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Lower Bounds in Quantum Computing and Applications to Cryptography

Avradip Mandal

Fujitsu Laboratories of America, Inc.

Software Quality and Security Lab

Power of Quantum Computing

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Quantum Computing can provide exponential speed up against certain class of problems

- Example: Factoring, Discrete Log

On the other hard on the problems like unordered search quantum algorithms provide only quadratic speed up

- Known to be optimal
- Focus of this talk: how to prove optimality or lower bounds

Goal of this talk

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- Proving Lower bounds of the query complexity of Quantum Algorithms is a rich research area.
- However, in crypto settings we are generally interested in average case lower bound as opposed worst case lower bound.
- In this talk we will go over two well known proof techniques Polynomial method and Adversarial method and their extension to cryptographic setting

Topics covered in this talk

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- Recap of Quantum Computing Basics
- Quantum Query Model
- Polynomial Method
  - Lower bound of Grover Search
  - Lower bound of Average Case Grover Search
- Adversary Method
  - Lower bound of Grover Search
  - Lower bound of Average Case Grover Search
- Random Permutation vs Random Function

Qubits

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- Classical bit takes $0, 1$ values; Quantum Bit (qubit) is a unit vector in $\mathbb{C}^2$.
  - $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ is a state which is even super position of $|0\rangle$ and $|1\rangle$
- State of n qubits can be represented by a unit vector in $\mathbb{C}^{2^n}$
  - Different vectors with same global phase corresponds to exact same qubit state
  - i.e. $|\Psi\rangle$ and $c|\Psi\rangle$ represent the same qubit state for any $c \in \mathbb{C}$, with $|c| = 1$
- Quantum States allow (only) two kinds of operations: Measurement and Unitary transformation

Measurement

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- Measuring $a|0\rangle + b|1\rangle$ with respect to standard basis $\{|0\rangle, |1\rangle\}$
  - Collapses to state $|0\rangle$ with probability $|a|^2$
  - Collapses to state $|1\rangle$ with probability $|b|^2$
- Measuring $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$ in standard basis results in states $|0\rangle$ or $|1\rangle$ with probability $\frac{1}{2}$
- Hadamard basis $\{|+\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, |-\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}}\}$ measurement results in $|+\rangle$ with probability 1.

Projective Measurement

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- Suppose a vector space $\mathcal{H}$ decomposes into k orthogonal subspaces S_i s.t. $\mathcal{H} = S_1 \oplus S_2 \oplus \cdots \oplus S_k$
- There are k projective operators $P_1, \cdots, P_k$ s.t. $P_i : \mathcal{H} \to S_i$
- Projective measurement on $|\psi\rangle \in \mathcal{H}$ with this operator set would result in state $\frac{P_i|\Psi\rangle}{|P_i|\Psi\rangle|}$ with probability $|P_i|\Psi\rangle|^2$
- A single measurement is always enough.

Unitary Transformation

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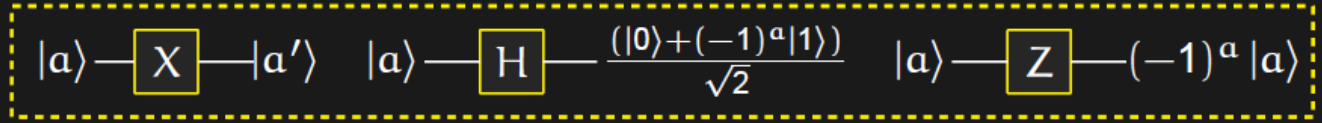
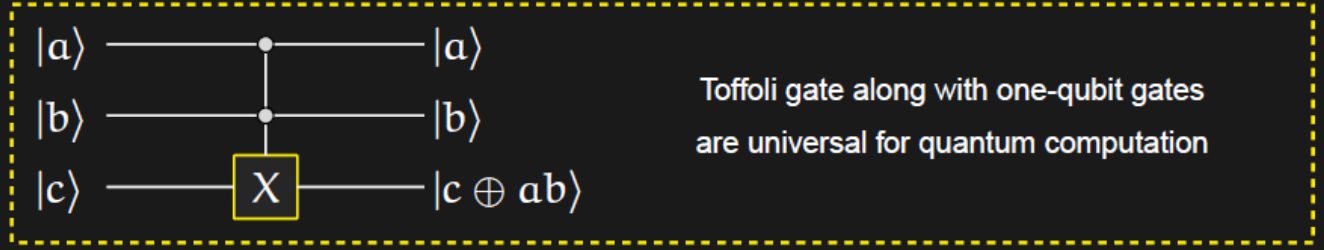
- $U: \mathcal{H} \rightarrow \mathcal{H}$ is unitary if $U^{+}U = I$
  - $U^{+} = (\overline{U})^{T} = \overline{U^{T}}$
- Transform then measurement, is same as measurement then transform
- In classical sense unitary transformations corresponds to reversible circuits.

