Integral Value Transformations: A Class of Affine Discrete Dynamical Systems and an Application

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Abstract

In this paper, the notion *Affine Discrete Dynamical Systems* (ADDS) in terms of *Integral Value Transformations* (IVTs), is introduced and their mathematical properties, particularly their equilibrium and stability as a dynamical system, are studied. It is shown that some ADDS are Collatz-like, which generate fractals. It is also shown that ADDS can model *Optimal Distributed and Parallel Environment* (ODPE).

Keywords: *Integral Value Transformations, Affine Discrete Dynamical System, Fractal, Collatz-like IVTs and Optimal Distributed Parallel Computing Environment.*

1. Introduction: In our earlier studies [1, 2, and 3] we have introduced the concept of *Integral Value Transformation* (IVT) by defining it as p-adic, k-dimensional map denoted by $\mathrm{IVT}^{p,k}{}_{j}$. with domain $\mathbb{N}_0{}^{K}$ and range $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that

$IVT^{p,k}{}_{j}((n_1, n_2, \dots . n_k) =$

$(f_j(a_0{}^{n_1}, a_0{}^{n_2}, \dots, a_0{}^{n_k})\ f_j(a_1{}^{n_1}, a_1{}^{n_2}, \dots, a_1{}^{n_k}) \dots \dots f_j(a_{l-1}{}^{n_1}, a_{l-1}{}^{n_2}, \dots, a_{l-1}{}^{n_k}))_p = m$

where $n_1 = (a_0{}^{n_1} a_1{}^{n_1} \dots . a_{l-1}{}^{n_1})_p$, $n_2 = (a_0{}^{n_2} a_1{}^{n_2} \dots . a_{l-1}{}^{n_2})_p, \dots \dots$ $n_k = (a_0{}^{n_k} a_1{}^{n_k} \dots . a_{l-1}{}^{n_k})_p$

$f_j : \{0, 1, 2, \dots, p-1\}^k \to \{0,1,2, \dots, p-1\}$ and m is the decimal conversion from the p adic number.

If k=1 the domain becomes $\mathbb{N}$ and definition boils down to the following:

$$\mathrm{IVT}^{p,1}{}_{j}\ (x) = \left(f_j(x_n)\ f_j(x_{n-1}) \dots \dots \dots f_j(x_1)\right)_p \ = m$$

where m is the decimal conversion from the p adic number, and $x = \ (x_1\ x_2 \dots \dots x_n\ )_p$

In our earlier work [1], we have also mentioned, in the background of Collatz conjecture, the notion of Collatz-like function f with domain $\mathbb{N}_0$ and range $\mathbb{N}_0$, which being iteratively takes any point of $\mathbb{N}_0$ to a fixed point. That is $\exists\ m\ \in\ \mathbb{N}_0$ such that $f^m(n) = c$, c is a fixed point over the iterations for any choice of n.

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In the same work it is shown that some $\mathrm{IVT^{p,1}}_j$ are Collatz-like Integral Value Transformations as the iterative scheme $X_{n+1} = \mathrm{IVT^{p,1}}_j\,(X_n), n = 0, 1, \ldots n, \ldots\ldots$ for such $\mathrm{IVT^{p,1}}_j$ converges to the fixed point c for any given X_0 (belongs to $\mathbb{N}_0$). Further this iterative scheme also serves as a one-dimensional discrete dynamical system (DDS) [3,4]. In the present study, a particular class of such one-dimensional discrete dynamical system, named as affine discrete dynamical system (ADDS) is introduced in section 2 and its various characteristics including its stability are studied in subsequent sections of 3 and 4. In section 5, it is shown that an *Optimal Distributed Parallel Computing Environment* (ODPE) can be designed with the help of Collatz like ADDS.

2. Affine Discrete Dynamical System (ADDS) and their characterization:

2.1 Definitions of ADDS

Definition 2.1: The *ADDS of Type-I* is defined as

$$Y_{t+1} = A\,\mathrm{IVT^{p,1}}_j\,(Y_t) + B;\; A = 1, 2, \ldots., p-1.\; B = 0, 1, 2, .., p-1. \ldots\ldots (2)$$

With y_0 as the initial state with $t = 0$.

Hence,

$$Y_1 = A\,\mathrm{IVT^{p,1}}_j\,(Y_0) + \mathrm{B}, Y_2 = A\,\mathrm{IVT^{p,1}}_j\,(Y_1) + \mathrm{B} = A\,\mathrm{IVT^{p,1}}_j\,\big(A\,\mathrm{IVT^{p,1}}_j\,(Y_0) + B\big) + \mathrm{B},$$

Hence, the generalized form of the above iterative definition can be summarized as

$$Y_t = A\,\mathrm{IVT^{p,1}}_j\,(A\,\mathrm{IVT^{p,1}}_j\,(A\,\mathrm{IVT^{p,1}}_j\,(\ldots\ldots.\mathrm{IVT^{p,1}}_j\,(A\,\mathrm{IVT^{p,1}}_j\,(Y_0) + B) + \ldots\ldots.. + B) + B$$

Definition 2.2: The *ADDS of Type-II* is defined as

$$Y_{t+1} = \mathrm{IVT^{p,1}}_j\,(a\,Y_t + b);\; a = 1,2, \ldots., p-1.\; b = 0,1,2, .., p-1 \ldots\ldots (3)$$

