# Application of Integral Value Transformations in Computer Network Design

Prof. Pabitra Pal Choudhury
Applied Statistics Unit
Indian Statistical Institute, Kolkata
Email: pabitra@isical.ac.in

Collaborators: **Souvik Naskar, Avishek Ghosh**

# Network Design Aspects

Following considerations are of significant importance while designing a network.

- The address resolution between source and destination machine.
- The path selection between the source and destination machine. Some routing algorithms select only the shortest path between source and destination machines.
- In some routing algorithms the cost of sending the data to a machine is included while considering the traffic in a selected path.

# What are to be addressed in our Network Design

- A network design is proposed using some properties of Collatz like bijective Integral Value Transformations(IVT).

- Using IVT the address resolution between the source to destination machine is made. And it is also shown that once the connection is established, there is no need of further address resolution.

# Continued

- We used **Collatz like Integral Value Transformations (IVT)** as tool to explore the above idea.

- Let us explore what is **Collatz like IVTs**. Before coming to the detail of that, let us see what is *Collatz Conjecture*.

# Collatz Conjecture

Consider a function from $\mathbb{N}_0$ to $\mathbb{N}_0$, defined as follows:

$$\mathrm{T}(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (3n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

Next define the iterate of T as usual:

$$\begin{cases} \mathrm{T}^{(0)}(n) = n \\ \mathrm{T}^{(i+1)}(n) = \mathrm{T}(\mathrm{T}^{i}(n)) \end{cases}$$

The question is now to show that for every $n \in \mathbb{N}_0$, there is a finite $k$, such that

$$\mathrm{T}^{(k)}(n) = 1.$$

**This is the conjecture also known as the 3n + 1 conjecture**

# An illustration

*A straightforward example:* take $n = 7$, then we have the following sequence

$$
\begin{array}{cccccccccccc}
7 \rightarrow & 11 \rightarrow & 17 \rightarrow & 26 \rightarrow & 13 \rightarrow & 20 \rightarrow & 10 \rightarrow & 5 \rightarrow & 8 \rightarrow & 4 \rightarrow & 2 \rightarrow & 1 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}
$$

therefore $\mathrm{T}^{(11)}(7) = 1$ and $k = 11$.

**Now we are free to play with some other natural numbers as we wish.**

# It is worth to be aware of the facts:

- The conjecture remains *Unanswered,* although it has been shown to be valid for all natural numbers up to $5.764 * 10^{18}$ .
- Professor Paul ErdŐs once commented on the Collatz conjecture: "Mathematics is not yet ready for such problems”. He offered Rs. 2500000/- for a solution.

# So where is the challenge?

*Challenge*: To define an analogue Conjecture and to prove or disprove it!

This, in our view, will be one step forward towards the settlement of Collatz Conjecture in the sense that we may get some clue for settling the original conjecture.

Before coming to it, let us see what IVTs do.

# Notion of Integral Value Transformations

Now let us define the IVT in $\mathbb{N}_0^{K}$ as the following:

$IVT^{p,k}{}_j : \mathbb{N}_0^{K} \rightarrow \mathbb{N}_0$

$IVT^{p,k}{}_j((n_1, n_2, \ldots . n_k)=$

$(f_j(a_0{}^{n_1}, a_0{}^{n_2}, \ldots, a_0{}^{n_k})\ f_j(a_1{}^{n_1}, a_1{}^{n_2}, \ldots, a_1{}^{n_k}) \ldots \ldots f_j(a_{l-1}{}^{n_1}, a_{l-1}{}^{n_2}, \ldots, a_{l-1}{}^{n_k}))_p = m$

where $n_1 = (a_0{}^{n_1} a_1{}^{n_1} \ldots . a_{l-1}{}^{n_1})_p$, $n_2 = (a_0{}^{n_2} a_1{}^{n_2} \ldots . a_{l-1}{}^{n_2})_p, \ldots \ldots n_k = (a_0{}^{n_k} a_1{}^{n_k} \ldots . a_{l-1}{}^{n_k})_p$

$f_j$: $\{0, 1, 2, \ldots, p-1\}^k \rightarrow \{0,1,2, \ldots, p-1\}$.

m is the decimal conversion from the p adic number.