Quantum operations: Few examples

Single qubit operations: NOT, Hadamard and Phase-flip gates

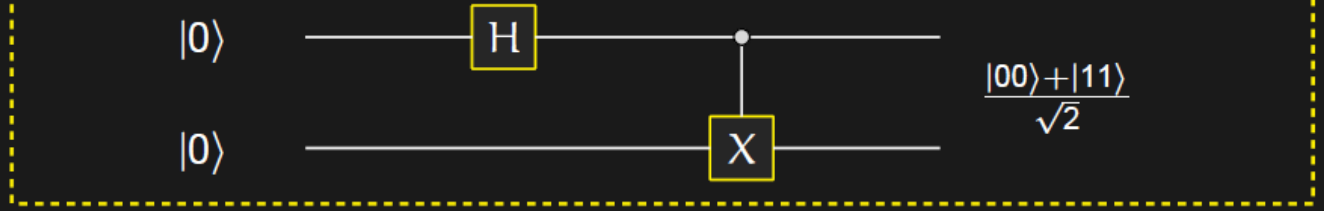


Simulating classical gates: Toffoli gate



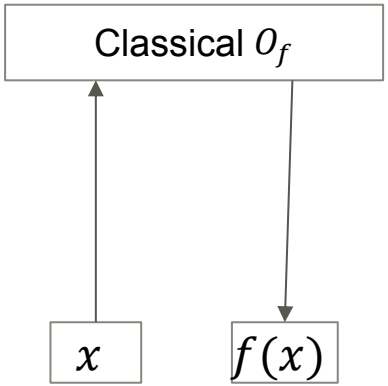
Toffoli gate along with one-qubit gates are universal for quantum computation

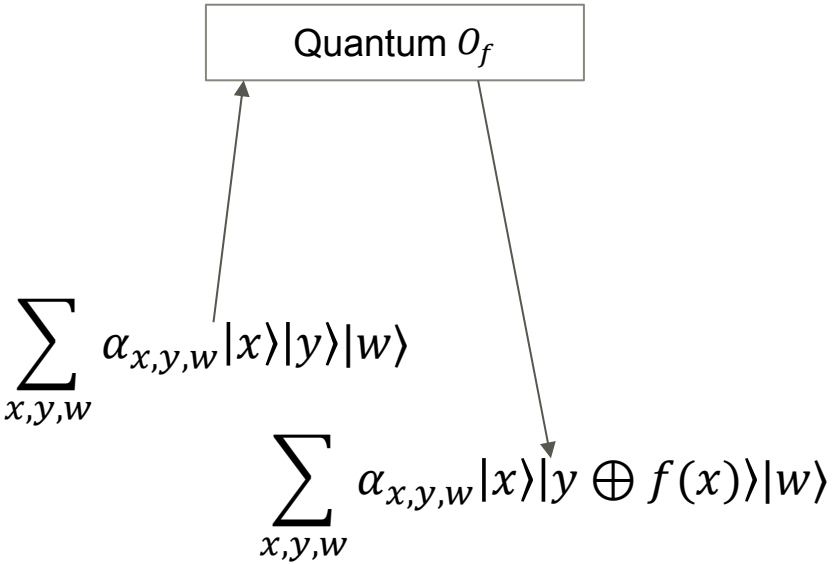
Creating entanglement: Using Hadamard and CNOT gate



