COMPRESSIVE SAMPLING USING EM ALGORITHM

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ABSTRACT. Conventional approaches of sampling signals follow the celebrated theorem of Nyquist and Shannon. Compressive sampling, introduced by Donoho, Romberg and Tao, is a new paradigm that goes against the conventional methods in data acquisition and provides a way of recovering signals using fewer samples than the traditional methods use. Here we suggest an alternative way of reconstructing the original signals in compressive sampling using EM algorithm. We first propose a naive approach which has certain computational difficulties and subsequently modify it to a new approach which performs better than the conventional methods of compressive sampling. The comparison of the different approaches and the performance of the new approach has been studied using simulated data.

1. INTRODUCTION

In recent years there has been a huge explosion in the variety of sensors and the dimensionality of the data produced by these sensors and this has been in a large number of applications ranging from imaging to other scientific applications.The total amount of data produced by the sensors is much more than the available storage. So we often need to store a subset of the data. We want to reconstruct the entire data from it. The famous Nyquist-Shannon sampling theorem [5] tells us that if we can sample a signal at twice its highest frequency we can recover it exactly. In applications this often results in too many samples which must be compressed in order to store or transmit. An alternative is compressive sampling (CS) which provides a more general data acquisition protocol by reducing the signal directly into a compressed representation by taking linear combinations. In this paper we present a brief of the conventional approach of compressive sampling and propose a new approach that makes use of the EM algorithm to reconstruct the entire signal from the compressed signals.

2. SETUP

When a signal is sparse in some basis , a few well chosen observations suffice to reconstruct the most significant nonzero components.
Consider a signal $\mathbf{x}$ represented in terms of a basis expansion as.

$$\mathbf{x} = \sum_{i=1}^{n} s_i \psi_i = \psi \mathbf{s}$$

The basis $\mathbf{s}$ is such that only $k << n$ coefficients ψ_i have significant magnitude. Many natural and artificial signals are sparse in the sense that there exists a basis where the above representation has just a few large coefficients and other small coefficients. As an example natural images are likely to be compressible in discrete cosine transform(DCT) and wavelet bases [1]. In general we do not know apriori

which coefficients are significant. The data collected by a measurement system consists of some linear combinations of the signals

$$\mathbf{y} = \phi\mathbf{x} + \mathbf{e} = \phi\psi\mathbf{s} + \mathbf{e} = \mathbf{A}\mathbf{s} + \mathbf{e}$$

where $A = \phi\psi$ is a measurement matrix (also called sensing matrix) which is chosen by the statistician. The measurement process is non-adaptive as ϕ(and hence A) does not depend in any way on the signal $\mathbf{x}$. $\mathbf{e}$ is the error which is assumed to be bounded or bounded with high probability.

Our aim here is to :

- design a stable measurement matrix that preserves the information in any k-sparse signal during the dimensionality reduction from $\mathbb{R}^n$ to $\mathbb{R}^m$.
- design a reconstruction algorithm to recover the original data $\mathbf{x}$ from the measurements $\mathbf{y}$.

We note that the recovery algorithm addresses the problem of solving for $\mathbf{x}$ when the number of unknowns (i.e. n) is much larger than the number of observations (i.e. m) . In general this is an ill-posed problem but CS theory provides a condition on ϕ which allows accurate estimation. One such popularly used property is Restricted Isometry Property (RIP) [2].

Definition 1. The matrix A satisfies the restricted isometry property of order k with parameters $\delta_k \in [0, 1)$ if

$$(1 - \delta_k) \parallel \theta \parallel_2^2 \leq \parallel A\theta \parallel_2^2 \leq (1 + \delta_k) \parallel \theta \parallel_2^2$$

holds simultaneously for all sparse vectors θ having no more than k nonzero entries. Matrices with this property are denoted by $\text{RIP}(K, \delta_k)$

3. CONVENTIONAL APPROACH

The following theorem shows that matrices satisfying RIP will yield accurate estimates of $\mathbf{x}$ with the help of recovery algorithms.

