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Sharp Bounds on DMRL and IMRL Classes of Life Distributions with Specified Mean

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Abstract. The class of life distributions with decreasing mean residual life (DMRL) is well known in the field of reliability modeling. In this article, we obtain sharp upper and lower bounds for the reliability function of the DMRL class in terms of its mean. The constructive proofs also establish that both the upper and lower bounds are sharp. We also provide a similar pair of bounds for the reliability function when the distribution is increasing mean residual life (IMRL) with a fixed mean. These inequalities fill a long-standing void in the literature.

Key words: IFR, NBUE, Reliability Bound, Mean Residual Life Function, Reliability Function.

1 Introduction

Suppose F is a distribution function defined over the interval $[0, \infty)$ and $\bar{F} = 1 - F$ is the corresponding reliability/survival function. The distribution F is said to be "decreasing mean residual life" (DMRL) if its mean residual life function, defined as

$$m(t) = \frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)}, t \geq 0 \tag{1}$$

is a non-increasing function. The DMRL property of a distribution is an intuitively meaningful notion of ageing. Therefore, the class of DMRL life distributions, introduced by Bryson and Siddiqui ([9]) in 1969, has been considered by reliability theorists, demographers, actuarial scientists and biometrists as an important class of ageing distributions. The DMRL class of life distributions contains the "increasing failure rate" (IFR) class of distributions ([9]) and is contained in the "new better than used in expectation" (NBUE) class of distributions ([26]). It has partial overlap with the "increasing failure rate average" (IFRA) ([9]) and the "new better than used in" (NBU) classes of distributions. The DMRL class is closed under formation of parallel systems of independent and identically distributed components ([1]), but not closed under convolution, formation of general coherent systems or mixtures ([24]). Klefsjo ([17]) has provided a characterization of the DMRL class through the total time on test (TTT) transform. Various useful properties of the DMRL class of life distributions may be found in Balaban and Singpurwalla ([6]), Willmot and Lin ([27]), Korczak ([19]). Definitions, properties and interrelations of the different ageing classes may be found in Barlow and Proschan ([7]), Gertsbakh ([12]), Lai and Xie ([21]) and Gupta, Zeng and Wu ([13]). Many researchers have proposed tests of exponentiality of a distribution against the DMRL alternative (see [15], [18], [8], [3], [22], [2], [4], [5]). On the other hand the dual of the DMRL class, i.e., the "increasing mean residual life" (IMRL) class, contains the "decreasing failure rate" (DFR) class of distributions and is contained in the "new worse than used in expectation" (NWUE) class of distributions. It has partial overlap with the "decreasing failure rate average" (DFRA) and the "new worse than used" (NWU) classes of distributions and is closed under formation of mixtures of life distributions ([24]).

The main motivation for consideration of a nonparametric class of distributions with an ageing property (such as IFR or DMRL) lies in the common properties enjoyed by members of that class. In particular, bounds on the reliability of an ageing unit, whose life distribution belongs to such a class, has been a topic of great interest among reliability theorists. The interest lies in finding the sharpest possible upper and lower bounds on $\bar{F}(t)$ for a given t, assuming that the lifetime distribution belongs to a class such as IFR or NBU with a specified mean (or a specified quantile or higher moment). Various results in this area are available in the literature. Upper and lower bounds of IFR and IFRA life distributions with a specified mean are available in the classic textbook by Barlow and Proschan ([7]). Lower bounds on the reliability function for the NBU and the NBUE classes of life distributions can be found in Marshall and Proschan ([23]). Korzeniowski and Opawski ([20]) derived an upper bound on the reliability function for the NBU class. Haines and Singpurwalla ([14]) provided an upper bound of the NWUE class of reliability functions. Klefsjo ([16]) has provided both the upper and lower bounds of the reliability functions of the "harmonically new better than used in expectation" (HNBUE) class and its dual, the "harmonically new worse than used in expectation" (HNWUE) class. These results are proved using different techniques and the sharpness of these bounds have been established. Sengupta ([25]) derived the sharpest upper and lower bounds on the reliability of IFR, IFRA, NBU, DFR, DFRA and NWU classes of life distributions, in a unified way. Bounds based on other moments and quantiles are also available. One can see ([21]) for a compilation of results on this topic.