With y_0 as the initial state with $t = 0$ Hence,

$$Y_1 = \mathrm{IVT^{p,1}}_j\,(a\,Y_0 + b), Y_2 = \mathrm{IVT^{p,1}}_j\,(a\,Y_1 + b) = \mathrm{IVT^{p,1}}_j\,(a\,\mathrm{IVT^{p,1}}_j\,(a\,Y_0 + b) + b),$$

Hence, the generalized form of the above iterative definition can be summarized as

$$Y_t = \mathrm{IVT^{p,1}}_j\,(a\,\mathrm{IVT^{p,1}}_j\,(a\,\mathrm{IVT^{p,1}}_j\,(\ldots\ldots.\mathrm{IVT^{p,1}}_j\,(a\,\mathrm{IVT^{p,1}}_j\,(a\,Y_0 + b) + b) + \ldots\ldots.. + b)$$

2.2 Steady State Equilibrium of ADDS

Definition 2.3: A system is said to reach a steady state equilibrium iff, once the state is reached the system will remain in that state for future iterations [4].

i.e. $Y_{t+1} = Y_t$ Also, …. = $Y_{t+2} = Y_{t+1} = Y_t$ (successive iterative value after t remains the same). For the above definition of affine class, let $\bar{Y}$ be the steady state equilibrium. Then the ADDS of type-I and ADDS of type-II become $\bar{Y} = A\,\mathrm{IVT^{p,1}}_j\,(\bar{Y}) + B$ and $\bar{Y} = \mathrm{IVT^{p,1}}_j\,(a\,\bar{Y} + b)$ respectively.

Solution of the above equation will yield steady state equilibrium points. It may or may not be unique depending upon $\mathrm{IVT^{p,1}}_j$, $a, b,$ A and B.

It may be mentioned that about steady state equilibrium point $\bar{Y}$, $\mathrm{IVT^{p,1}}_j\,(Y_t)$ and $\mathrm{IVT^{p,1}}_j\,(a\,Y_t + b)$ can, respectively, be expanded as Taylor series as

$$\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(Y_t) \;=\; \mathrm{IVT^{p,1}}_{\mathrm{j}}(\bar{Y}) \;+ \mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})\,(Y_{\,t} - \bar{Y}) \;+\; (\mathrm{D^2IVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})\,(Y_{\,t} - \bar{Y})2)/\,2! \;+\; \ldots\ldots..$$

and

$\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(a\,Y_t \;+\; b) \;=\; \mathrm{IVT^{p,1}}_{\mathrm{j}}(a\,\bar{Y} \;+\; b) \;+ a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b)\,(Y_t - \bar{Y}) \;+\; \ldots\ldots\ldots$

which can be linearized, by neglecting higher order terms, to obtain, respectively, the relations

$\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(Y_t) \;=\; \mathrm{IVT^{p,1}}_{\mathrm{j}}\,(\,\bar{y}) \;+\; \mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{y})\,(Y_t - \bar{y})$

and ,

$\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(a\,Y_t \;+\; b) \;=\; \mathrm{IVT^{p,1}}_{\mathrm{j}}(a\,\bar{Y} \;+\; b) \;+ a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b)\,(Y_t - \bar{Y}) \;+\; \ldots\ldots..,$

These on substitution in (2) and (3) lead respectively to

$Y_{t+1} \;=\; A\,\left(\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(\,\bar{Y}) +\; \mathrm{DIVT^{p,1}}_{\mathrm{j}}(\bar{Y})(Y_{\,t} - \bar{Y})\right) +\; B$

$\quad = \; A\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})Y_t \;+\; A\,\mathrm{IVT^{p,1}}_{\mathrm{j}}(\bar{Y}) + B - A\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})\bar{y} \;=\; \mathcal{A}Y_t \;+\; \mathcal{B}$

Where, $\mathcal{A} \;=\; A\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})$ & $\mathcal{B} =\; A\,\mathrm{IVT^{p,1}}_{\mathrm{j}}(\bar{Y}) - A\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})\bar{Y} + B.$

and

$$Y_{t+1} =\; \mathrm{IVT^{p,1}}_{\mathrm{j}}\,(\,a\,Y_t +\; b) =\; \mathrm{IVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b\,) +\; a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\bar{Y} +\; b)(Y_t - \bar{Y})$$

$$=\; a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b)\,Y_{\mathrm{t}} + \mathrm{IVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} +\; b) - a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b)\bar{Y} = \mathcal{A}\,Y_t +\; \mathcal{B}$$

where, $\mathcal{A} \;=\; a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}(a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}} \;+\; b)$ & $\mathcal{B} =\; \mathrm{IVT^{p,1}}_{\mathrm{j}}\,(a\,\bar{Y} \;+\; b) -\; a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}(a\,\bar{Y} \;+\; b)\bar{Y}.$

Thus both ADDS Type I and II, as defined in (2) and (3), respectively, on linearization, behave like a linear system with $Y_{t+1} \;=\; \mathcal{A}\,Y_t \;+\; \mathcal{B}$, in small neighborhood of $\bar{Y}$. The **local stability** condition requires [4] that $|\mathcal{A}| \;<\; 1$, which, for linearized ADDS of Type I and ADDS Type II, yields, respectively,