Theorem 2. *Let A be a matrix satisfying $RIP(2k, \delta_{2k})$ with $\delta_{2k} < \sqrt{2} - 1$ and let $\mathbf{y} = A\mathbf{s} + \mathbf{e}$ be a vector of noisy observations , where $\parallel \mathbf{e} \parallel_2 \leq \epsilon$. Let $\mathbf{s}_k$ be the best k-sparse approximation of $\mathbf{s}$, that is , $\mathbf{s_k}$ is the approximation obtained by keeping the k largest entries of s and setting others to zero. Then the estimate*

$$(3.1) \qquad \hat{\mathbf{s}} = \arg\min_{\mathbf{s}\in\mathbb{R}^n} \parallel \mathbf{s} \parallel_1 \;\; subject\; to \;\; \parallel \mathbf{y} - A\mathbf{s} \parallel_2 \leq \epsilon$$

obeys

$$(3.2) \qquad \parallel \mathbf{s} - \hat{\mathbf{s}} \parallel_2 \leq C_{1,k}\epsilon + C_{2,k}\frac{\parallel \mathbf{s} - \mathbf{s}_k \parallel_1}{\sqrt{k}}$$

where $C_{1,k}$ and $C_{2,k}$ are constants depending on k but not on n or m.

The reconstruction in (3.1) is equivalent to.

$$(3.3) \qquad \hat{\mathbf{s}} = \arg\min_{\mathbf{s}\in\mathbb{R}^n} \frac{1}{2} \parallel \mathbf{y} - A\mathbf{s} \parallel_2^2 + \lambda \parallel \mathbf{s} \parallel_1 \;\; \text{and} \; \hat{\mathbf{x}} = \psi\hat{\mathbf{s}}$$

where $\lambda > 0$ is a regularization parameter which depends on ϵ.

4. A NAIVE APPROACH

In this approach we apply EM algorithm for the reconstruction of the signal. Since we observe some linear combinations of the signals instead of the entire signals we can treat the observed linear combinations as our observed data and the entire signals as the complete data which is unobserved. Hence we apply EM algorithm as a most natural tool of missing data analysis to reconstruct the data. Here we assume that data are coming from a population with mean $\boldsymbol{\mu}$ and that $\boldsymbol{\mu}$ is sparse (w.r.t some basis). Without loss of generality we assume that $\boldsymbol{\mu}$ is sparse with respect to euclidean basis.We assume that at most k elements of $\boldsymbol{\mu}$ is nonzero. Let us assume that the parent population is normal viz. $N(\boldsymbol{\mu}, \sigma^2 I_n)$.
Then we have the signal as
$$\mathbf{x} = \boldsymbol{\mu} + \boldsymbol{\epsilon}$$
where $\boldsymbol{\epsilon} \sim N(0, I_n)$. Then with the help of the sensing matrix we have the observed data as

$$\mathbf{y} = \phi \mathbf{x} = \phi(\mu + \epsilon) = \phi \boldsymbol{\mu} + \mathbf{e}$$
where $\mathbf{e} = \phi \boldsymbol{\epsilon}$

Thus unlike the conventional approach here we assume that the signals themselves are subject to error and consequently the observed combinations of the signals are also subject to error. Here we try to reconstruct the unobserved true signals which are free from error. We then treat $\mathbf{x}$ as the complete data and $\mathbf{y}$ as the observed data and try to estimate $\boldsymbol{\mu}$ from the observed data using EM algorithm .Thus we have

$$\boldsymbol{\mu} = (\mu_1, \mu_2 ... \mu_n)^{'}$$

$$\mathbf{x} \sim N_n(\boldsymbol{\mu}, \sigma^2 I_n) : \text{Complete data}$$

$$\mathbf{y} = \phi \mathbf{x} \sim N_m(\phi \boldsymbol{\mu}, \sigma^2 \phi \phi^{'}) : \text{Observed data}$$