Unfortunately, there is no analogous result for the DMRL and IMRL classes. Cheng and He ([10]) has provided a bound on the gap between a DMRL reliability function and an exponential reliability function with equal mean. This inequality may be used to obtain bounds on a DMRL reliability function given in ([23]). The lower bound on NBUE reliability function and the upper bound given in ([16]) for HNBUE reliability functions are applicable to DMRL reliability functions. However, these bounds may not be sharp.

In this paper, the sharp upper and lower bounds of a reliability function $\bar{F}$ belonging to the DMRL class are obtained in terms of its mean $\mu = m(0) = \int_0^\infty \bar{F}(t)dt$. This is achieved by following the strategy of Sengupta (1994), i.e., by identifying a tractable and parametric subclass over which the optimization would suffice. Results for the negatively ageing class of IMRL distributions are also derived in the same manner.

2 Lower Bound on DMRL Reliability Function

The reliability function $\bar{F}(t)$ can be expressed in terms of the corresponding mean residual life (MRL) function of a life distribution through the relation

$$\bar{F}(t) = \frac{m(0)}{m(t)} e^{-\int_0^t \frac{du}{m(u)}}.$$

This representation and the following property of the MRL function would be useful in the sequel.

Lemma 1. If $m(t)$ is the MRL of a life distribution, then $m'(t) \geq -1$.

Proof. By differentiating (1), we get

$$\begin{aligned} m'(t) &= \frac{-\bar{F}^2(t) + f(t)\int_t^\infty \bar{F}(u)du}{\bar{F}^2(t)} \\ &= -1 + r(t)m(t), \end{aligned}$$

where $r(t)$ is the hazard rate of the distribution F. The stated result follows. □

Let $\mathcal{D}$ be the set of all DMRL reliability functions with mean μ, that is,

$$\mathcal{D} = \left\{ \bar{F} : m(t) \downarrow t, \int_0^\infty \bar{F}(u)du = \mu \right\}. \tag{2}$$

We consider the family of reliability functions $\bar{F}_a(t)$ defined as

$$\bar{F}_a(t) = \begin{cases} 1 & \text{for } 0 \leq t < a, \\ e^{-\frac{t-a}{\mu-a}} & \text{for } a \leq t, \end{cases} \tag{3}$$

for $a \geq 0$. It may be easily verified that the corresponding mean residual life function is

$$m_a(t) = \begin{cases} \mu - t & \text{for } 0 \leq t < a, \\ \mu - a & \text{for } a \leq t. \end{cases}$$

Clearly, $m_a(0) = \mu$. Therefore, $\bar{F}_a$ is a member of $\mathcal{D}$ for all $a \in [0, (t \wedge \mu)]$.

The next lemma shows that minimization of $\bar{F}(t)$ over $\mathcal{D}$ is equivalent to its minimization over the sub-class of reliability functions of the form $\bar{F}_a$.

Lemma 2. Let $\mathcal{D}$ and $\bar{F}_a$ be as defined in (2) and (3), respectively. Then for all $t \geq 0$,

$$\inf_{\bar{F}(t) \in \mathcal{D}} \bar{F} = \inf_{a \in [0,(t \wedge \mu)]} \bar{F}_a(t).$$

Proof. Let $\bar{F}$ be any member of the class $\mathcal{D}$ and m_F be the corresponding MRL function. We shall show that

$$\bar{F}(t) \geq \bar{F}_{a_0}(t), \text{ for } a_0 = \mu - m_F(t).$$

Note that

$$\begin{aligned} m_F(t) &= m_F(0) + \int_0^t m_F'(u)du \\ &\geq \mu + \int_0^t (-1)du \text{ (by Lemma 1)} \\ &= \mu - t, \end{aligned}$$

i.e., $a_0 = \mu - m_F(t) \leq t$. Therefore, $a_0 \leq t \wedge \mu$.