Let us fix the domain of IVTs as $\mathbb{N}_0$ (k=1) and thus the above definition boils down to the following:

$$\mathrm{IVT}^{p,1}{}_j\,(x) = \left(f_j(x_n)\ f_j(x_{n-1}) \ldots \ldots \ldots f_j(x_1)\right)_p = m$$

where m is the decimal conversion from the p adic number, and $x = (x_n\ x_{n-1} \ldots \ldots x_1)_p$.

Let us denote the set of $\mathrm{IVT}^{p,1}{}_j$ as

$$T^{p,1} = \left\{ \mathrm{IVT}^{p,1}{}_j : \mathbb{N} \rightarrow \mathbb{N} \;\middle|\; \begin{array}{c} 0 \leq j < p^p,\ \mathrm{IVT}^{p,1}{}_j\,(x) = \left(f_j(x_n)\ f_j(x_{n-1}) \ldots \ldots \ldots f_j(x_1)\right)_p = m \\ \text{where m is the decimal conversion from the p adic number} \\ \text{and } x = (x_n\ x_{n-1} \ldots \ldots x_1)_p \end{array} \right\}$$

# $IVT^{2,1}{}_{\#}$

Let us define the IVT in $\mathbb{N}_0$ in 2-adic number systems. There are 4 ($2^{2^1}$) one variable two state cellular automata rules. These are as follows:

| **Variable** | $\mathbf{f_0}$ | $\mathbf{f_1}$ | $\mathbf{f_2}$ | $\mathbf{f_3}$ |
|---|---|---|---|---|
| **0** | **0** | **1** | **0** | **1** |
| **1** | **0** | **0** | **1** | **1** |

$IVT^{2,1}{}_{\#}$ mapping a non-negative integers to a non-negative integers.

$IVT^{2,1}{}_0(a) = \big((f_0(a_n)f_0(a_{n-1}) \ldots f_0(a_1)\big)_2 = b$

$IVT^{2,1}{}_1(a) = \big((f_1(a_n)f_1(a_{n-1}) \ldots f_1(a_1)\big)_2 = b$

$IVT^{2,1}{}_2(a) = \big((f_2(a_n)f_2(a_{n-1}) \ldots f_2(a_1)\big)_2 = b$

and $IVT^{2,1}{}_3(a) = \big((f_3(a_n)f_3(a_{n-1}) \ldots f_3(a_1)\big)_2 = b,$

where ‘a’ is a non-negative integer and $a = (a_n a_{n-1} \ldots a_1)_2$ and ‘b’ is the decimal value corresponding to the binary number.

# An Illustration : p=4, k=1 & j=120

$IVT^{4,1}{}_{\#}$ is mapping a non-negative integer to a non-negative integer.

$$IVT^{4,1}{}_{\#}(a) = \left((f_{\#}(a_n) f_{\#}(a_{n-1}) \ldots f_{\#}(a_1)\right)_4 = b$$

Where 'a' is a non-negative integer and $a = (a_n a_{n-1} \ldots a_1)_4$ and 'b' is the decimal value corresponding to the 4-adic number.

For an example, let us consider $a = 225 = (3201)_4$ and $\# = 120$ so $f_{\#}(0) = 0$; $f_{\#}(1) = 2$; $f_{\#}(2) = 3$ *and* $f_{\#}(3) = 1$

Therefore, $IVT^{4,1}{}_{120}(225) = (f_{120}(3)(f_{120}(2)(f_{120}(0)(f_{120}(1))_4 = (1302)_4 = 114.$

Consequently, $IVT^{4,1}{}_{120}(225) = 114.$

# Collatz like Conjecture (Problem)

Let us consider the iterative scheme $(IVT^{p,1}{}_{j})^{i}(n) = IVT^{p,1}{}_{j}\left((IVT^{p,1}{}_{j})^{i-1}(n)\right)$