The complete data likelihood is given by
$$f(\mathbf{x}) = \frac{1}{(\sigma \sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}(\mathbf{x} - \boldsymbol{\mu})^{'}(\mathbf{x} - \boldsymbol{\mu})}, \mathbf{x} \in \mathbb{R}^n, \boldsymbol{\mu} \in \mathbb{R}^n, \sigma > 0$$

The conditional distribution of the complete data given the observed data is.
$$\mathbf{x}|\mathbf{y}, \boldsymbol{\mu} \sim \mathbf{N_n}(\boldsymbol{\mu} + \phi^{'}(\phi \phi^{'})^{-1}(\mathbf{y} - \phi \boldsymbol{\mu}), \ \sigma^2(\mathbf{I_n} - \phi^{'}(\phi \phi^{'})^{-1}\phi))$$

After t iterations in EM algorithm we have,

- **E Step**:We compute the expected complete data log-likelihood w.r.t the conditional distribution of $\mathbf{x}|\mathbf{y}, \boldsymbol{\mu}^{(t)}$.Now
$$\ell(\boldsymbol{\mu}) = \ln(f(\mathbf{x})) = constant - \frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^{'}(\mathbf{x} - \boldsymbol{\mu})$$

$$\Rightarrow \ell(\boldsymbol{\mu}) = constant - \frac{1}{2}\sum_{i=1}^{n}(x_i - \mu_i)^2$$

Also
$$\mathbf{x}|\mathbf{y}, \boldsymbol{\mu}^{(\mathbf{t})} \sim \mathbf{N_n}(\mu^{(\mathbf{t})} + \phi^{'}(\phi \phi^{'})^{-1}(\mathbf{y} - \phi \boldsymbol{\mu}^{(\mathbf{t})}), \ \sigma^2(\mathbf{I_n} - \phi^{'}(\phi \phi^{'})^{-1}\phi))$$

Define
$$Q(\boldsymbol{\mu}) = E(\ell(\boldsymbol{\mu})|y, \boldsymbol{\mu}^{(t)})$$

- **M Step**:Here we try to maximize $Q(\boldsymbol{\mu})$ with respect to $\boldsymbol{\mu}$.We know that $\boldsymbol{\mu}$ is sparse i.e. some of the μ_i are zero. So we need to maximize $Q(\boldsymbol{\mu})$ w.r.t. $\boldsymbol{\mu}$ belonging to a subset

$$S = \{\boldsymbol{\mu} : \text{at most } k \text{elements of } \boldsymbol{\mu} \text{ are nonzero}\}$$

Thus we find
$$\arg\max_{\boldsymbol{\mu} \in S} Q(\mu)$$

For this we note that $S = \cup_{i=1}^{\binom{n}{k}} S_i$ where

$$S_i = \{\boldsymbol{\mu} : \text{at most } i \text{specific elements of } \boldsymbol{\mu} \text{ are nonzero}\}$$

We then find
$$\arg\max_{\boldsymbol{\mu} \in S_i} Q(\boldsymbol{\mu})$$

for each i and call the estimate as

$$\hat{\mu}^{(t+1)}(S_i) = (\hat{\mu}_1^{(t+1)}(S_i), \hat{\mu}_2^{(t+1)}(S_i)..., \hat{\mu}_k^{(t+1)}(S_i))'$$

Now the $\arg\max_{\boldsymbol{\mu} \in S_i} Q(\boldsymbol{\mu})$ is found out in the following way:

Setting $\frac{\partial}{\partial \mu_j} Q(\boldsymbol{\mu}) = 0$ for those j such that $\mu_j \neq 0$ we find that

$$\hat{\mu}_j^{(t+1)}(S_i) = \mu_j^{(t)} + \alpha_j + \beta_j$$

where $\alpha_j = j^{th}$element of $\phi'(\phi\phi')^{-1}y$ and $\beta_j = j^{th}$element of $\phi'(\phi\phi')^{-1}\phi\boldsymbol{\mu}^{(t)}$.