Let m_{a_0} be the MRL corresponding to $\bar{F}_{a_0}$. For $u \in [0, a_0)$,

$$\begin{aligned} m_F(u) &= \mu + m_F(u) - \mu \\ &= \mu + \int_0^u m_F'(v)dv \\ &\geq \mu + \int_0^u (-1)dv \text{ (by Lemma 1)} \\ &= \mu - u \\ &= m_{a_0}(u). \end{aligned}$$

For $u \in [a_0, t]$, $m_{a_0}(u) = \mu - a_0 = m_F(t) \leq m_F(u)$. Therefore,

$$m_{a_0}(u) \leq m_F(u) \ \forall \ u \in [0, t].$$

Further, $m_{a_0}(0) = m_F(0) = \mu$ and $m_{a_0}(t) = \mu - a_0 = m_F(t)$. Now,

$$\begin{aligned} \bar{F}(t) &= \frac{\mu}{m_F(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\ &= \frac{\mu}{m_{a_0}(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\ &\geq \frac{\mu}{m_{a_0}(t)} e^{-\int_0^t \frac{du}{m_{a_0}(u)}} \\ &= \bar{F}_{a_0}(t). \end{aligned}$$

Thus, we have shown that for every $\bar{F} \in \mathcal{D}$, there exists $a_0 \in [0, t \wedge \mu]$ such that $\bar{F}(t) \geq \bar{F}_{a_0}(t)$. Therefore, we conclude that

$$\inf_{\bar{F} \in \mathcal{D}} \bar{F}(t) \geq \inf_{a \in [0, (t \wedge \mu)]} \bar{F}_a(t).$$

The reverse inequality follows from the fact that $\bar{F}_{a_0}$ is a member of $\mathcal{D}$. This completes the proof. ☐

We are now ready for the main result of this section.

Theorem 1. Suppose F is a DMRL life distribution with mean μ. Then

$$\inf_{\bar{F} \in \mathcal{D}} \bar{F}(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } 0 \leq t \leq \mu, \\ 0 & \text{for } t > \mu. \end{cases}$$

Proof. By Lemma 2,

$$\begin{aligned} \inf_{\bar{F} \in \mathcal{D}} \bar{F}(t) &= \inf_{a \in [0, t \wedge \mu]} \bar{F}_a(t) \\ &= \inf_{a \in [0, t \wedge \mu]} \left(1_{[0,a)}(t) + e^{-\frac{t-a}{\mu-a}} 1_{[a,\infty)}(t) \right) \\ &\quad \text{(where } 1_A(t) \text{ is the indicator of the event } t \in A\text{)} \\ &= \inf_{a \in [0, t \wedge \mu]} e^{-\frac{t-a}{\mu-a}} \\ &= \begin{cases} e^{-\frac{t}{\mu}} & \text{for } t \leq \mu, \\ 0 & \text{for } t > \mu. \end{cases} \quad \square \end{aligned}$$

Remark 1. For $t > \mu$, the lower bound given in Theorem 1 is attained by the degenerate distribution at μ. On the other hand, for $t \leq \mu$, the lower bound is attained by th exponential distribution with mean μ. Therefore, the lower bound is sharp.

Remark 2. The DMRL lower bound coincides with the IFR lower bound given in [7], and is strictly larger than the NBUE lower bound given by [23], in the range $[0, \mu]$. These bounds are plotted in Figure 1 for comparison.

3 Upper Bound on DMRL Reliability Function

Consider the family of reliability functions $\bar{F}_b(t)$ defined as

$$\bar{F}_b(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } 0 \le t < b, \\ e^{-\frac{b}{\mu}} & \text{for } b \le t < b + \mu, \\ 0 & \text{for } b + \mu \le t \end{cases}, \tag{4}$$

for $b \ge 0$. The corresponding mean residual life function is

$$m_b(t) = \begin{cases} \mu & \text{for } 0 \le t < b, \\ b + \mu - t & \text{for } b \le t < b + \mu, \\ 0 & \text{for } b + \mu \le t. \end{cases}$$

It is clear that $\bar{F}_b(t)$ is a member of $\mathcal{D}$, for all $b \in [(t-\mu) \vee 0, t]$.

The next lemma shows that maximization of $\bar{F}(t)$ over $\mathcal{D}$ is equivalent to its maximization over the sub-class of reliability functions of the form $\bar{F}_b$.

