Introduction to **NP**-Completeness

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Organization

1. Preliminaries
2. Polynomial-time reductions
3. Polynomial-time Reduction: SAT $\leq_P$ 3SAT
4. Polynomial-time Reduction: SAT $\leq_P$ CLIQUE
5. Polynomial-time Reduction: SAT $\leq_P$ VERTEX COVER
6. Efficient certification, Nondeterministic Algorithm and the class NP
7. NP-Complete problems

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Preliminaries

Contrapositive statement

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Understanding a problem and an algorithm to solve that problem.

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Deterministic polynomial time algorithm

What are they?

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- Similarly, we have for independent set, vertex cover, chromatic number, etc.
- Notice that the optimization versions and decision versions are polynomially equivalent.
- Consider any decision problem Π and any instance I of Π. The problem is to classify any such instance I to either a yes or no instance.

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Beyond the class **P**

How do we grade problems based on their difficulty level? From here, starts the notion of polynomial-time reductions.

Polynomial-time reduction

Polynomial-time reducibility between decision problems Π and Π'

We say that Π reduces to Π' in deterministic polynomial-time, symbolized as $\Pi \leq_P \Pi'$, if there exists a deterministic polynomial time algorithm A that takes I, an instance of Π, as input and transforms it into I', an instance of Π' such that I is a yes-instance if and only if I' is an yes-instance.

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The contrapositive statement

Suppose $\Pi \leq_P \Pi'$. If Π cannot be solved in polynomial time, then Π' cannot be solved in polynomial time.

Polynomial-time reduction

Relative hardness among problems

Suppose $\Pi \leq_P \Pi'$. Then Π' is a harder problem than Π. Precisely, Π' is at least as hard as Π.

Polynomial reducibility is transitive

Transitivity of reductions

If $\Pi_1 \leq_P \Pi_2$, and $\Pi_2 \leq_P \Pi_3$, then $\Pi_1 \leq_P \Pi_3$.

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- An example. $f = (x_1 \vee x_2) \wedge (\overline{x_1} \vee x_3 \vee x_4 \vee \overline{x_5}) \wedge (x_1 \vee \overline{x_3} \vee x_4)$. If x_1 and x_3 are set to TRUE, then f is TRUE.

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- If the number of literals per clause is 3, then the boolean formula is a 3SAT formula.

SAT $\leq_P$ 3SAT

Lemma

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Proof

- Map a CNF formula ϕ into a 3CNF formula ψ such that ψ is satisfiable iff ϕ is.

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- Any clause C of size $k > 3$ can be changed to an equivalent pair of clauses C_1 of size $k-1$ and C_2 of size 3 by using an additional auxiliary variable.
- Say $C = \overline{x_1} \vee x_2 \vee x_3 \vee \overline{x_4}$. Let $C_1 = \overline{x_1} \vee x_2 \vee z$ and $C_2 = x_3 \vee \overline{x_4} \vee \overline{z}$. Clearly, if C is true, then there is an assignment to z that satisfies both C_1 and C_2 and vice versa.