$\quad |\;A\;D\mathrm{IVT^{p,1}}_{\mathrm{j}}\,(\bar{Y})\;| \;<\; 1$ or $|\;D\mathrm{IVT^{p,1}}_{\mathrm{j}}{}'\,(\bar{Y})\;| \;<\; 1/A$ (As A is positive) ,

and

$\quad \left|a\,\mathrm{DIVT^{p,1}}_{\mathrm{j}}\,(a\;\;\bar{Y}\;\;+\;b)\right| <\; 1 \Rightarrow\;\; |\;\mathrm{DIVT^{p,1}}_{\mathrm{j}}(a\,\bar{Y} \;+ b)\;| \;<\; 1/a$ (As a is positive)

Further, the **global stability** condition requires that the $\mathrm{IVT^{p,1}}_{\mathrm{j}}$ has to be a contraction or Lipchitz function, and ADDS Type I and Type II should, respectively, satisfy the conditions

$$\Rightarrow |A\mathrm{DIVT^{p,1}}_{\mathrm{j}}(Y_t)| < 1 \;\; \forall\, t = 0,1,2, \ldots\ldots, \infty$$

and

$$|a\mathrm{DIVT^{p,1}}_{\mathrm{j}}(aY_t + b)\;| < 1 \;\; \forall\; t = 0,1,2, \ldots\ldots, \infty$$

3. Stability Analysis of ADDS

3.1. Illustration of Local Stability

3.1.1. ADDS of Type I:

Consider $Y_{t+1} \;=\; A\,\mathrm{IVT^{2,1}}_{\mathrm{j}}\,(Y_t) \;+\; B$ where A=1 and B is either 0 or 1.

There are only four IVTs for p=2, namely, $\mathrm{IVT^{2,1}}_0$, $\mathrm{IVT^{2,1}}_1$, $\mathrm{IVT^{2,1}}_2$ & $\mathrm{IVT^{2,1}}_3$.

(I) In case of $\mathrm{IVT}^{p,1}{}_0\ (x) = 0$ (Zero function)

Steady state equilibrium point $Y_{t+1} = Y_t = \bar{Y}$

$$\bar{Y} = A\ \mathrm{IVT}^{p,1}{}_0\ (\bar{Y}) + B \Rightarrow \quad \bar{y} = B \quad (As\ \mathrm{IVT}^{p,1}{}_0(x) = 0)$$

It is unique as it is independent of the initial point y_0.

Further $\mathrm{DIVT}_0^{2,1}(\mathrm{x}) = 0$

$So,$ $\mathrm{DIVT}^{2,1}{}_0\ (\bar{y}) = 0$ and So, $|\ A\ D\mathrm{IVT}^{2,1}{}_0\ (\bar{Y})\ | = 0 < 1$

Thus at $\bar{Y} = B$ points are locally stable as well as globally stable because $|AD\mathrm{IVT}^{2,1}{}_0(Y_t)| = 0 < 1\ \forall\ t = 0,1,2, \dots \dots, \infty$

(II) For $\mathrm{IVT}^{2,1}{}_1 = \left(2^k - 1\right) - x$ where x is of k bit, the steady state equilibrium points

$\bar{Y} = A\ \mathrm{IVT}^{2,1}{}_1(\bar{Y}) + B = A(2^k - 1 - \bar{Y}) + B \Rightarrow \bar{Y}(1 + A) = A(2^k - 1) + B \Rightarrow \bar{Y} = \frac{A(2^k-1)+B}{(1+A)}$. (4).

Here $\bar{Y}$ too is represented in k bits.

As A=1, necessarily, two cases arise, viz. B=0 & B=1.

When B=0, $\bar{Y} = \frac{(2^k-1)}{2}$ which does not a natural number for all non-zero values of k. Therefore, no steady state point exists for this case. But the function is Collatz like which converges to Zero.

On the other hand when B=1, $\bar{Y} = 2^{k-1}$ which implies that there exist steady state points, for every k though the function is non-Collatz like.

The function is not differentiable (i.e.D [$\mathrm{IVT}_1^{2,1}\ (x)$] does not exist). So no question of stability (local or global) arises for this function.

(III) For $\mathrm{IVT}^{2,1}{}_2(x) = x,$ (Identity function), the steady state equilibrium equation $Y_{t+1} = Y_t \Rightarrow \bar{y} = A\,\bar{y} + B$, reduces to $\mathrm{Y} = \mathrm{Y} + \mathrm{B}$ as A=1, necessarily. This equation shall hold good for all values of Y, when B=0, hence no unique steady state solution exists.

Further as $\mathrm{DIVT}_2^{2,1}\ (\mathrm{x}) = 1$ and hence $\mathrm{DIVT}_2^{2,1}\ (\bar{Y}) = 1$, the condition for local stability, $|\ A\ \mathrm{DIVT}_2^{2,1}\ (\bar{Y})| < 1$ does not hold good as A=1 necessarily.

(IV) For $\mathrm{IVT}_3^{2,1}(x) = 2^k - 1$ where x is of k bit, the steady state equilibrium points $\bar{Y} = A\ \mathrm{IVT}_3^{2,1}(\bar{Y}) + B = A(2^k - 1) + B$ where $\bar{y}$ is represented in k bits.

Here also two cases arise. For B=0, $\bar{Y} = 2^k - 1$ and for B=1, $\bar{Y} = 2^k$. Hence steady states points exist.