Slide stolen from Sarvagya Upadhyay

Classical vs Quantum Oracle





Generic Quantum Algorithm

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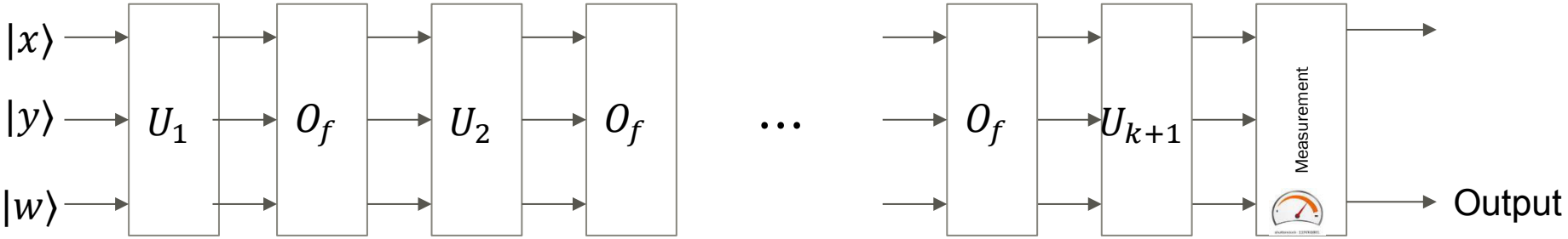
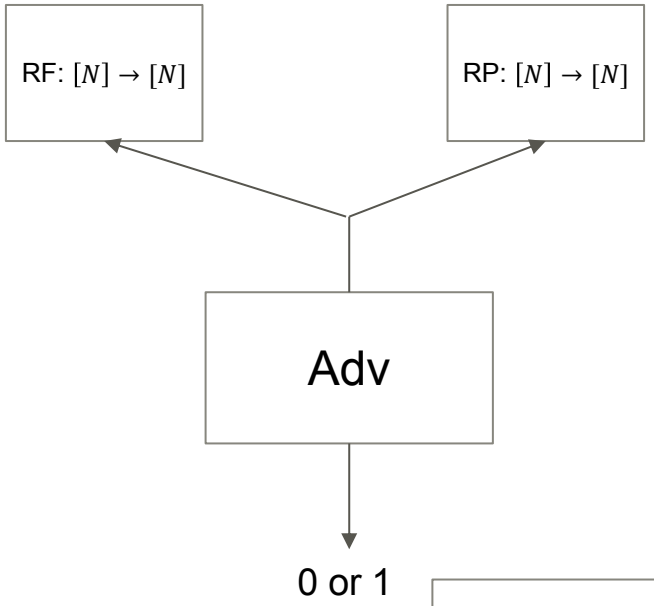


Classical Lower Bound: RP vs RF



Suppose the adversary makes q queries $x_1, \dots, x_q$

$$Adv = SD\left(\left(RF(x_1), \dots RF(x_q)\right), \dots \left(RP(x_1), \dots RP(x_q)\right)\right)$$

$$\leq 2 \sum_{i \in [0, q-1]} (N-i)\left(\frac{1}{N-i} - \frac{1}{N}\right) = \frac{2q(q-1)}{N}$$

This style of argument does not really work against quantum adversaries which make queries in superposition

Grover Search Polynomial Lower Bound[Beals et al. 01]

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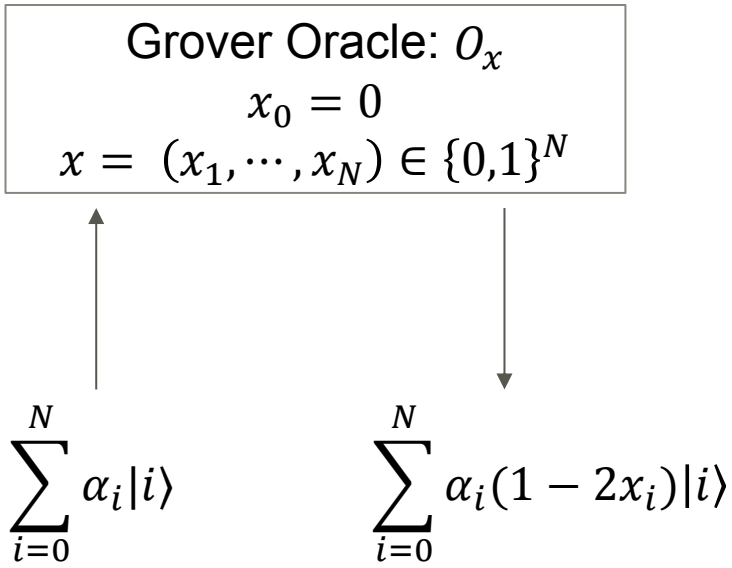