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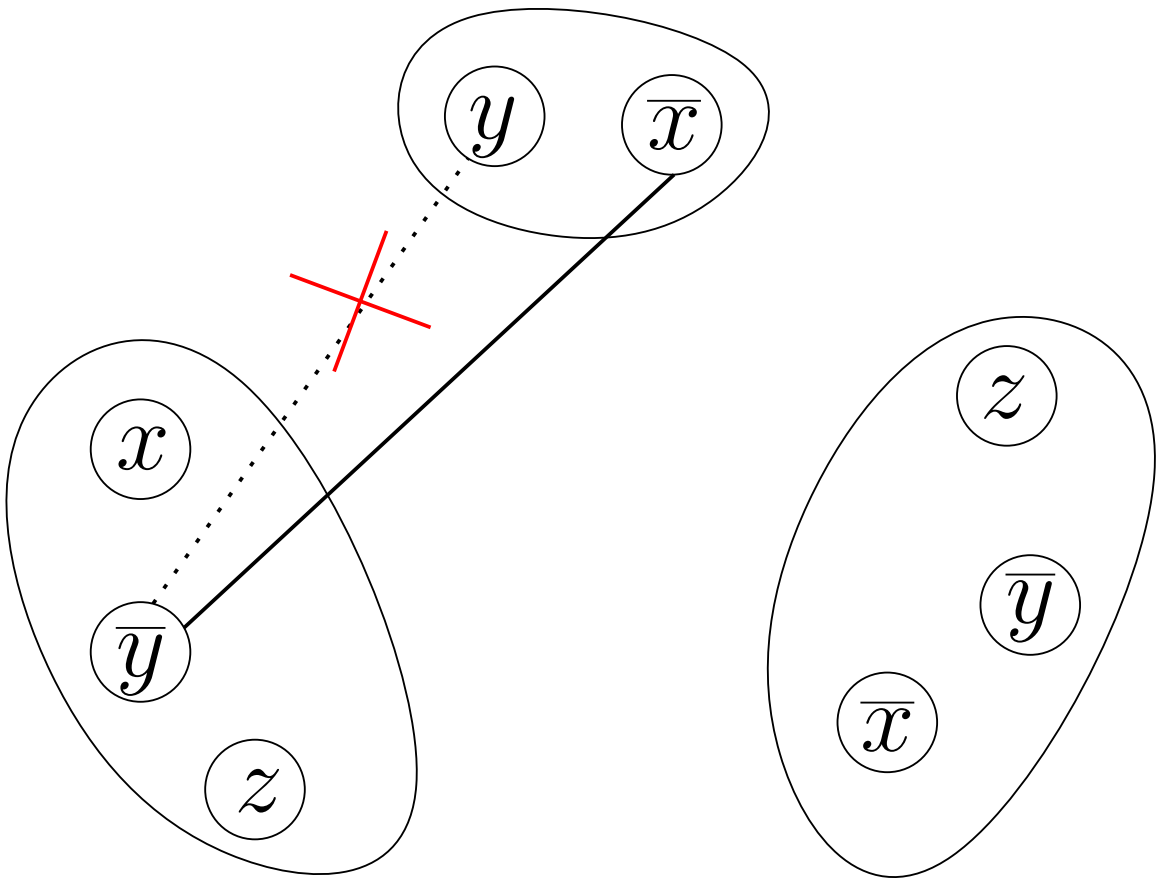

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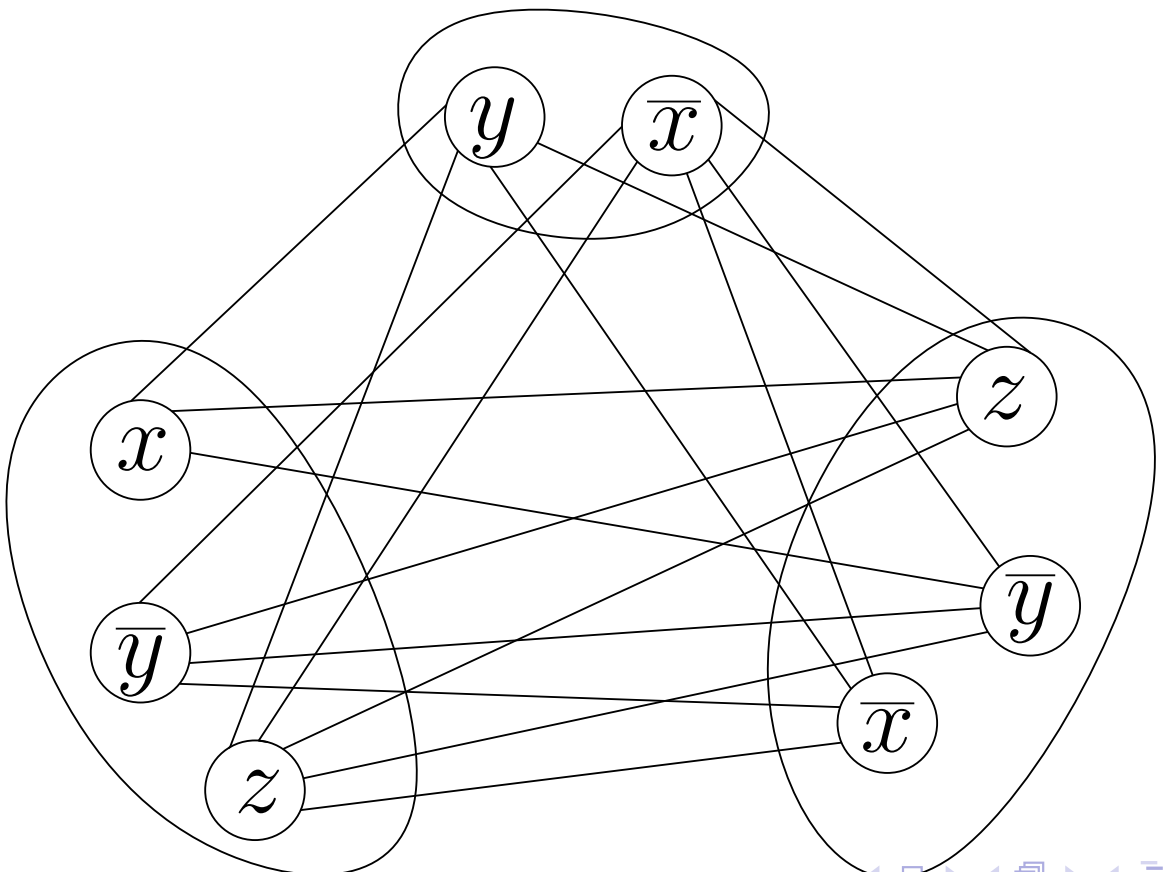