Then we choose the $\hat{\boldsymbol{\mu}}^{(t+1)}(S_i)$ for which $Q(\hat{\boldsymbol{\mu}}^{(t+1)}(S_i))$ is maximum as the new estimate of $\boldsymbol{\mu}$ at $(t+1)^{th}$ iteration.Thus the estimate of $\boldsymbol{\mu}$ is

$$\hat{\boldsymbol{\mu}}^{(t+1)} = \hat{\boldsymbol{\mu}}^{(t+1)}(S_i)$$

such that

$$Q(\hat{\boldsymbol{\mu}}^{(t+1)}(S_i)) \geq Q(\hat{\boldsymbol{\mu}}^{(t+1)}(S_j)) \,\forall j \neq i$$

We iterate until convergence.

5. NEW APPROACH

The new approach discussed in the previous section requires the maximization of $Q(\boldsymbol{\mu})$ over $\binom{n}{k}$ subspaces and then choose the one for which it is maximum at the M step of each EM iteration. This is computationally expensive and practically impossible to implement for large n. Hence we suggest an alternative way which instead of maximization over $\binom{n}{k}$ subspaces in each EM iteration identifies a particular subspace where $\boldsymbol{\mu}$ is most likely to belong , and then finds the maximum over that subspace in each M step.

Let $S_{\boldsymbol{\mu}}$ be the subspace where $\boldsymbol{\mu}$ lies , that is
$$S_{\boldsymbol{\mu}} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \forall i \; \mu_i = 0 \Rightarrow x_i = 0\}$$
We note that if we find the unrestricted maximizer of $Q(\boldsymbol{\mu})$ in each M step of the EM algorithm (henceforth call unrestricted EM) , that is if we find
$$\hat{\boldsymbol{\mu}}^{un} = \arg \max_{\boldsymbol{\mu} \in \mathbb{R}} Q(\boldsymbol{\mu})$$
then the unrestricted EM estimate $\hat{\boldsymbol{\mu}}^{un}$ should lie close to $S_{\boldsymbol{\mu}}$. Hence the unrestricted estimate should provide an indication of the subspace in which the original parameter lies. Hence we find which components of $\hat{\boldsymbol{\mu}}^{un}$ are significant so that we can take the other insignificant components to be zero and take the corresponding subspace thus formed to be the one in which our estimate should lie. We test which components of $\hat{\boldsymbol{\mu}}^{un}$ are significantly different from zero. Now for the unrestricted EM algorithm the estimate of $\boldsymbol{\mu}$ should converge to the maximizer of the observed log-likelihood. The observed log-likelihood is
$$\ell_{obs}(\boldsymbol{\mu}) = constant - \frac{1}{2}(\mathbf{y} - \phi\boldsymbol{\mu})^{'}(\sigma^2 \phi\phi^{'})^{-1}(\mathbf{y} - \phi\boldsymbol{\mu})$$
Setting $\frac{\partial}{\partial \boldsymbol{\mu}} \ell_{obs}(\boldsymbol{\mu}) = 0$ we get
$$(5.1) \qquad (\phi^{'} V^{-1} \phi)\boldsymbol{\mu} = \phi^{'} V^{-1} \phi \mathbf{y}$$
where $V = \phi\phi^{'}$.
The above equation (5.1) does not have a unique solution as $rank[(\phi^{'} V^{-1} \phi)_{n \times n}] = m \ll n$. Hence the observed likelihood does not have a unique maximum and our unrestricted EM algorithm will produce many estimates of $\boldsymbol{\mu}$. Among these many estimates we choose the sparsest solution. This is taken care of by taking the initial estimate of $\boldsymbol{\mu}$ as $\mathbf{0}$ in the iterative process as then the estimate will hopefully converge to nearest solution which will be the sparest one. We will justify this later with the help of simulation.
We have
$$\hat{\boldsymbol{\mu}}^{un} = (\phi^{'} V^{-1} \phi)^{+} \phi^{'} V^{-1} y = P\mathbf{y}$$
where $P = (\phi^{'} V^{-1} \phi)^{+} \phi^{'} V^{-1}$.