Lemma 3. Let $\mathcal{D}$ and F_b be as defined in (2) and (4), respectively. Then for all $t \ge 0$,

$$\sup_{\bar{F} \in \mathcal{D}} \bar{F}(t) = \sup_{b \in [(t-\mu) \vee 0, t]} \bar{F}_b(t)$$

Proof. Let $\bar{F} \in \mathcal{D}$ and m_F be the corresponding MRL function. We shall show that

$$\bar{F}(t) \le \bar{F}_{b_0}(t), \text{ for } b_0 = m_F(t) + t - \mu.$$

Since $\bar{F} \in \mathcal{D}$, $m_F(t) \le \mu$. Therefore, we have $b_0 \le t \le b_0 + \mu$. Let $m_{b_0}(t)$ be the MRL function corresponding to $\bar{F}_{b_0}(t)$. For $u \in [0, b_0)$,

$$m_{b_0}(u) = \mu \ge m_F(u).$$

For $u \in [b_0, b_0 + \mu]$,

$$\begin{aligned} m_{b_0}(u) &= b_0 + \mu - u \\ &= m_F(t) + t - u \\ &= m_F(u) + \int_u^t m'_F(v)dv + t - u \\ &\ge m_F(u) + \int_u^t (-1)dv + t - u \text{ (by Lemma 1)} \\ &= m_F(u) - t + u + t - u \\ &= m_F(u). \end{aligned}$$

It follows that $m_{b_0}(u) \geq m_F(u)$ for $u \in [0, b_0 + \mu]$. Further, $m_{b_0}(0) = m_F(0) = \mu$, and $m_{b_0}(t) = b_0 + \mu - t = m_F(t)$. It follows that

$$\begin{aligned}
\bar{F}(t) &= \frac{\mu}{m_F(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\
&= \frac{\mu}{m_{b_0}(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\
&\leq \frac{\mu}{m_{b_0}(t)} e^{-\int_0^t \frac{du}{m_{b_0}(u)}} \\
&= \bar{F}_{b_0}(t).
\end{aligned}$$

Thus, we have proved that for every $\bar{F} \in \mathcal{D}$, there exists $b_0 \in [(t-\mu) \vee 0, t]$ such that $\bar{F}(t) \leq \bar{F}_{b_0}(t)$. We conclude that

$$\sup_{\bar{F} \in \mathcal{D}} \bar{F}(t) \leq \sup_{b \in [(t-\mu) \vee 0, t]} \bar{F}_b(t).$$

The reverse inequality follows from the fact that $\bar{F}_{b_0}$ is a member of $\mathcal{D}$. This completes the proof. □

We are now ready to present the upper bound of a DMRL reliability function.

Theorem 2. Suppose F is a life distribution which is DMRL with mean μ. Then

$$\sup_{\bar{F} \in \mathcal{D}} \bar{F}(t) = \begin{cases} 1 & \text{for } t \leq \mu, \\ e^{-\frac{t-\mu}{\mu}} & \text{for } t > \mu \end{cases}$$

Proof. By Lemma 3,

$$\begin{aligned}
\sup_{\bar{F} \in \mathcal{D}} \bar{F}(t) &= \sup_{b \in [(t-\mu) \vee 0, t]} \bar{F}_b(t) \\
&= \sup_{b \in [(t-\mu) \vee 0, t]} \left(e^{-\frac{t}{\mu}} 1_{[0,b)}(t) + e^{-\frac{b}{\mu}} 1_{[b,b+\mu)}(t) \right) \\
&= \sup_{b \in [(t-\mu) \vee 0, t]} e^{-\frac{b}{\mu}} \\
&= \begin{cases} 1 & \text{for } t \leq \mu, \\ e^{-\frac{t-\mu}{\mu}} & \text{for } t > \mu. \end{cases} \quad \square
\end{aligned}$$

Remark 3. For $t < \mu$, the upper bound of Theorem 2 is attained by the degenerate distribution at μ. For $t \geq \mu$, the upper bound is attained by the DMRL distribution G_t