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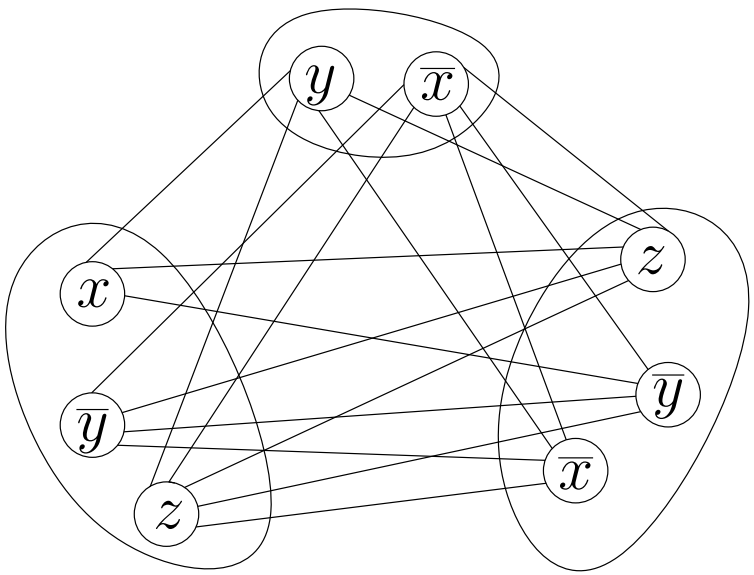

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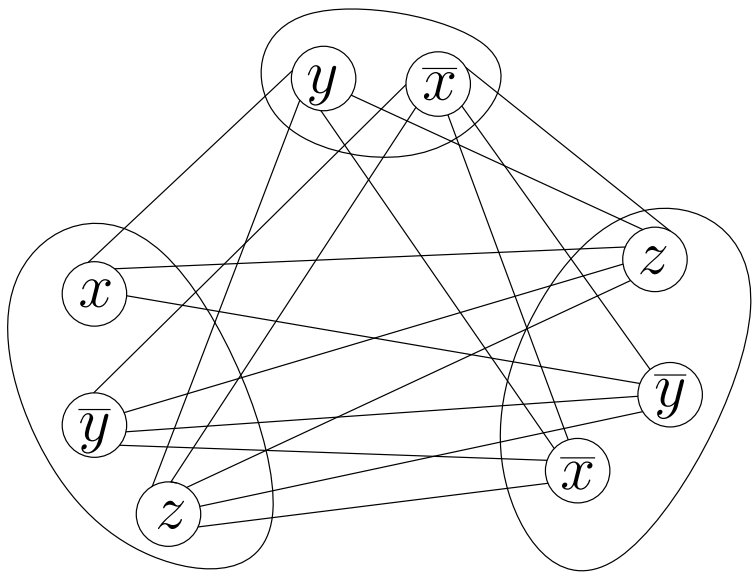