Evaluate $F(x_1, .., x_N): \{0,1\}^N \to [N]$

Output i s.t. $x_i = 1$

Observation:

- If we make T queries to Grover Oracle, Coefficient of $|i\rangle$ is a degree T multinomial in $x_1, \cdots, x_n$
- Probability of accepting $|i\rangle$ is a degree $2T$ multinomial

Fact: if $p(x_1, \ldots, x_N)$ approximates $F(x_1, \ldots x_N)$,
then degree of p is $\Omega(\sqrt{N})$ [Nissan-Szegedy 94]

AvgSearch Lower bound (polynomial)

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- In crypto we are interested in average case lower bounds (not worst case)

Oracle D_λ: Initialize Grover Oracle with

$(x_1, \cdots, x_N) \leftarrow \{0,1\}^N$, where $\Pr[x_i = 1] = \lambda$

AvgSearch Lower bound (polynomial) [Hulsing, Rijneveld, Song PKC16, Zhandry12]

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- One can transform Avgsearch adversary to distinguishing adversary between D_λ and D_0
- For any q-query adversary Probability of outputting any state $|j\rangle$ is a linear combination of $\Pr[x_{i_1} = b_1, x_{i_2} = b_2, \dots, x_{i_{2q}} = b_{2q}]$ for all possible $\left((i_1, b_1), \dots, (i_{2q}, b_{2q})\right)$ tuples
  - $\Pr[x_{i_1} = b_1, x_{i_2} = b_2, \dots, x_{i_{2q}} = b_{2q}]$ is a degree $2q$ polynomial in λ
- $Adv(D_\lambda, D_0) = p(\lambda) - p\,(0)$, where p is a degree $2q$ polynomial and $0 \le \lambda \le 1 \Rightarrow 0 \le p(\lambda) \le 1$
- By markov inequality, for $\lambda \in [0,1]$ we have

$$\max_{\lambda \in [0,1]} |p'(\lambda)| \le \deg(p)^2 \max_{\lambda \in [0,1]} |p(\lambda)| \le 4q^2$$

  - Shadrin, Aleksei. "Twelve proofs of the Markov inequality." *Approximation theory: a volume dedicated to Borislav Bojanov* (2004): 233-298.

- For $\lambda \in [0,1]$ we have $Adv(D_\lambda, D_0) = p(\lambda) - p\,(0) \le \lambda \max_{\lambda \in [0,1]} |p'(\lambda)| \le 4\lambda q^2$

Density Matrix and Mixed states

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- Until now we have seen only pure states
- Mixed states are probabilistic ensemble of mixed states
- $|\psi_{mixed}\rangle = \{(p_1, |\Psi_1\rangle), \dots, (p_n, |\Psi\rangle), \text{where}$
  - p_i's are probabilities which sum to 1
  - $|\Psi_1\rangle, |\Psi_2\rangle, \dots, |\Psi_n\rangle$ are pure states
- Can be represented by density matrix
  - $\rho_{mixed} = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$

Density Matrices

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- Density matrices are used to describe quantum subsystems
- ρ is a density matrix, if
  - ρ is hermitian : $\rho^+ = \rho$
  - ρ is positive semi definite: for any $|\Psi\rangle$ we have $\langle\Psi|\rho|\Psi\rangle$
  - $Tr(\rho) = 1$: diagonal elements should sum to 1
- Unitary Transformation: $\rho \xrightarrow{U} U\rho U^+$

Adversary Method : basic idea [Ambainis02]

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$$\rho_{start} = \rho_0 \xrightarrow{U_1 O_x U_0} \rho_1 \xrightarrow{U_2 O_x} \dots \xrightarrow{U_q O_x} \rho_q = \rho_{end}$$

- Fix ρ_{start} and find some nice properties (depending on notion of successful adversary) of ρ_{end}
- Define a function $S: Density\ Matrix \to \mathbb{R}$, s.t.
  - $S(\rho_{start}) = s_0$
  - $S(\rho_{end}) \leq s_q$ (use the nice properties of ρ_{end})
  - $S(\rho_{i-1}) - S(\rho_i) \leq \Delta$
- We have lower bound $q \geq \frac{s_0 - s_q}{\Delta}$
- Why Density Matrices, not pure states?