But the above IVT is neither Collatz like nor a differentiable. So the dynamical system for this IVT, is neither locally stable nor globally stable and admits no attractor.

3.1.2. ADDS of Type II

Consider $Y_{t+1} = \mathrm{IVT}_j^{2,1}\left(a\,Y_t + b\right)$, where a=1 and b=0 or 1.

As for ADDS Type I, there are only four IVTs, for p=2, namely, $\mathrm{IVT}^{2,1}{}_0$, $\mathrm{IVT}^{2,1}{}_1$, $\mathrm{IVT}^{2,1}{}_2$ & $\mathrm{IVT}^{2,1}{}_3$.

(I) For $\mathrm{IVT}_0^{2,1}$, $\mathrm{IVT}_0^{2,1}\ (x) = 0$, the steady state equilibrium points $Y_{t+1} = Y_t = \bar{Y}$

$$\bar{Y} = 2^{k-1} = \text{IVT}_0^{2,1}\ (a\,\bar{Y}\ + b\ ) \Rightarrow \quad \bar{Y} = 0$$

It is unique as it is independent of the initial point y_0.

Further as $\text{DIVT}_0^{2,1}(\text{x}) = 0$ So, $\text{DIVT}_0^{2,1}\ (a\,\bar{Y} + b) = 0$ and

$$So, |\ a\ \text{IVT}_0^{2,1}\ (\ a\bar{Y} + b)\ | = 0 < 1$$

$$\text{Also}, |a\ \text{DIVT}_0^{2,1}(a\ Y_t + \text{b})| < 1\ \ \forall\ t = 0,1,2, \ldots \ldots, \infty$$

Therefore the solution $\bar{Y} = 0$ is unique, locally and globally stable.

(ii) For $\text{IVT}_1^{2,1}(x) = (2^k - 1) - x$ where x has k bit representation, the steady state equilibrium points $\bar{y} = \text{IVT}_1^{2,1}(a\,\bar{Y} + b) = (2^k - 1 - a\,\bar{Y} - b)$ where $a\bar{Y} + b$ is represented in k bits.

As a=1, necessarily, two cases arise, viz. $\text{b} = 0$ & $\text{b} = 1$.

When b=0, $\bar{Y} = \frac{(2^k - 1)}{2}$ which does not a natural number for all non-zero values of k. Therefore, no steady state point exists for this case. But the function is Collatz like which converges to Zero.

On the other hand when b=1, $\bar{Y} = 2^{k-1} - 1$ which implies that there exist steady state points, for every k though the function is non-Collatz like.

The function is not differentiable (i.e.D [$\text{IVT}_1^{2,1}\ (x)$] does not exist). So no question of stability (local or global) arises for this function.

(III) For $\text{IVT}_2^{2,1}\ (x) = x$, the Steady state equilibrium equation $Y_{t+1} = Y_t \Rightarrow \bar{Y} = a\,\bar{Y} + b$

$\Rightarrow \bar{Y}\ (1 - a) = b \Rightarrow \bar{Y} + b$, the conclusions are same as in the corresponding case for ADDS of type I.

(IV) For $\text{IVT}_3^{2,1}(x) = 2^k - 1$ where x is k −bit representation, the steady state equilibrium points

$\bar{Y} = \text{IVT}_3^{2,1}(a\,\bar{Y} + b) = (2^k - 1)$ where $a\bar{Y} + b$ is represented in k bits.

But the above IVT is neither Collatz like nor a differentiable. So the dynamical system for this IVT, is neither locally stable nor globally stable and admits no attractor.

3.2 Illustration of Global Stability:

In the previous section, it has been stated in general terms that the global stability of the ADDS requires that $\text{IVT}^{\text{p},1}{}_{\text{j}}$ has to be a contraction that is $\left|\text{IVT}_{\text{j}}^{\text{p},1}(x) - \text{IVT}_j^{p,1}(y)\right| \leq \lambda|x - y|$ where $\lambda \in (0,1)$. It may be observed below that all $\text{IVT}^{2,1}{}_0, \text{IVT}^{2,1}{}_1, \text{IVT}^{2,1}{}_2$ & $\text{IVT}^{2,1}{}_3$ do not satisfy the contraction property, when in 2-adic linearized system is assumed where $A = 1$ & $B = 0$.

(I) For the function, $\text{IVT}_0^{2,1}(x) = 0$,

The contraction condition becomes $\frac{|\text{IVT}_0^{2,1}(x) - \text{IVT}_0^{2,1}(y)|}{|x-y|} = 0 < \lambda\,; \lambda \in (0,1)$

Therefore $\text{IVT}_0^{2,1}$ is a contraction.

(II) For the function $\text{IVT}_1^{2,1}(x) = (2^k - 1) - x$ where x is of k bit number.

Since $\frac{d(\text{IVT}_1^{2,1}(x), \text{IVT}_1^{2,1}(y))}{d(x,y)} = \left|\frac{(2^k - x) - (2^l - y)}{(x-y)}\right|$ where x and y has k and l bit representation respectively. If

x and y has same number of bits i.e. k=l, then $\frac{d(\mathrm{IVT}_1^{2,1}(x),\mathrm{IVT}_1^{2,1}(y))}{d(x,y)} = 1 \nleq \lambda;\ \lambda \in (0,1)$.
So $d\left(\mathrm{IVT}_1^{2,1}(x), \mathrm{IVT}_1^{2,1}(y)\right) \nleq \lambda\, d(x,y)\ \forall\, x, y \in X$. Thus$\mathrm{IVT}_1^{2,1}$ is not a contraction.