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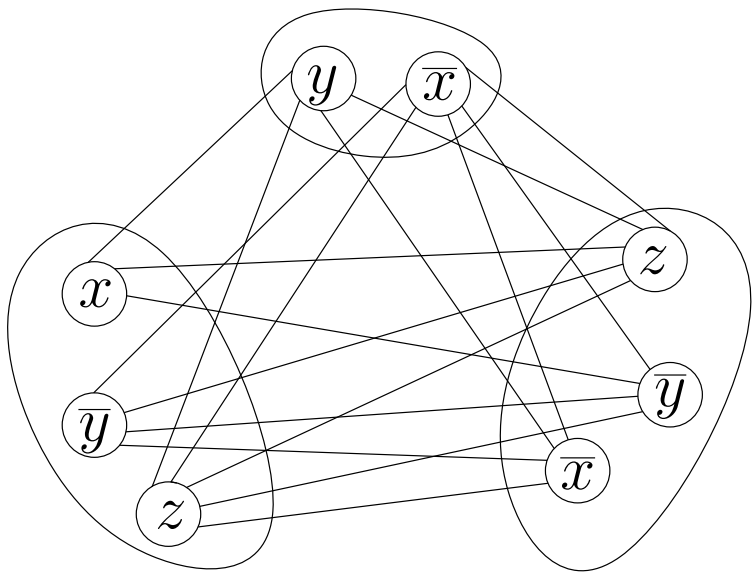
- By construction, there are edges between all pairs of these m different vertices, and hence, a clique of size m.



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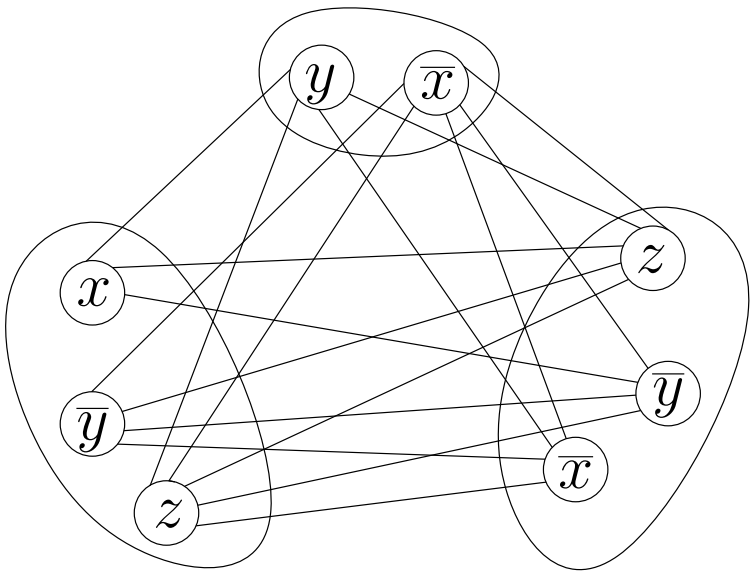

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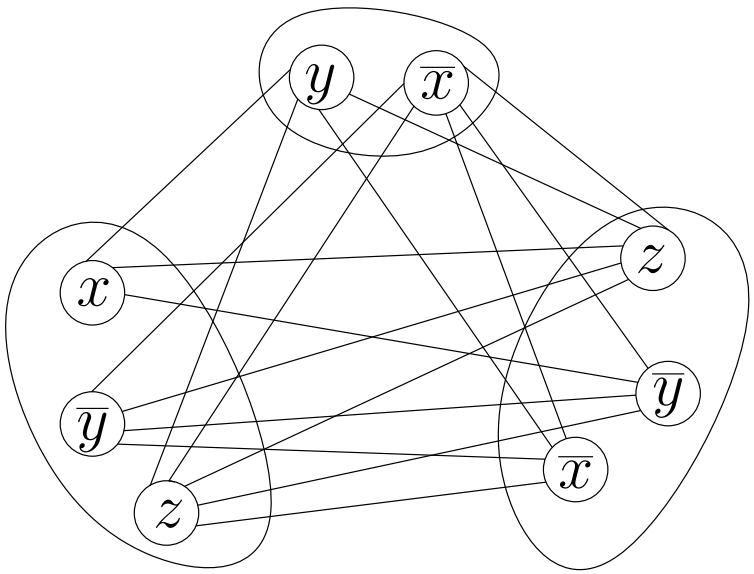

