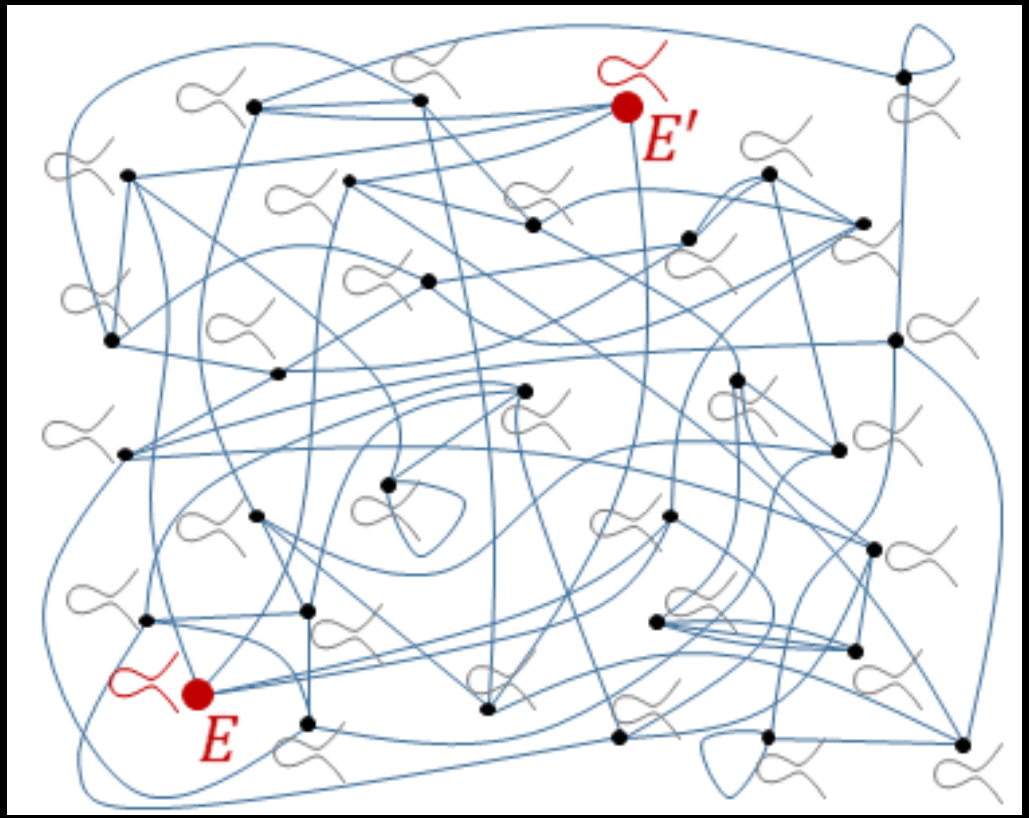

Isogeny graph (source: W. Castryck)

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- **Set**: isomorphism classes of elliptic curves having the same endomorphism ring $\mathcal{O}$.
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$$G = \{[\mathfrak{a}] : \mathfrak{a} \text{ is an ideal of } \mathcal{O}\},$$

where $[\mathfrak{a}] = [\mathfrak{b}] \Leftrightarrow \exists\, 0 \neq \alpha \in \mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $\mathfrak{a} = (\alpha)\mathfrak{b}$.

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$$\varphi_{\mathfrak{a}} : E \to E'$$

with $\deg(\varphi_{\mathfrak{a}}) = \mathcal{N}(\mathfrak{a})$

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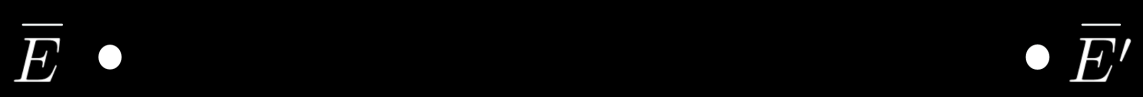
Problem: Given $[\mathfrak{a}] * \overline{E}$ find $[\mathfrak{a}]$.

Traveling the isogeny graph

Suppose that $G = \langle [\mathfrak{p}_1], [\mathfrak{p}_2], [\mathfrak{p}_3] \rangle$



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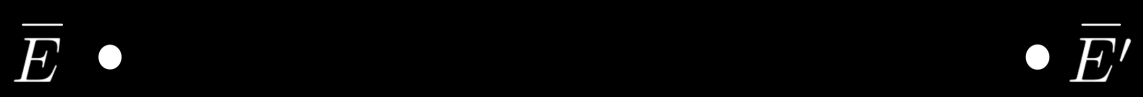

We want to find (short) (e_1, e_2, e_3) such that $[\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\mathfrak{p}_3^{e_3}] * \overline{E} = \overline{E'}$.