Adversarial lower bound [Ambainis02]

- $|\Psi_{start}\rangle = \sum_x \alpha_x |0\rangle \otimes |x\rangle \xrightarrow{U_q O \ldots U_1 O U_0} \alpha_x |\Psi_x\rangle \otimes |x\rangle$
- $\mathcal{H}_A \otimes \mathcal{H}_I$ total workspace of the algorithm
- Initially $\mathcal{H}_I$ and $\mathcal{H}_A$ are not entangled. In the end they become highly entangled
- The goal is to have some bound on this entanglement propagation.

Grover Search Adversarial Lower Bound [Ambainis02]

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$$|\Psi_{start}\rangle = \frac{|0\rangle \otimes (\,|100\ldots0\rangle + |010\ldots0\rangle + \cdots + |000\ldots1\rangle)}{\sqrt{N}}$$

$$|\Psi_{end}\rangle = \frac{|1\rangle|100\ldots0\rangle + |2\rangle|010\ldots0\rangle + \cdots + |N\rangle|000\ldots1\rangle)}{\sqrt{N}}$$

- ρ_{start} and ρ_{end} be the density matrices corresponding to $\mathcal{H}_I$ part of the workspace

$$\rho_{start} = \begin{pmatrix} \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{N} & \frac{1}{N} & \cdots & \frac{1}{N} \end{pmatrix} \qquad \rho_{end} = \begin{pmatrix} \frac{1}{N} & 0 & \cdots & 0 \\ 0 & \frac{1}{N} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{N} \end{pmatrix}$$

Grover Search Adversarial Lower Bound

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- The function S: sum of absolute values of off diagonal entries.
- $S(\rho_{start}) = N - 1, S(\rho_{end}) = 0$
- Observation: if measurement of $\mathcal{H}_A$ results in $|i\rangle$, then only i^{th} column and i^{th} row gets affected by every Oracle call O
- With some linear algebra and Cauchy Schwartz:
  - $|S(\rho_{i+1}) - S(\rho_i)| \leq 2\sqrt{N-1}$
- Hence, $q \geq 0.5\sqrt{N-1}$

AvgSearch: Adversarial Bound [Ambainis14]

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- a, k, v be the state, input and output register of the adversary.
- $O_f|a\rangle|k\rangle|v\rangle = |a\rangle|k\rangle|v + f(k)\rangle$
- Adversary interacts either with O_f or O_N
  - $f \leftarrow D_\lambda : \Pr[f(k) = 1] = \lambda$
  - all zero function $N \leftarrow D_0 : N(k) = 0 \; \forall k$

Trace Distance TD

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- ρ and σ be any two density matrices
- $TD(\rho,\sigma) = \frac{1}{2}Tr(\sqrt{(\rho-\sigma)^{+}(\rho-\sigma)})$
  - $A^{+}A$ is hermitian for any A. Hence it has square root.
- $TD(\rho,\sigma) \leq TD(\rho,\tau) + TD(\tau,\sigma)$
- If U is unitary, then $TD(U\rho U^{+}, U\sigma U^{+}) = TD(\rho,\sigma)$
- $Adv(\rho,\sigma) \leq TD(\rho,\sigma)$

AvgSearch: Adversarial Bound

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$$|\Psi\rangle \xrightarrow{UO_f} |\Psi_f^1\rangle \xrightarrow{UO_f} \dots \xrightarrow{UO_f} |\Psi_f^q\rangle$$

$$|\Psi\rangle \xrightarrow{UO_N} |\Psi_N^1\rangle \xrightarrow{UO_N} \dots \xrightarrow{UO_N} |\Psi_N^q\rangle$$

$$|\Psi_f^i\rangle = (UO_f)^i |\Psi\rangle$$

$$|\Psi_N^i\rangle = U^i |\Psi\rangle$$

$$D_f^i = TD(|\Psi_f^i\rangle, |\Psi_N^i\rangle) = TD(O_f|\Psi_f^{i-1}\rangle, |\Psi_N^{i-1}\rangle)$$

$$\leq TD(O_f|\Psi_f^{i-1}\rangle, O_f|\Psi_N^{i-1}\rangle) + TD(O_f|\Psi_N^{i-1}\rangle, |\Psi_N^{i-1}\rangle)$$

$$\leq D_f^{i-1} + TD(O_f|\Psi_N^{i-1}\rangle, |\Psi_N^{i-1}\rangle)$$

$$D_f^0 = TD(|\Psi_f^i\rangle, |\Psi_N^i\rangle) = TD(|\Psi\rangle, |\Psi\rangle) = 0$$