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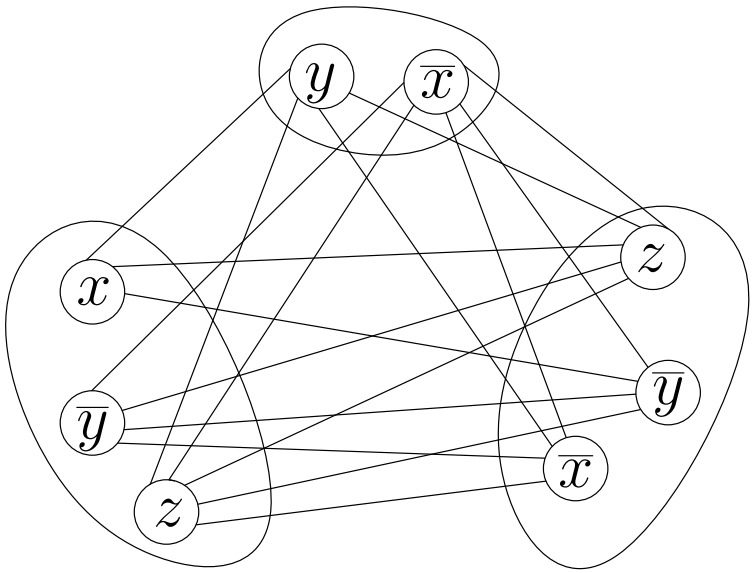

Theorem

f is satisfiable if and only if G has a clique of size m.

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VERTEX COVER

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Given an undirected graph $G = (V, E)$ and a positive integer k, is there a subset $C \subseteq V$ of size k such that each edge in E is incident to at least one vertex in C?
(Such a set C is a vertex cover of G.)

Exercise

For a clique of n vertices, what is the size of minimum vertex cover?

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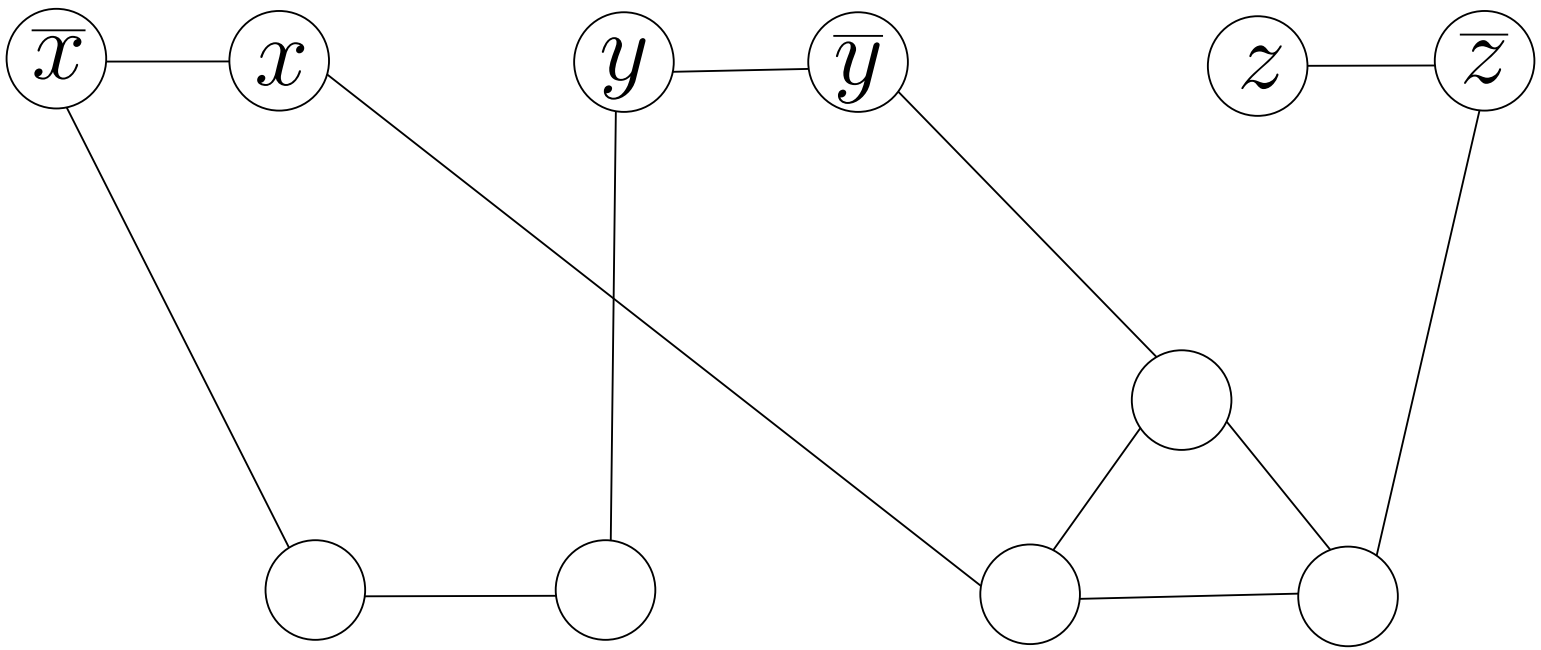