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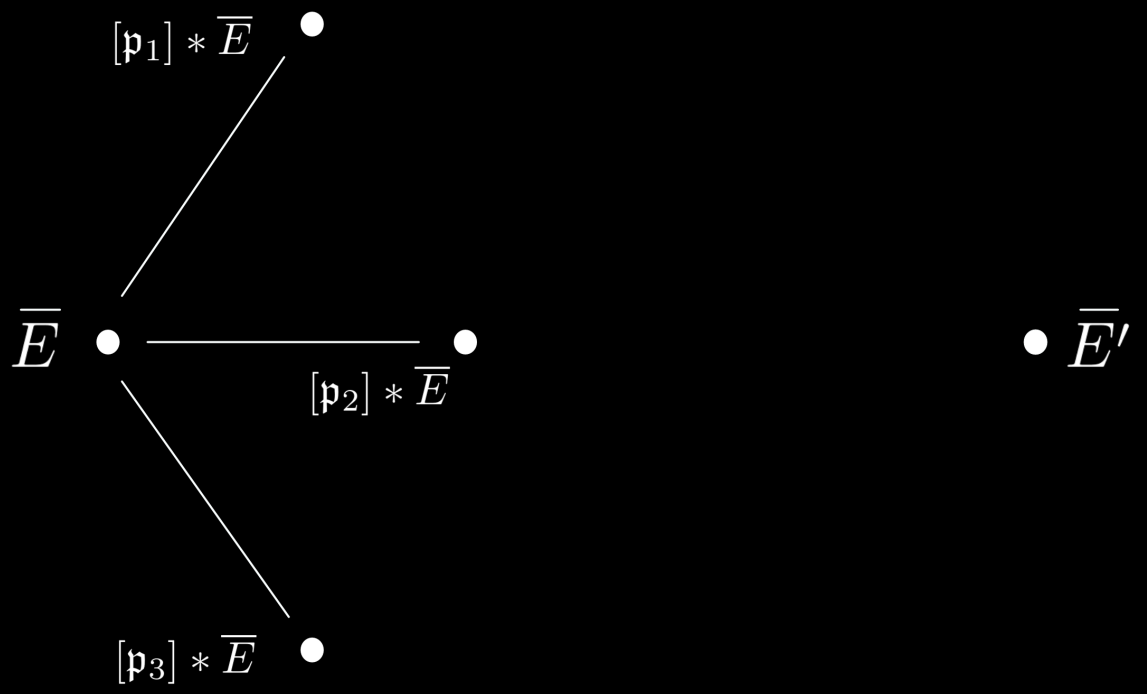

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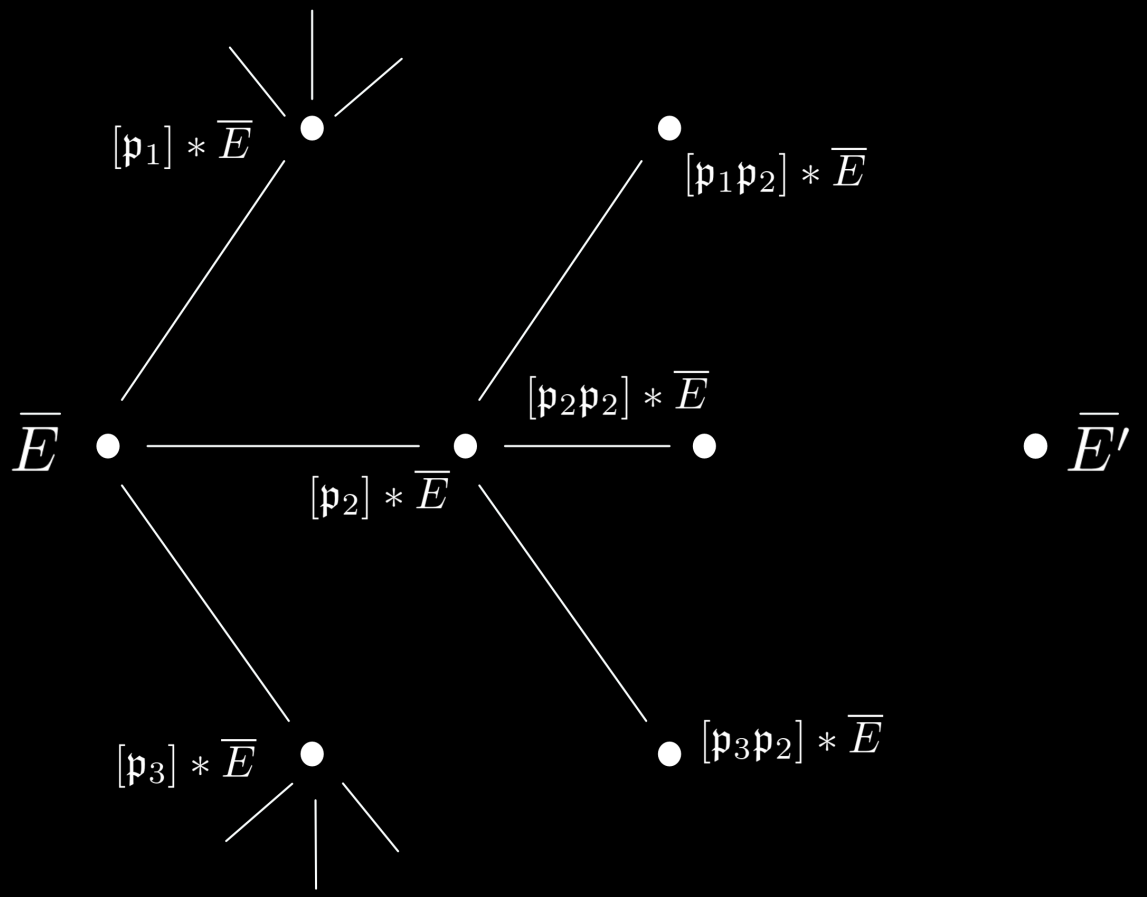