(III) It is trivial that $\mathrm{IVT}_2^{2,1}(x) = x$ is not a contraction as $\frac{d(f(x),f(y))}{d(x,y)} = 1\ \forall\, x, y \in X$.

(IV) For the function $\mathrm{IVT}_3^{2,1}(x) = 2^k - 1$ where x is k bit number the contraction condition as follows:

Let $x > y$ and x has higher number of bits in its representation than that of y.

In that case $\frac{d(\mathrm{IVT}_3^{2,1}(x),\mathrm{IVT}_3^{2,1}(y))}{d(x,y)} = \left|\frac{(2^k)-(2^l)}{(x-y)}\right|$

Now, $2^k > x, 2^{k1} > y \Rightarrow |2^k - 2^{k1}| > |x - y|$.$\frac{d(\mathrm{IVT}_3^{2,1}(x),\mathrm{IVT}_3^{2,1}(y))}{d(x,y)} > 1$. Thus$\mathrm{IVT}_3^{2,1}$ is not a contraction.

Therefore, $\mathrm{IVT}_0^{2,1}$ is the only function, which satisfies the contraction condition.

For rest of the functions, the ADDS Type I and ADDS Type II do not exhibit global stability.

4. Properties of Collatz like ADDS

4.1. Relation between Attractors

Theorem 4.1. If Â is the attractor of Collatz-like ADDS of type-I, then $\hat{\mathrm{A}} - \frac{\mathrm{B}}{\mathrm{A}}$ is the attractor of Collatz like ADDS of Type II.

Proof: Let K be the attractor of ADDS of Type II defined by $\mathrm{Y}_{t+1} = \mathrm{IVT}_j{}^{p,1}(\mathrm{A\,Y}_t + \mathrm{B})$ assumed to be Collatz-like. If $\mathrm{Y}_0{}'$ is the initial point and after iterations K is reached, then

$$\mathrm{K} = \mathrm{IVT_j}{}^{\mathrm{p},1}\,(\mathrm{A\,IVT_j}{}^{\mathrm{p},1}\,(\mathrm{AIVT_j}{}^{\mathrm{p},1}\,(\mathrm{A}\dots\dots\dots\dots\dots..\mathrm{AIVT_j}{}^{\mathrm{p},1}\,(\mathrm{A\,Y_0}{}' + \mathrm{B}) + \mathrm{B}) + \mathrm{B}) + \mathrm{B}) + \cdots + \mathrm{B})\dots\dots(5)$$

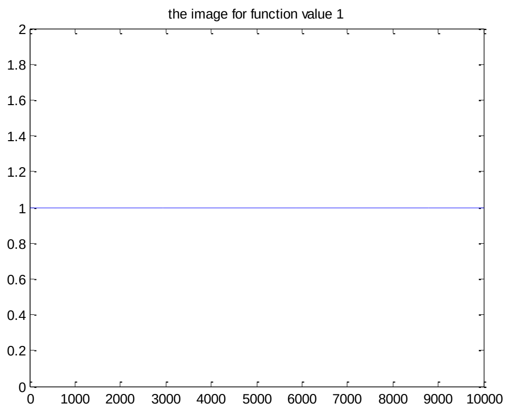
On the other hand if Y_0 is the initial point for the Collatz like ADDS of Type-I defined by $\mathrm{Y}_{t+1} = \mathrm{A\,IVT_j}{}^{\mathrm{p},1}(\mathrm{Y}_t) + \mathrm{B}$.

$$\hat{\mathrm{A}} = \mathrm{A\,IVT_j}{}^{\mathrm{p},1}(\mathrm{A\,IVT_j}{}^{\mathrm{p},1}(\mathrm{A\,IVT_j}{}^{\mathrm{p},1}(\mathrm{A}\dots\dots\dots\dots\dots..\mathrm{A\,IVT_j}{}^{\mathrm{p},1}\,(\mathrm{Y_0}) + \mathrm{B}) + \mathrm{B}) + \mathrm{B}) + \mathrm{B}) + \cdots + \mathrm{B}) + \mathrm{B}\dots.(6)$$

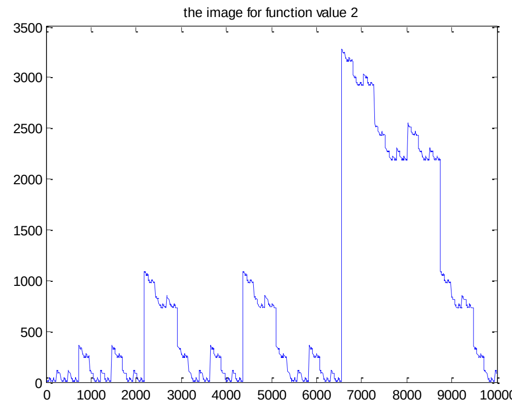
If $\mathrm{Y_0}{}' = \mathrm{Y_0} - \frac{\mathrm{B}}{\mathrm{A}}$, then it may easily be verified that $\hat{\mathrm{A}} = \mathrm{A\,K} + \mathrm{B}$, i.e, $\mathrm{K} = \hat{\mathrm{A}} - \frac{\mathrm{B}}{\mathrm{A}}$.