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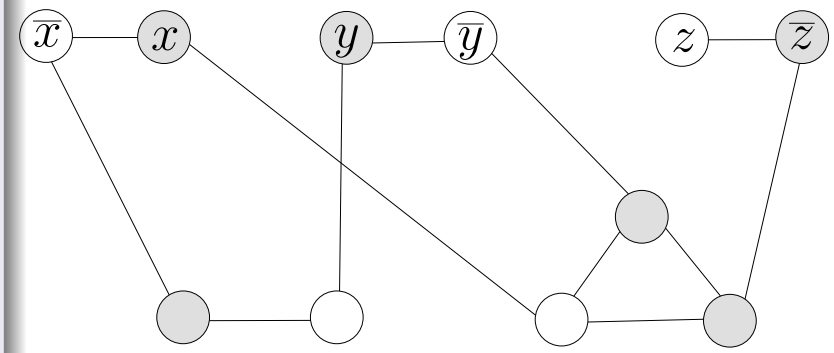

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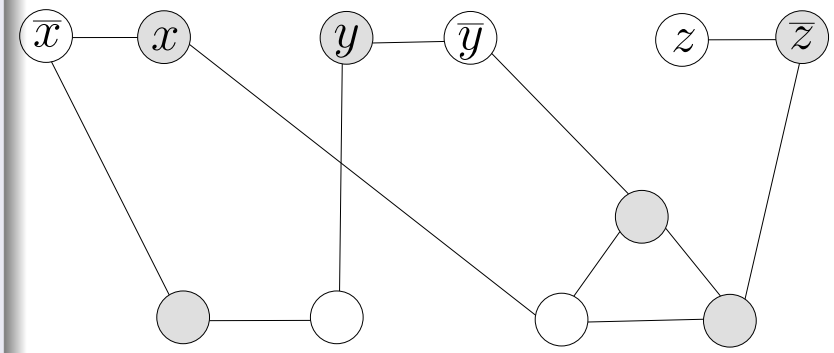
- If x_i is assigned TRUE, add vertex x_i to the VC; otherwise, add $\overline{x_i}$ to the VC.
- Since f is satisfiable, in each clique C_j there is a vertex w whose corresponding literal is TRUE; so a connection edge is covered. We add other $n_j - 1$ vertices in each C_j to the VC.