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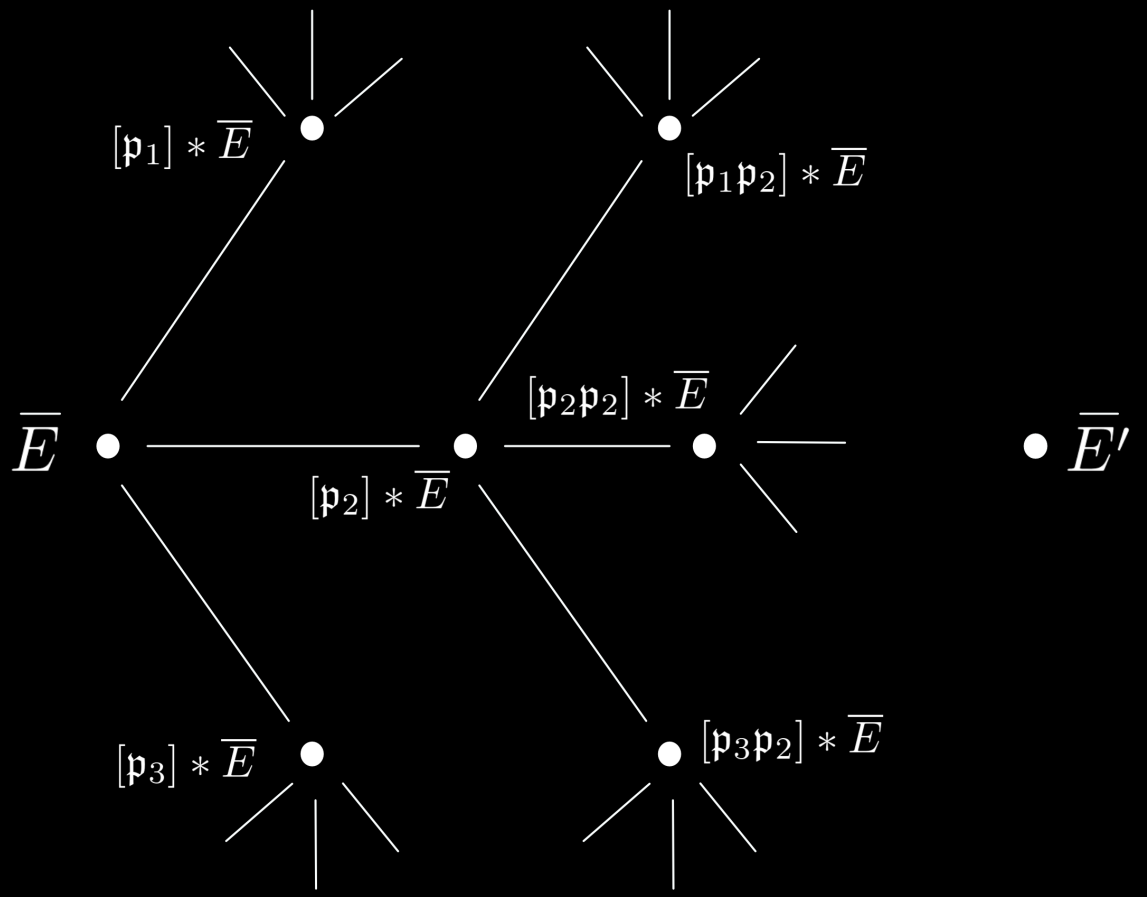

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This can be rephrased as follows.