Here we take the Moore-Penrose inverse of $(\phi^{'} V^{-1} \phi)$ as we want to find the least norm solution of (5.1) .
Now
$$\hat{\boldsymbol{\mu}}^{un} \sim \mathbf{N_n}(P\phi\boldsymbol{\mu} \, , \, PVP^{'})$$
Thus $E(\hat{\boldsymbol{\mu}}^{un}) = P\phi\boldsymbol{\mu}$ and $\hat{\boldsymbol{\mu}}^{un}$ should lie close to the sparse $\boldsymbol{\mu}$. Hence $P\phi\boldsymbol{\mu}$ should be close to $\boldsymbol{\mu}$ and $\hat{\boldsymbol{\mu}}^{un}$ is used to test hypotheses regarding $\boldsymbol{\mu}$.
We want to test n hypotheses
$$H_{0i} : \mu_i = 0 \qquad \forall i = 1(1)n$$
Let
$$\hat{\boldsymbol{\mu}}^{un} = (\hat{\mu}_1^{un}, \hat{\mu}_2^{un}, \ldots \hat{\mu}_n^{un})^{'}$$
Then the test statistics for testing H_{0i} is
$$\tau_i = |\frac{\hat{\mu}_i^{un}}{\sqrt{s_{ii}}}| \sim N(0,1) \quad \text{under } H_{0i} \qquad \forall i = 1(1)n$$
where $s_{ii} = i^{th}$ diagonal element of $PVP^{'}$.

Thus we estimate the subspace where $\boldsymbol{\mu}$ lies as

$$\hat{S}_{\boldsymbol{\mu}} = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : \forall i \ \tau_i \leq z_{\alpha/2} \Rightarrow x_i = 0\}$$

With this new estimated subspace we apply our original restricted EM algorithm as in the previous section as follows:

After t iterations in EM algorithm we have,

- **E step**: Compute $Q(\boldsymbol{\mu}) = E(\ell(\boldsymbol{\mu})|y, \boldsymbol{\mu}^{(t)})$
- **M-step**: We find

$$\arg\max_{\boldsymbol{\mu} \in \hat{S}_{\boldsymbol{\mu}}} Q(\boldsymbol{\mu})$$

and take the maximizer as the new estimate of $\boldsymbol{\mu}$, that is , $\hat{\boldsymbol{\mu}}^{(t+1)}$.

We iterate until convergence.

6. SIMULATION STUDY

In this section we compare the different approaches with the help of simulation. We will also verify the convergence of $\hat{\boldsymbol{\mu}}^{un}$ to the sparsest solution as claimed in the previous section. The performance of the new proposed algorithm will be studied using simulation technique where we will investigate to what extent we can reduce the dimension of the observed data using the proposed approach in order to have a fair reconstruction of the parameter.

6.1. **Convergence of the Unrestricted EM estimate:** Here we see that in the unrestricted EM algorithm the EM estimate of $\boldsymbol{\mu}$ converge to the sparsest solution of equation (5.1) if we take our initial estimate as $\mathbf{0}$ (or very close to $\mathbf{0}$). We take different initial estimates of $\boldsymbol{\mu}$ randomly and check the L_1 norm of the final estimates $\hat{\boldsymbol{\mu}}^{un}$ in each case. For demonstration we work with $n = 4$. We find that we reach the minimum norm solution if the initial estimate of $\boldsymbol{\mu}$ is taken close to $\mathbf{0}$.