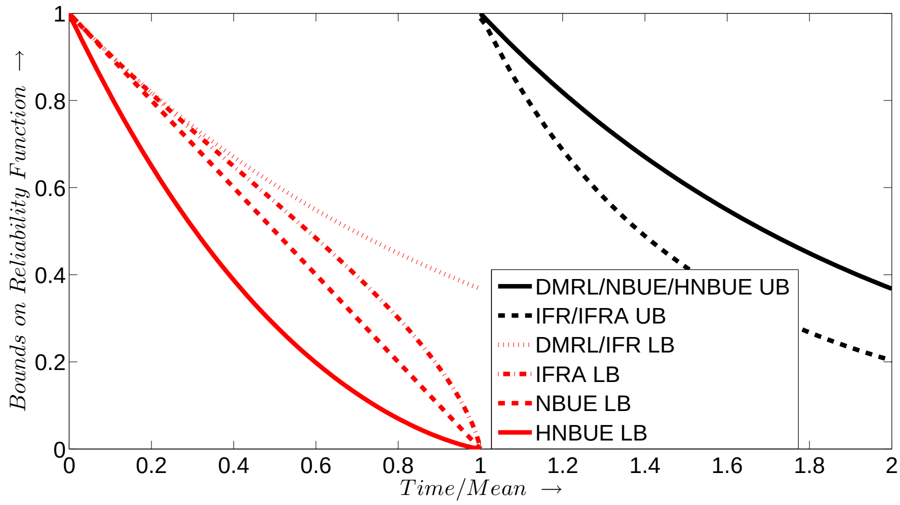

Figure 1: The Upper and Lower Bounds for the Positively Ageing Classes

given by

$$\bar{G}_t(u) = \begin{cases} e^{-\frac{u}{\mu}} & \text{for } 0 \leq u < t-\mu, \\ e^{-\frac{t-\mu}{\mu}} & \text{for } t-\mu \leq u < t \\ 0 & \text{for } u \geq t. \end{cases}$$

Therefore, the upper bound is sharp.

Remark 4. The DMRL upper bound coincides with the HNBUE upper bound [16]. Since the NBUE class contains the DMRL class and is contained in the HNBUE class [16], it follows that the bound given in Theorem 2 is also the sharp upper bound for an NBUE reliability function. This result complements the lower bound given by Marshall [23] and other available inequalities for different ageing classes [11]. The DMRL upper bound is found to be strictly above the IFR/IFRA upper bound in the range $[\mu, \infty)$ ([25]). Figure 1 shows the plots of all the upper and lower bounds.

4 Bounds on IMRL Reliability Function

Let $\mathcal{I}$ be the set of all IMRL distributions, with mean μ, that is,

$$\mathcal{I} = \left\{ \bar{F} : m(t) \uparrow t, \int_0^\infty \bar{F}(v)dv = \mu \right\}. \tag{5}$$

We consider the family of reliability functions $\bar{F}_c(t)$ defined as

$$\bar{F}_c(t) = \begin{cases} 1 & \text{for } t = 0, \\ \frac{\mu}{c} e^{-\frac{t}{c}} & \text{for } t > 0, \end{cases} \tag{6}$$

for $c \geq 0$. The corresponding mean residual life function is

$$m_c(t) = \begin{cases} \mu & \text{for } t = 0, \\ c & \text{for } t > 0. \end{cases}$$

Therefore, $\bar{F}_c(t)$ is a member of $\mathcal{I}$, for all $c \geq \mu$.

The next lemma shows that maximization of $\bar{F}(t)$ over $\mathcal{I}$ is equivalent to its maximization over the sub-class of reliability functions of the form $\bar{F}_c$.

Lemma 4. Let $\mathcal{I}$ and $\bar{F}_c$ be as defined in (5) and (6), respectively. Then for all $t \geq 0$,

$$\sup_{\bar{F} \in \mathcal{I}} \bar{F}(t) = \sup_{c \geq \mu} \bar{F}_c(t)$$

Proof. Let $\bar{F} \in \mathcal{I}$ and m_F be the corresponding MRL function. We shall show that

$$\bar{F}(t) \leq \bar{F}_{c_0}(t), \text{ for } c_0 = m_F(t).$$

Let m_{c_0} be the MRL function corresponding to $\bar{F}_{c_0}$. For $u = 0$,

$$m_{c_0}(u) = \mu = m_F(u).$$

For $u \in (0, t]$,

$$m_{c_0}(u) = c_0 = m_F(t) \geq m_F(u),$$

since $\bar{F} \in \mathcal{I}$. It follows that, $m_{c_0}(u) \geq m_F(u)$ for $u \in [0, t]$. Further, $m_{c_0}(t) = c_0 = m_F(t)$. Hence,

$$\begin{aligned} \bar{F}(t) &= \frac{\mu}{m_F(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\ &= \frac{\mu}{m_{c_0}(t)} e^{-\int_0^t \frac{du}{m_F(u)}} \\ &\leq \frac{\mu}{m_{c_0}(t)} e^{-\int_0^t \frac{du}{m_{c_0}(u)}} \\ &= \bar{F}_{c_0}(t). \end{aligned}$$

Since for every $\bar{F} \in \mathcal{I}$, there exists $c_0 \geq \mu$ such that $\bar{F}(t) \leq \bar{F}_{c_0}(t)$, we conclude that

$$\sup_{\bar{F} \in \mathcal{I}} \bar{F}(t) \leq \sup_{c \geq \mu} \bar{F}_c(t).$$

The reverse inequality follows from the fact that $\bar{F}_{c_0}$ is a member of $\mathcal{I}$. This completes the proof. $\square$

We are now ready for the main result of this section.