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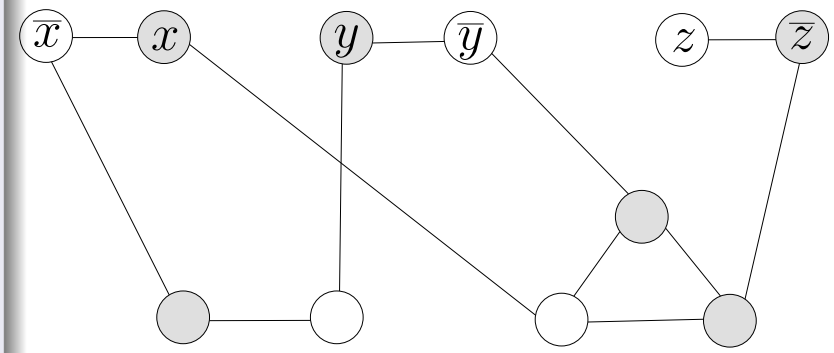
- Since f is satisfiable, in each clique C_j there is a vertex w whose corresponding literal is TRUE; so a connection edge is covered. We add other $n_j - 1$ vertices in each C_j to the VC.
- The size of the VC is $k = n + \sum_{j=1}^{m}(n_j - 1)$.



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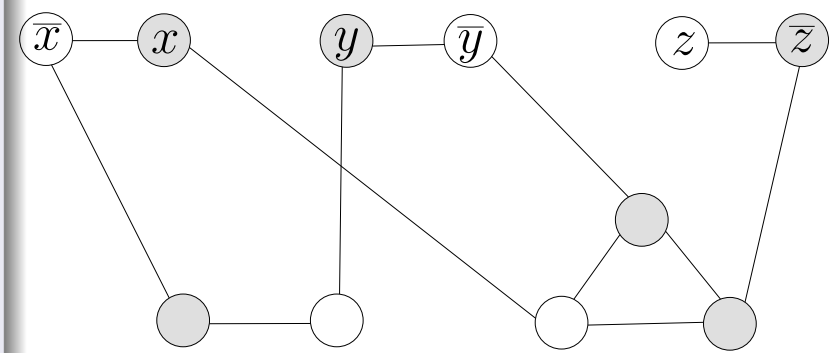

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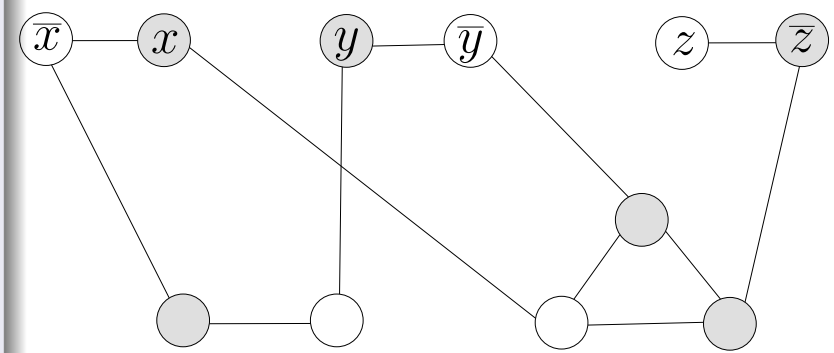
- At least one vertex of each edge $(x_i, \overline{x_i})$ must be in the VC. We are left with $k - n = \sum_{j=1}^{m}(n_j - 1)$ vertices.



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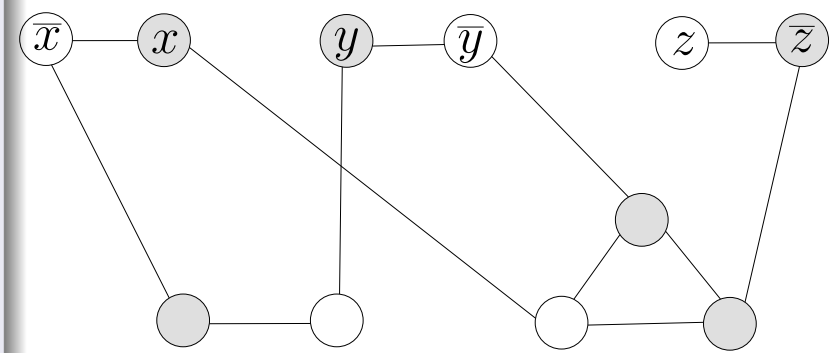