Claim: $\mathrm{Exp}[TD(O_f|\Psi_N^i\rangle, |\Psi_N^i\rangle)] \leq 2\sqrt{\lambda}$

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- Q_f be the projector that projects input register to $|k\rangle$ to a subspace where $f(z) = 1$
  - $Q_f = \sum_{z,f(z)=1} Q_z$, where $Q_z = (I \otimes |\mathrm{z}\rangle \otimes \mathrm{I})(I \otimes \langle \mathrm{z}| \otimes \mathrm{I})$
- $Exp\left[\left\|Q_f|\Psi_N^i\rangle\right\|^2\right] = \lambda$
- $\mathrm{Exp}\left[TD(O_f|\Psi_N^i\rangle, |\Psi_N^i\rangle)\right] \leq 2Exp\left[\left\|Q_f|\Psi_N^i\rangle\right\|\right]$

$$\leq 2\sqrt{Exp\left[\left\|Q_f|\Psi_N^i\rangle\right\|^2\right]} = 2\sqrt{\lambda}$$

AvgSearch: Adversarial Bound

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$$\text{Adv} \leq Exp\left[D_f^q\right] = Exp\left[D_f^{q-1}\right] + \text{Exp}[TD\left(O_f\left|\Psi_N^i\right\rangle, \left|\Psi_N^i\right\rangle\right)]$$

$$\leq 2q\sqrt{\lambda}$$

Random Function vs Random Permutation

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Zhandry's theorem[Zha12]:

D_r is a distribution on $X \to Y$ indexed by $\mathrm{r} \in \mathbb{Z} \cup \{\infty\}$, s.t. for every k pairs $(x_i, y_i) \in X \times Y$ the function $p(r) = \Pr_{f \leftarrow D_r}[f(x_i) = y_i, \forall [1,k]]$ is a polynomial of degree at most k in $\frac{1}{r}$; then $Adv(D_r, D_\infty) \leq \frac{Cq^3}{r}$ for some universal constant C

Random Function vs Random Permutation[Zhandry15]

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Define D_r as follows:

$$[N] \xrightarrow{F \leftarrow RF} [r] \xrightarrow{\pi \leftarrow RI} [N]$$

RI is a partial Random Injective function defined over range of $F([N])$

Observation:

D_N is a $[N] \rightarrow [N]$ Random Function

D_∞ is a $[N] \rightarrow [N]$ Random Permutation

By Zhandry's theorem:

$$\text{Adv(RF,RP)} \leq \frac{Cq^3}{N}$$

Conclusion

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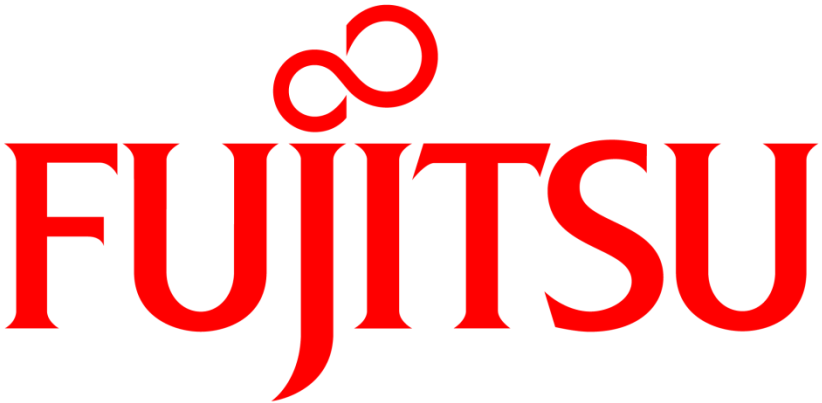
- Adversary Methods and Polynomial methods have a long line of research for proving lower bounds for Quantum Algorithms
- Polynomial method requires symmetry in the problem domain
- Depending on problem in worst case setting sometime
  - Adversary method gives better lower bound
  - Polynomial method gives better lower bound
  - They are equivalent
- However, the average case setting is still a relatively new line of research and not much is known on how these two methods compare against each other.
- Open Problem: Can we use adversary method to show RF , RP equivalence?



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