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Kuperberg's sieve

Finding $[\mathfrak{s}]$ boils down to finding the periods H of an oracle f defined over $\frac{\mathbb{Z}}{2\mathbb{Z}} \ltimes G$ by:

$$f(x,[\mathfrak{a}]) := \begin{cases} |[\mathfrak{a}] * \overline{E}\rangle = |f_1([\mathfrak{a}])\rangle, & x = 0 \\ |[\mathfrak{a}]^{-1} * \overline{E}'\rangle = |f_2([\mathfrak{a}]^{-1})\rangle, & x = 1 \end{cases}$$

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Kuperberg's sieve finds H by using:

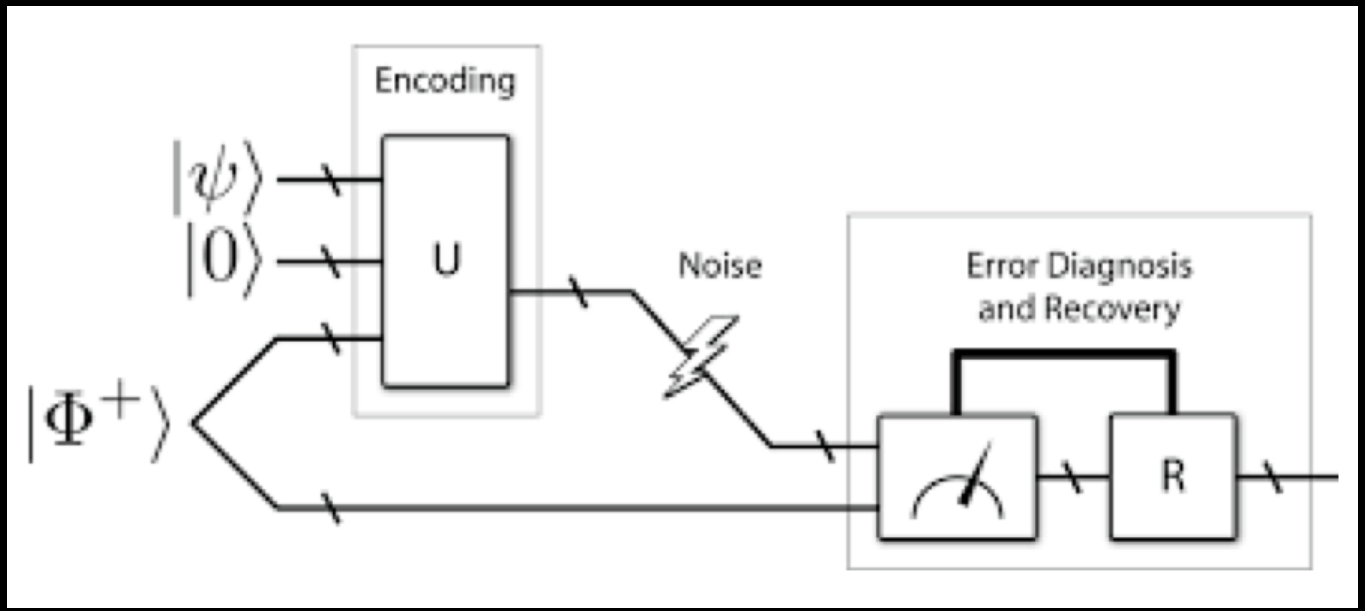
- $2^{O\left(\sqrt{\log(N)}\right)}$ calls to f and $2^{O\left(\sqrt{\log(N)}\right)}$ quantum memory.
- $2^{O\left(\sqrt{\log(N)\log\log(N)}\right)}$ calls to f and polynomial quantum memory.

On the importance of quantum memory

Everyone agrees:

Quantum memory won't be cheap.

It will take a long time to build a quantum computer handling a large amount of memory.



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BDD: Find $\vec{e} \in \mathcal{L}$ such that $\|\vec{y}\| = \|\vec{x} - \vec{e}\|$ is small.

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All steps have polynomial classical and quantum memory requirement.

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Open problems

- Remove heuristics
- Concrete security estimates

ध्यान देने के लिए आपका धन्यवाद

Acknowledgment: research funded by