Theorem 3. Suppose F is a life distribution which is IMRL with mean μ. Then

$$\sup_{\bar{F} \in \mathcal{I}} \bar{F}(t) = \begin{cases} e^{-\frac{t}{\mu}} & \text{for } t \leq \mu, \\ \frac{\mu}{t} e^{-1} & \text{for } t > \mu \end{cases}$$

Proof. By Lemma 4,

$$\begin{aligned}
\sup_{\bar{F}\in\mathcal{I}} \bar{F}(t) &= \sup_{c\geq\mu} \bar{F}_c(t) \\
&= \sup_{c\geq\mu} \left(1_{[0]}(t) + \frac{\mu}{c} e^{-\frac{t}{c}} 1_{(0,\infty)}(t)\right) \\
&= \sup_{c\geq\mu} \frac{\mu}{c} e^{-\frac{t}{c}} \\
&= \begin{cases} e^{-\frac{t}{\mu}} & \text{for } t \leq \mu, \\ \frac{\mu}{t} e^{-1} & \text{for } t > \mu. \end{cases} \quad \square
\end{aligned}$$

Remark 5. The upper bound given in Theorem 3 is sharp. For $t \leq \mu$, the bound is attained by the exponential distribution with mean μ. For $t > \mu$, the bound is attained by the IMRL distribution G_t with reliability function given by

$$\bar{G}_t(u) = \begin{cases} 1 & \text{for } u = 0 \\ \frac{\mu}{t} e^{-\frac{u}{t}} & \text{for } u > 0. \end{cases}$$

Remark 6. The sharp upper bound given in Theorem 3 coincides with the upper bound of DFR reliability function with mean μ. It has to be noted that the upper bound corresponding to the NWUE class [14] is higher (See Figure 2).

Remark 7. Since the lower bound for DFR reliability function with mean μ is 0 [16], the same lower bound holds for an IMRL reliability function with mean μ (a larger class).

References

[1] A. Abouammoh and E. El-Neweihi. Closure of the NBUE and DMRL classes under formation of parallel systems. *Statistics & Probability Letters*, 4(5):223–225, 1986.

[2] S. E. Abu-Youssef. A moment inequality for decreasing (increasing) mean residual life distributions with hypothesis testing application. *Statistics & Probability Letters*, 57(2):171–177, 2002.

[3] I.A. Ahmad. A new test for mean residual life times. *Biomterika*, 79(2):416–419, 1992.

[4] Ibrahim A. Ahmad and A. R. Mugdadi. Further moments inequalities of life distributions with hypothesis testing applications: the IFRA, NBUE and DMRL classes. *Journal of Statistical Planning and Inference*, 120(1):1–12, 2004.

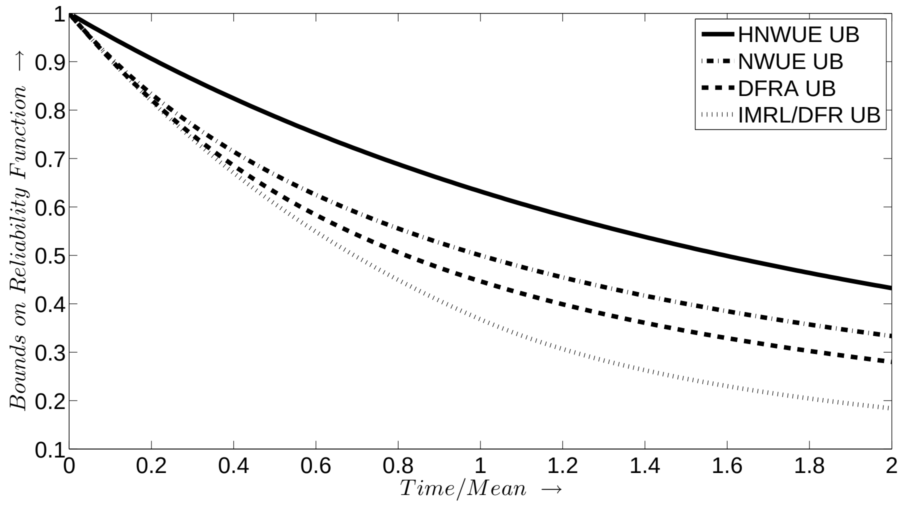

Figure 2: The Upper Bounds for the Negatively Ageing Classes.