4.2 Graphical Representation of Collatz like ADS:

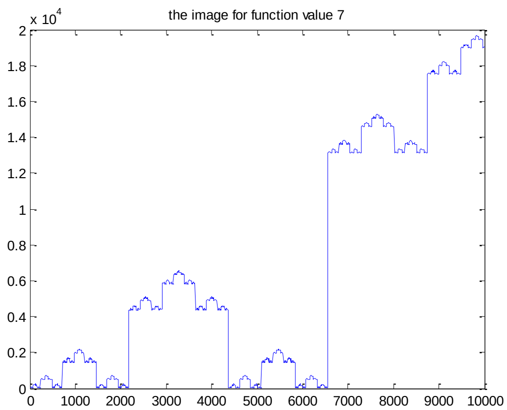
Here we represent a graphical view of dynamics of the ADDS of type-I (similar representation can be done for ADDS of type-II).The graph here depicts the relation $Y_1 = A\, IVT_j^{p,1}(Y_0) + B$, for only four functions are Collatz like $\mathrm{IVT}_0^{3,1}, \mathrm{IVT}_1^{3,1}, \mathrm{IVT}_6^{3,1}$ and $\mathrm{IVT}_7^{3,1}$ when $A = 1, B = 1$ .The following graphs show $Y_1 \sim Y_0$ relationship.



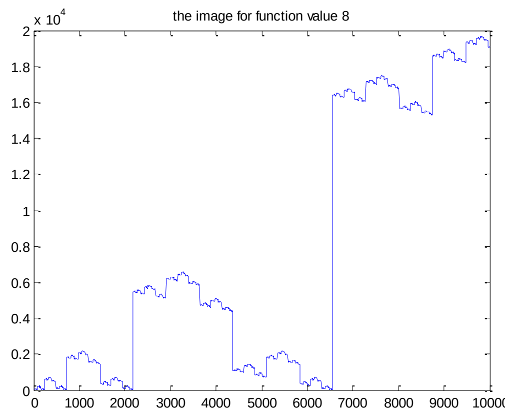
$Y_1 \sim Y_0$ of $\mathrm{IVT}_0^{3,1}$



$Y_1 \sim Y_0 \sim Y_0 \mathrm{IVT}_1^{3,1}$



$Y_1 \sim Y_0$ of $\mathrm{IVT}_6^{3,1}$



$Y_1 \sim Y_0$ of $\mathrm{IVT}_7^{3,1}$

The above graphs being nowhere differentiable and self-repeating (self-similar), imply that Collatz-like ADDS generate *fractals.* By using method of Box Counting Dimension, through the software Benoit, the fractal dimensions of the graphs of the ADDS for four $IVT_j^{p,1}$s (j = 0, 1, 6, 7) are found to be 1, 1.94006, 1.94012 and 1.94016 respectively.

The graphs above which depict the case for single iteration, appear as follows for multiple iterations, where ordinate denotes Y_{i+1} and Y_i is denoted by abscissa.

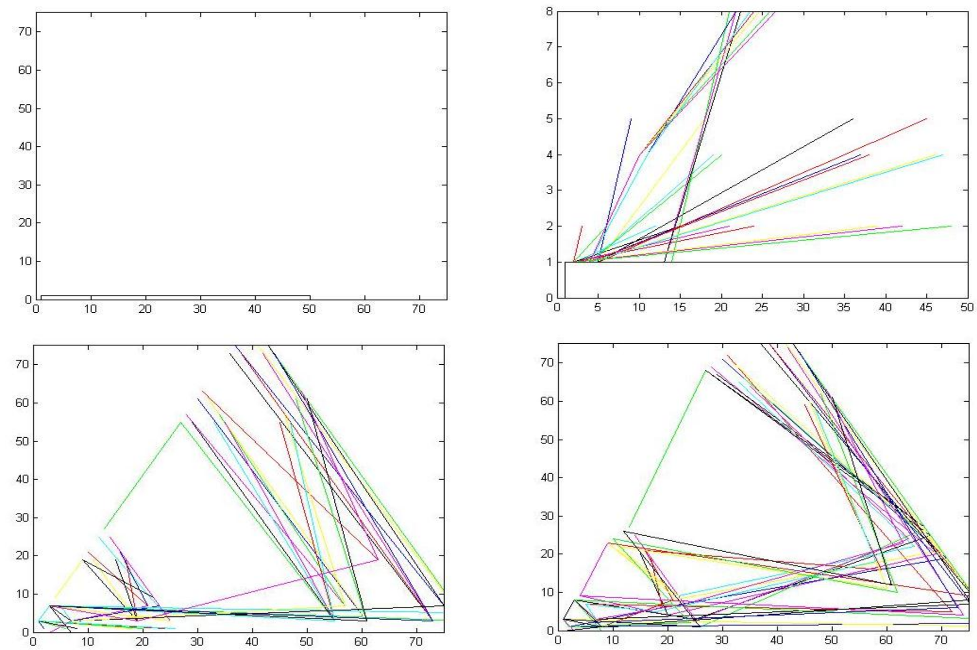


Fig. 1*: Graph of* $Y_{i+1} \sim Y_i$ *of* $IVT_0^{3,1},\ IVT_1^{3,1}, IVT_6^{3,1}$ *and* $IVT_7^{3,1}$

From the above Y_{i+1} vs Y_i graphs, it is intuitively clear that the above Collatz-like IVTs are non-periodic.