**Conjecture**: there exists an integer i such that $(IVT^{p,1}{}_{j})^{i}(n) = 0$

Let us see the above conjecture in case of $p = 2, j = 1$

# Iterative convergence for few numbers

| $X_0$ | *Iterative sequences* |
|---|---|
| 0 | 0 |
| 1 | 0 |
| 2 | 1, 0 |
| 3 | 0 |
| 4 | 3, 0 |
| 5 | 2, 1, 0 |
| 6 | 1, 0 |
| 7 | 0 |
| 8 | 7, 0 |
| 9 | 6, 1, 0 |

| | |
|---|---|
| 10 | 5, 2, 1, 0 |
| 11 | 4, 3, 0 |
| 12 | 3, 0 |
| 13 | 2, 1, 0 |
| 14 | 1, 0 |
| 15 | 0 |
| 16 | 15, 0 |
| 17 | 14, 1, 0 |
| 18 | 13, 2, 1, 0 |
| 19 | 12, 3, 0 |
| 20 | 11, 4, 3, 0 |

# Main Result of the Collatz like Problem in IVT

The iterative scheme $\{X_n\}$ converges to 0 for any given $X_0$ where $X_{n+1}=IVT^{2,1}{}_1(X_n)$.

# Collatz behavioral IVTs in $T^{3,1}$

$IVT^{3,1}{}_{0}$

$IVT^{3,1}{}_{1}$

$IVT^{3,1}{}_{2}$

$IVT^{3,1}{}_{6}$

$IVT^{3,1}{}_{7}$

$IVT^{3,1}{}_{8}$

$IVT^{3,1}{}_{9}$

$IVT^{3,1}{}_{10}$

$IVT^{3,1}{}_{11}$

As a general remark,there are $p^{p-1}-1$ number of Collatz like functions in $T^{p,1}$

# Main Results

Given any natural number N, the p$^{th}$ pre image A can be calculated as follows:

*In other words,*

*For every natural number N, there exists A (A>N) such that* $IVT_{\#}^{P,1}(A)=N$.

*Where,*

$$A = N+(p^{|N|+1}) \times \gamma \text{ , if } IVT_{\#}^{P,1}(0) = \alpha \text{ , } IVT_{\#}^{P,1}(\gamma) = 0$$

$$= N+(p^{|N|}) \times \gamma \text{ , if } IVT_{\#}^{P,1}(\gamma) = \alpha$$

$\alpha$ is MSD of N in p-adic representation |N|. is number of digits in in p-adic representation of N ,$0 < \gamma \leq$ p-1 and $IVT_{\#}^{P,1} \in \Gamma^{P,1}$.Here the # that is used should be Collatz like and bijective.

# An illustration

We take N=486 and applying the above method its $4^{th}$ pre image in $IVT^{4,1}{}_{57}$ system is :

$486+4^{5+1}\times 3=12774$.

If we notice the trajectory of 12774, we find after applying $IVT^{4,1}{}_{57}$ 4 times we get 486 again. The trajectory of 12774 is:

12774 → 1595 → 2892 → 3217 → 486 → 571 → 844 → 145 → 230 → 59 → 12 → 1 → 2 → 3 → 0

# Network Design based on main result

- Consider a network where a number of source machine are trying to send data to a single destination machine.
- The target machine is designated by the number 0.
- Each source machine sends data to the immediate machine with address $IVT^{p,1}{}_{\#}(N)$where N is the address of the source machine and # denotes bijective Collatz like IVT function in p-adic domain.
- The source machine will store the number $IVT^{p,1}{}_{\#}(N)$ along with some information regarding the machine with address $IVT^{p,1}{}_{\#}(N)$, so that for further communications it will not compute or search for the machine to send its data. It will straightly forward data to the machine designated as $IVT^{p,1}{}_{\#}(N)$. So a virtual link (path) is established between machines N and $IVT^{p,1}{}_{\#}(N)$.