[5] M. Z. Anis. On testing exponentiality against DMRL alternatives. *Economic Quality Control*, 25(2):281–299, 2010.

[6] H.S. Balaban and N.D. Singpurwalla. Stochastic properties of a sequence of inter-failure times under minimal repair and under revival. In Abdel-Hamid Mohamed, editor, *Reliabiity Theory and Models*, pages 65–80. Academic Press, Orlando, FL, USA, 1984.

[7] Richard E. Barlow and Frank Proschan. *Statistical Theory of Reliability and Life Testing: Probability Models.* To Begin With, Silver Spring, MD, USA, 1981.

[8] B. Bergman and Bengt Klefsjö. A family of test statistics for detecting monotone mean residual life. *Journal of Statistical Planning and Inference*, 21(2):161–178, 1989.

[9] Maurice C. Bryson and M. M. Siddiqui. Some criteria for aging. *Journal of the American Statistical Association*, 64(328):1472–1483, 1969.

[10] K. Cheng and Z. He. On proximity between exponential and dmrl life distributions. *Statistics and Probability Letters*, 8:55–57, 1989.

[11] K. Cheng and Z. He. Reliability bounds for NBUE and NWUE distributions. *Acta Mathematicae Applicatae Sinica*, 5(1):81–88, 1989.

[12] I.B. Gertsbakh. *Statistical Reliability Theory.* New York., 1989.

[13] A.K. Gupta, W.B. Zeng, and Y. Wu. *Probability and Statistical Models: Foundations for Problems in Reliability and Financial Mathematics.* Springer, New York., 2010.

[14] Andrew L. Haines and Nozer D. Singpurwalla. Some contributions to the stochastic characterization of wear. In Proschan F. and Serfling R.J., editors, *Reliability and Biometry*, pages 47–80. Society for Industrial and Applied Mathematics, Philadelphia, 1974.

[15] Myles Hollander and Proschan Frank. Tests for the mean residual life. *Biometrika*, 62(3):585–593, 1975.

[16] Bengt Klefsjö. The HNBUE and HNWUE classes of life distributions. *Naval Research Logistics Quarterly*, 29(2):331–345, 1982.

[17] Bengt Klefsjö. On aging properties and total time on test transforms. *Scandinavian Journal of Statistics*, 9(1):37–41, 1982.

[18] Bengt Klefsjö. Some tests against aging based on the total time on test transform. *Communications in Statistics: Theory and Methods*, 12(8):907 – 927, 1983.

[19] Edward Korczak. Upper bounds on the integrated tail of the reliability function, with applications. *Control and Cybernetics*, 30(3):355–366, 2001.

[20] A. Korzeniowski and A. Opawski. Bounds for reliability in the nbu, nwu, nbue and nwue classes. *Zast. Mat. Appl. Math*, 15:1–5, 1976.

[21] Chin-Diew Lai and Min Xie. *Stochastic ageing and dependence for reliability.* Springer Science & Business Media, 2006.

[22] J.H. Lim and D. H. Park. A family of test statistics for dmrl (imrl) alternatives. *Journal of Nonparametric Statistics*, 8(4):379–392., 1997.

[23] A. W. Marshall and F. Proschan. Classes of distributions applicable in replacement with renewal theory implications. In Le Cam (ed.), editor, *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability.*, volume 11, pages 395–415, 1972.

[24] Dong Ho Park. *Class of NBU-t_0 life distribution. Handbook of Reliability Engineering.* Springer, London, 2003.

[25] Debasis Sengupta. Another look at the moment bounds on reliability. *Journal of Applied Probability*, 31:777–787, 1994.

[26] Deshpande Jayant V., Kochar Subhash C., and Singh Harshinder. Aspects of positive ageing. *Journal of Applied Probability*, 23(3):748758, 1986.

[27] G.E. Willmot and X. S. Lin. *Lundberg Approximations for Compound Distributions with Insurance Applications.* Number 156 in Lecture Notes in Statistics. Springer Science & Business Media, New York., 2001.