5. Application in Distributed and Parallel Environment (DPE):

The essence of distributed Parallel Computing (DPC) [5, 6] is that the central authority (often called the Super Controlling Agent) distributes a set of tasks to its immediate subordinate authorities, which re-distribute their tasks to their corresponding subordinates and this process continues downward finitely. Finally, the agents residing in the lowermost level of the described architecture in parallel computing, the subtasks they are assigned are performed and submitted to their immediate higher authorities through the path where from they got the task. This way the process moves upward and submission continues, and finally the set of completed tasks are submitted to the central authority.

Collatz like ADDS of type-I, namely $Y_{n+1} = \mathrm{IVT}^{p,1}{}_{j}(Y_n)$ is shown here as a model of this process where its attractor represents the central authority.

5.1 Architecture and Requirements for ODPE:

The main constraints, which are considered here for ODPE are the following:

I. *SCA* will communicate with minimum number of *stations*.

II. The *sub-stations* cannot directly communicate with the *SCA*.

III. There is no interaction between the *stations*.

IV. The interaction between sub-stations (i.e. hopping) is minimum.

An example of such design (the 3-layed architecture) has been made as shown in Fig. 3, where Collatz like $IVT_8^{3,1}$ is used, with total number of nodes being 27, to draw a three layer architecture of an ODPE.

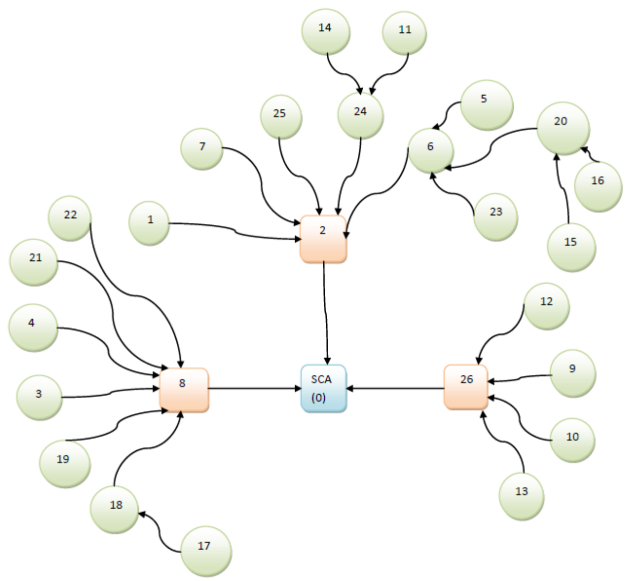


Fig.-3 Scenario of three layered architecture using $IVT_8^{3,1}$ considering 27 nodes.

Here the node designated by 0, the attractor of the ADDS $Y_{n+1} = \mathrm{IVT}^{p,1}{}_{j}(Y_n)$, represents *SCA* of the system. As 2, 8, 16, on single iteration of $IVT_8^{3,1}$ go to 0, the nodes designated by 2, 8, and 16 belong to the middle layer of the architecture and they are called *stations*. Further the numbers, which go to 2, 8 and 16 on single iteration of $IVT_8^{3,1}$ denote the nodes that belong to the outer most layer, and they are called substations. But they are also called substations, which communicate with the stations indirectly i.e., by a number of hops. For example, in Fig. above, *sub-stations* designated by 1, 7, and 25 communicates with 2 directly whereas *sub-station* 16 makes 3 hops ($16 \rightarrow 20 \rightarrow 6 \rightarrow 2$). The *stations* always communicate directly with the *SCA*.

5. 2. Results

The Collatz like IVTs, when p=3 and k=1 are $IVT_8^{3,1}$, $IVT_0^{3,1}, IVT_1^{3,1}, IVT_2^{3,1}, IVT_6^{3,1}, IVT_7^{3,1}, IVT_8^{3,1}, IVT_9^{3,1}, IVT_{10}^{3,1}$ and $IVT_{10}^{3,1}$. Since, the ADDS is linearized domain, so the SCA (attractor of the ADDS of type-I and type-II) is represented by the node designated by 0. The underlying f_js for such Collatz like IVTs are the following.

p	f_0	f_1	f_2	f_6	f_7	f_8	f_9	f_{10}	f_{11}
0	**0**	1	2	**0**	1	2	**0**	1	2
1	**0**	**0**	**0**	2	2	2	**0**	0	0
2	**0**	**0**	**0**	**0**	0	0	1	1	1