# Continued..

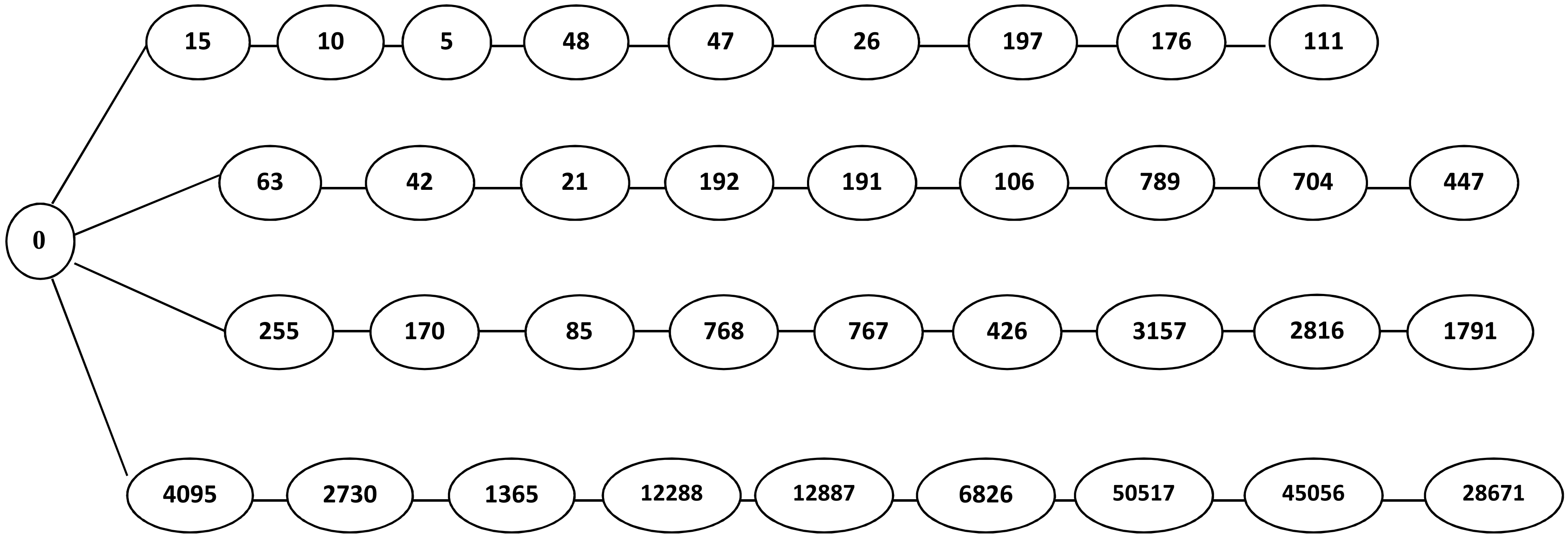
- As the # function  is Collatz like, it is assured that there exists a path from any source machine to the target machine as the destination machine is designated by 0 which is the attractor of the Collatz like IVT functions.
- Assume machine M sends data to destination machine in finite number of hops. So a virtual link is established between M and destination machine. For a given M, a natural number A can be found whose trajectory includes M.
- When A tries to communicate and send its data to the destination, after p iterations (hops) it reaches M. Once it reaches M, the path from M to destination is already established in the sense that no computing and searching is needed to route data from M to the target machine.

# Network Design

- The destination node will be connected to some parallel lines. Each parallel line will be formed by a number of phases. Each phase will accommodate p number of machines.
- For designing the network, three parameters are used: the number of machines that are connected to the destination node, denoted by v, the number of phases in each parallel line, denoted by n , and the p for which the $IVT^{p,1}_{\#}(N)$ will be computed.
- Using these parameters, we can accommodate at most $(n \times p+1) \times v+1$ number of nodes.

# An illustration

As an example, for p=4 and # =57, the network looks like this:



# Inference Drawn

We found the expression of finding the pth pre-image of a natural number in Collatz like bijective $IVT^{p,1}_{\#}$.Using that we designed a network that reduces the number of address calculations.

# Proposed design Aspects

In our proposed design,

- The address resolution is done by using the Collatz like bijective IVT functions on the address of the source machine.
- By applying the IVT functions iteratively on a source node address, we will get a set of node addresses. And the path which contains all these node address will be the selected path.
- We consider that cost of each single connection is constant.
- Since there is only one path from the source to destination machine, the traffic in the network is not considered.

# Concluding remarks

- In the network design, the choice of p depends on the total number of nodes in the network. The designer may chose the value of n and v in such a way that the value of p doesn't exceed 5. By doing so, we not only make the network design feasible but also reduce the number of unused nodes.
- In most cases number of parallel paths may be restricted by the network topology, so v is restricted. The parameter n, which denotes the number of phases in the network is unrestricted and for large n, the number of optimal pairs is increased.
- A further extension of the design can be done by varying the number of nodes in a parallel line or using variable p for each parallel line.

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Thank You !!!.