Initial estimate $\hat{\boldsymbol{\mu}}^{(1)}$	L_1 norm of $\hat{\boldsymbol{\mu}}^{un}$
(0.0001,0.0001,0.0001,0.0001)	10.5667
(12.52,22.76,35.98,67.72)	38.9358
(10.5,11.25,25.62,19.74)	27.8503

6.2. **Comparison of Approaches:** Next we compare the accuracy of the different approaches discussed in the paper. From **Theorem 2** we find that the accuracy of the reconstructed signal is shown by (3.1) . Hence we take $\| \mathbf{x} - \hat{\mathbf{x}} \|_{l_2}$ as measure of closeness between the original and the reconstructed signal. We note that there is difference in the setup of the data in the approaches (4). The conventional approach reconstruct the signal $\mathbf{x}$ whereas the new approaches reconstruct what is called true signal (free from noise) $\boldsymbol{\mu}$. Hence for comparison we reconstruct signals from same population using conventional approach and average out the residuals to remove the effect of the noise.

For the comparison of approaches we adopted the following technique:

- We set the actual number of observations n and the observed number of observations m. k, the maximum number of nonzero components in $\boldsymbol{\mu}$, is taken to be equal to m (maximum possible value), that is, we do not use any prior information about the number of nonzero components in $\boldsymbol{\mu}$.

- We fix a $\boldsymbol{\mu}$ such that its first 4 components are 5 and the rest are zero.

- We start with a value of σ between 0.1 and 1.

- **Assessing Conventional Approach:** We generate data $\mathbf{x}$ from $N_n(\boldsymbol{\mu}, \sigma^2 I_n)$ and reconstruct $\widehat{\mathbf{x}}$ using (3.3) from the conventional approach and find $\| \mathbf{x} - \widehat{\mathbf{x}} \|_{l_2}$. This process is repeated 1000 times to find the residuals in each case and then we compute the mean residual $\frac{1}{1000} \sum_{i=1}^{1000} \| \mathbf{x_i} - \widehat{\mathbf{x_i}} \|_{l_2}$ to remove the effect of randomness and get a measure of closeness among the original and reconstructed $\boldsymbol{\mu}$.

- **Assessing New Approaches:** We again generate data $\mathbf{x}$ from $N_n(\boldsymbol{\mu}, \sigma^2 I_n)$ .We apply the naive approach (wherever possible) and the new approach to reconstruct $\boldsymbol{\mu}$ and find $\| \boldsymbol{\mu} - \hat{\boldsymbol{\mu}} \|_{l_2}$ as a measure of closeness between the original and estimated values.

- For each value of σ in we repeat the process of assessing the conventional and new approaches 10 times each to get the average residual and standard error of the residuals for each of the conventional and the proposed algorithms.

- We repeat the above procedures for different values of σ in $[0.1, 1.0]$ and plot the mean residuals along with the standard error bars.

For small values of n we plot the average residuals for the three approaches discussed earlier.


Average residuals
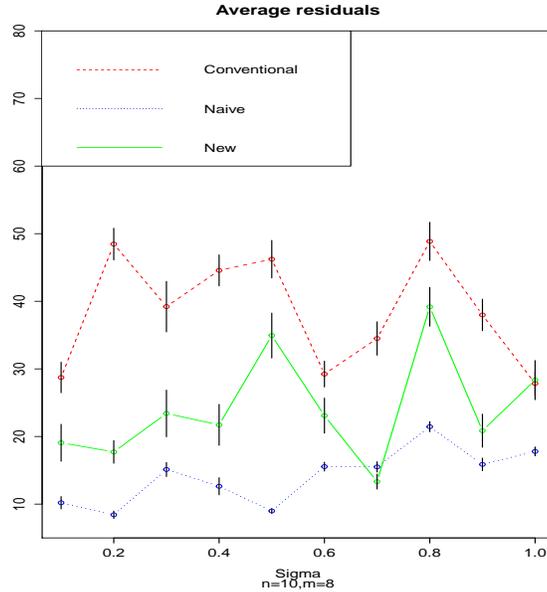
For $n = 10$ we find that the naive approach works uniformly best for different values of σ. Thus it would have been nice if we can apply this naive approach for all values of n , but unfortunately due to the inapplicability of this procedure we turn our attention towards the new approach.