Table-2. *The function definitions for Collatz like functions in 3-adic system*

It is quite obvious from the Table-2 that, since more than one entry for functions f_0, f_1, f_2, f_6, f_9 map to 0, $IVT_0^{3,1}, IVT_1^{3,1}, IVT_2^{3,1}, IVT_6^{3,1}$ and $IVT_9^{3,1}$ map more number of natural numbers directly to 0 than the IVTs $\mathrm{IVT}_7^{3,1}, \mathrm{IVT}_8^{3,1}, \mathrm{IVT}_{10}^{3,1}, \mathrm{IVT}_{11}^{3,1}$. In other words the architectures of DPE obtained by using IVTs $IVT_0^{3,1}, IVT_1^{3,1}, IVT_2^{3,1}, IVT_6^{3,1}$ and $IVT_9^{3,1}$ will have more number of stations communicating with SCA than the number of stations communicating with the SCA in the architectures of DPE obtained using IVTsIVT$_7^{3,1}$, $\mathrm{IVT}_8^{3,1}, \mathrm{IVT}_{10}^{3,1}, \mathrm{IVT}_{11}^{3,1}$. The constraint that *SCA* should communicate with minimum number of *stations* for optimal DPE, therefore, makes the IVTs $\mathrm{IVT}_7^{3,1}, \mathrm{IVT}_8^{3,1}, \mathrm{IVT}_{10}^{3,1}, \mathrm{IVT}_{11}^{3,1}$ preferred over IVTs $IVT_0^{3,1}, IVT_1^{3,1}, IVT_2^{3,1}, IVT_6^{3,1}$ and $IVT_9^{3,1}$. The pairs $\{\mathrm{IVT}_7^{3,1}, \mathrm{IVT}_8^{3,1}\}$ and $\{\mathrm{IVT}_{10}^{3,1}, \mathrm{IVT}_{11}^{3,1}\}$.are quite interesting. The first pair takes any number 3^n-1 for n>0 to 0 in one iteration, as the underlying functions f_7 and f_8 always map 2 to 0, whereas the second pair takes any number ½(3^n-1) to 0 in one iteration , as the underlying functions f_{10} and f_{11} always map 1 to 0. So given any number, more numbers less than this given number is likely to go to 0 in one iteration of the IVTs in second pair than the ones in the first pair. For example, if the given number is 14, then 1, 4, and 13 go to 0 in one iteration of $IVT_{10}^{3,1}, IVT_{11}^{3,1}$, whereas only 2 and 8 go to 0 in one iteration of $IVT_7^{3,1}, IVT_8^{3,1}$. Thus given any number, more number of stations is likely in DPE architecture obtained using $IVT_{10}^{3,1}, IVT_{11}^{3,1}$ than the number of stations in DPE architecture obtained using $IVT_7^{3,1}, IVT_8^{3,1}$. So in view of the constraint that *SCA* will communicate with minimum number of *stations* in ODPE, the preferred IVTs are $IVT_7^{3,1}, IVT_8^{3,1}$. Between $IVT_7^{3,1}$ and $IVT_8^{3,1}$, the constraint that in ODPE the number of hopping should be minimum, makes $IVT_8^{3,1}$ more suitable for ODPE as its underlying function f_8 maps both 0 and 1 to 2, whereas f_7, the underlying function of $IVT_7^{3,1}$ maps 0 and 1 to 1 and 2 respectively. Further the average hopping, which is defined as

$$\text{Avg Hopping} = \frac{\text{(Total number of hops)}}{\text{(The natural number upto which hopping is checked)}}$$

is found for $IVT_8^{3,1}$ to be less than the average hopping for $IVT_7^{3,1}$. This is inferred when it is checked with Matlab Codes. The following Fig-3 shows hopping ~ argument (natural number) relationship (the argument is taken upto 100)

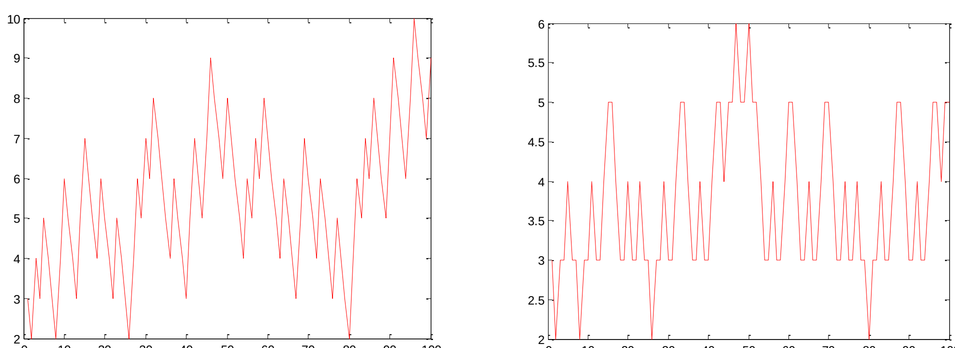


Fig. 4: f_7,$avg\ hop =$ 6.46 $\qquad$ f_8,$avg\ hop$ =4.73

Thus amongst all Collatz-like $IVT_8^{3,1}$, $IVT_0^{3,1}, IVT_1^{3,1}, IVT_2^{3,1}, IVT_6^{3,1}, IVT_7^{3,1}, IVT_8^{3,1}, IVT_9^{3,1}, IVT_{10}^{3,1}$ and $IVT_{10}^{3,1}$, conforming to the constraints for ODPE, $IVT_8^{3,1}$ is the most desirable IVTs for scheduling.

Thus Collatz like IVTs have a great potentiality in optimal designing of DPE.

6. Conclusion

In this paper, a primary study on Collatz like ADDS of type-I and type-II have been presented and using these ADDS an ODPE has been designed. The study of non-linear, multi-dimensional discrete dynamical systems and their characterization in the light of IVTs would be our future endeavor. In practice, the constraints, which are addressed here in designing an ODPE are not sufficient. So in our future research work, we envisage addressing more feasible and practical constraints depending upon nature of the DPE, where the mathematics of ADDS can be more effectively applied.

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