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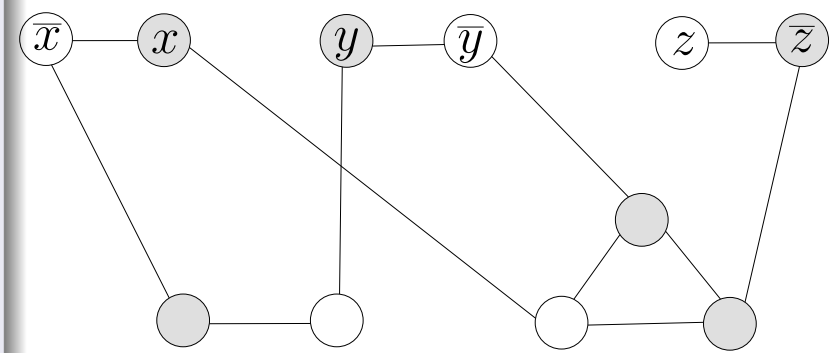
- At least one vertex of each edge $(x_i, \overline{x_i})$ must be in the VC. We are left with $k - n = \sum_{j=1}^{m}(n_j - 1)$ vertices.
- For each clique C_j, pick the rest $n_j - 1$ vertices in the VC.
- For each vertex x_i, if it is in the VC, let x_i be TRUE; else x_i be FALSE. Thus for each clique, there is a vertex having TRUE.



SAT $\leq_P$ VC

Theorem

f is satisfiable if and only if G has a VC of size k.

Outline

1. Preliminaries
2. Polynomial-time reductions
3. Polynomial-time Reduction: SAT $\leq_P$ 3SAT
4. Polynomial-time Reduction: SAT $\leq_P$ CLIQUE
5. Polynomial-time Reduction: SAT $\leq_P$ VERTEX COVER
6. Efficient certification, Nondeterministic Algorithm and the class NP
7. NP-Complete problems

Understanding Certifier

Decision problem and strings

We identify a decision problem L with the set of strings $x \in \{0,1\}^*$ on which the answer is YES.

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Definition: The Class **NP**

A decision problem L is in **NP** if $\exists$ a polynomial $p : \mathbb{N} \to \mathbb{N}$ and a polynomial-time algorithm M such that for every string x,

$$x \in L \iff \exists \text{ a short certificate } u \text{ such that } M(x,u) = \text{YES}$$

u is a short certificate if $|u| = p(|x|)$.
If $x \in L$ and $u \in \{0,1\}^{p(|x|)}$ satisfy $M(x,u) =$ YES, then we call u a certificate for x (w.r.t. problem L and algorithm M).

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Examples

Traveling Salesperson, Subset sum, IP, LP, Graph Isomorphism, etc.

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- It will not try to decide whether $x \in L$ on its own.
- It will rather try to efficiently evaluate proposed "proofs" u that $x \in L$ - provided they are not too long.

A Brute Force Algorithm

M's use in solving L

- On an input x, try all strings u, s.t. $|u| \leq p(|x|)$, and see if $M(x, u) = 1$ for any of these strings.

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- On an input x, try all strings u, s.t. $|u| \leq p(|x|)$, and see if $M(x, u) = 1$ for any of these strings.
- Existence of M does not provide an efficient solver for L.
- It is upto us to find a string u that will make $M(x, u) = 1$, and there are exponentially many possibilities for u.

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Relation between the class **P** and **NP**

Does $\mathbf{P} = \mathbf{NP}$? Or, is $\mathbf{P} \subset \mathbf{NP}$?

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  - $\forall L' \in \mathbf{NP}$, $L' \leq_P L$.

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- With $\mathbf{P} \subseteq \mathbf{NP}$ already known, we have $\mathbf{P} = \mathbf{NP}$.

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The first **NP**-complete problem

How do you get the first such problem Z? Cook Levin Theorem shows SAT to be such a problem.

SAT **NP**-complete

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  - If $x_{L'}$ is a YES instance of L', then x_L is a YES instance of L;
  - If x_L is a YES instance of L, then $x_{L'}$ is a YES instance of L';
  - The above two steps ensure that $x_{L'}$ and x_L have the same answer.

NP-complete problems

As SAT is **NP**-complete, and
3SAT $\in$ **NP** and SAT $\leq_P$ 3SAT,
so 3SAT is **NP**-complete.

Similarly, CLIQUE and VERTEX COVER are
NP-complete.

At Last!!!

Thank you