For moderate to large values of n we cannot plot the residuals of the naive approach as it is computationally impossible. Also the comparison between the new and the conventional approach cannot be performed for very large values of n because of computational time. We find that the new approach works uniformly better for different values of σ for both $n = 50$ and $n = 100$.
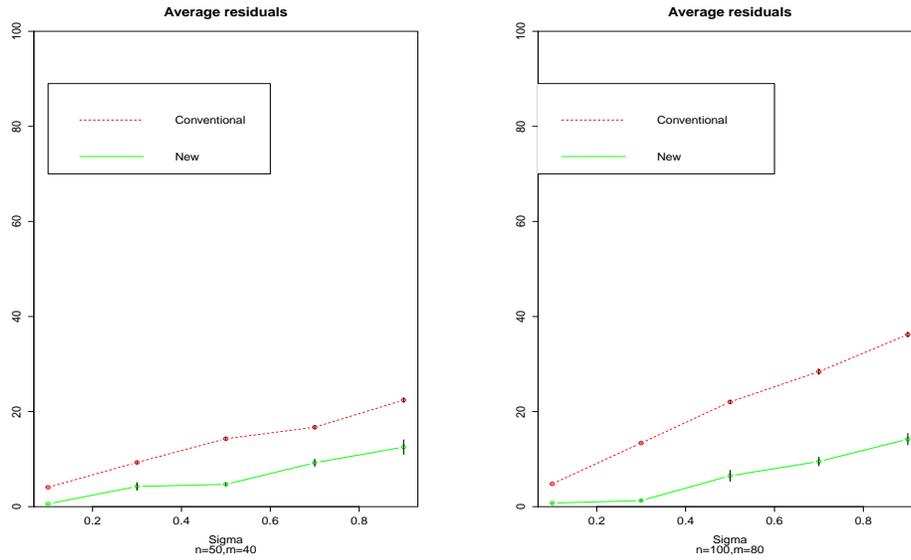
6.3. **Performance of the new approach:** The value of $\frac{m}{n}$ in the above procedures is an important point of consideration. It signifies the sampling fraction ,that is to what extent we can reduce the dimensionality of the problem. We fix $n = 1000$ and with $\sigma = 0.001$ we plot the average residuals for varying m.
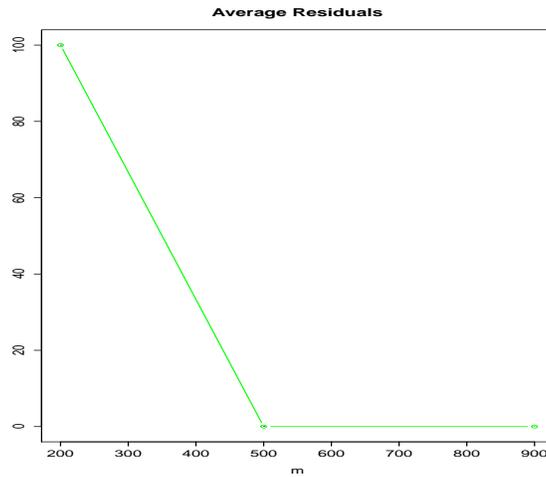
The procedure works good if we take $m = 500$,that is at this variance level we can afford 50% dimensionality reduction.

Thus we find that the new approach works better than the conventional method of signal reconstruction. The conventional method of reconstructing the signal assumes the noise to be bounded with high probability and thus fail to perform well for large error variance whereas the new approach allows the error variance to be large enough and thus make it applicable to other situations. Also the conventional approach assumes that the signal is sparse and sparsity is an essential ingredient in the reconstruction algorithm. The new proposed approach can easily be generalized to even situations where signals need not to be sparse. However we find that the naive approach we proposed earlier works best if it can be implemented. For moderate to large dimensional problems which are common in practice the new algorithm works better than the conventional approach.

7. Future work

The present paper treats observations or signals as iid samples from a population. This can be extended assuming a non-iid setup where the signals may be generated from a stochastic process. Further here we work with linear combinations of all signals. A further extension can be done where we builld the model with linear combinations of some signals and apply it for future signals in the process.

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