INDIAN STATISTICAL INSTITUTE

Mid Semestral Examination

M. Stat. – I Year, 2017-2018 (Semester – II)

Optimization Techniques

Date: 21.02.2018 Maximum Marks: 60 Duration: 2 hours 30 minutes

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Note: The question paper is of 75 marks. Answer as much as you can, but the maximum you can score is 60.

Vectors would be written in small letters with boldface, e.g. $\mathbf{b}$; matrices would be written in capital letters, e.g., A. Transpose of A would be denoted by A^T and transpose of $\mathbf{b}$ would be denoted by $\mathbf{b}^T$. Whenever we say that, $\mathcal{P}$ is a linear program, we mean $\mathcal{P}$ is of the form

$$\begin{array}{lrcl} \text{Maximize} & \mathbf{c}^T\mathbf{x} & & \\ \text{subject to} & A\mathbf{x} & \leq & \mathbf{b} \\ & \mathbf{x} & \geq & \mathbf{0} \end{array}$$

(Q1) Use SIMPLEX method to solve the following LP.

$$\begin{array}{llcl} \text{Minimize} & x_0 & & \\ \text{subject to} & 8x_1 - 7x_2 - x_0 & \leq & 0 \\ & -2x_1 + x_2 - x_0 & \leq & 0 \\ & x_1 + x_2 & = & 1 \\ & x_1, x_2 & \geq & 0 \end{array}$$

[10]

(Ans:) Since x_0 is an *unrestricted variable*, we first introduce two non-negative variables y_1 and y_2 such that $x_0 = y_1 - y_2$. Secondly, we introduce *slack variables* s_1 and s_2 and *artificial variable* s_3. The resulting LP can be rewritten as:

$$\begin{array}{llcl} \text{Maximize} & -y_1 + y_2 & & \\ \text{subject to} & -y_1 + y_2 + 8x_1 - 7x_2 + s_1 & = & 0 \\ & -y_1 + y_2 - 2x_2 + s_2 & = & 0 \\ & x_1 + x_2 + s_3 & = & 1 \\ & x_1, x_2, y_1, y_2 & \geq & 0 \end{array}$$

[10]

The following results can be found in successive iterations of SIMPLEX algorithm:

Iteration 1: entering variable y_2, departing variable s_1 and pivot element 1.
Iteration 2: entering variable x_2, departing variable s_2 and pivot element 8.

Iteration 3: entering variable x_1, departing variable s_3 and pivot element 2.25.

To get the exact optimal value, multiply by -1. The optimum value is $-\frac{1}{3}$ and $x_1 = \frac{4}{9}$ and $x_2 = \frac{5}{9}$.

(Q2) Let $f(x) = \max(\mathbf{c}_1{}^T\mathbf{x} + d_1, \mathbf{c}_2{}^T\mathbf{x} + d_2, \ldots, \mathbf{c}_p{}^T\mathbf{x} + d_p)$. For such a function f, consider the mathematical program

$$\begin{array}{lll} \text{Minimize} & f(x) & \\ \text{subject to} & A\mathbf{x} & \leq \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \end{array}$$

Can you convert this mathematical program to a linear program? Explain with proper arguments. [10]

(Ans:) The nonlinear function $f(x)$ can be converted to the following set of *linear* constraints: $y \geq \mathbf{c}_i{}^T\mathbf{x} + d_i$ where $i \in \{1, 2, \ldots, p\}$. As y is greater than each of the linear functions, y will be greater than the maximum. The resulting linear program can be written as:

$$\begin{array}{lll} \text{Minimize} & y & \\ \text{subject to} & y & \geq \mathbf{c}_i{}^T\mathbf{x} + d_i, \;\; i \in \{1, 2, \ldots p\} \\ & A\mathbf{x} & \leq \mathbf{b} \\ & \mathbf{x} & \geq \mathbf{0} \end{array}$$

(Q3) Let $P, Q \subseteq \mathbb{R}^n$ be convex sets and let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a strictly convex function. Suppose that x^* is an optimum solution to $\min\{f(x) \mid x \in P \cap Q\}$ and x^* lies in the interior of Q. Show that x^* is also an optimum solution to $\min\{f(x) \mid x \in P\}$. [10]

(Ans:) Suppose for the sake of contradiction that there is a $y^* \in P$ with $f(y^*) < f(x^*)$, then some convex combination $(1 - \lambda)y^* + \lambda x^*$ with $0 < \lambda < 1$ lies also in Q and has a better objective function than x^*, which is a contradiction.

(Q4) (a) Deduce the dual of the following LP, where A is an $m \times n$ matrix, $\mathbf{x}$ is an m-dimensional vector, and $\mathbf{y}$ is an n-dimensional vector:

$$\begin{array}{lll} \text{Minimize} & \mathbf{x}^T A\mathbf{y} & \\ \text{subject to} & \sum_{j=1}^{n} y_j & = 1 \\ & \mathbf{y} & \geq \mathbf{0} \end{array}$$

(Ans:) The dual is

$$\begin{array}{lll} \text{Maximize} & z_0 & \\ \text{subject to} & \mathbf{1}\, z_0 & \leq A^T\mathbf{x} \end{array}$$

(b) Deduce the conditions on A, $\mathbf{b}$ and $\mathbf{c}$ so that the primal linear program $\mathcal{P}$ and its dual $\mathcal{D}$ are the same linear program.

(Ans:) A should be skew symmetric, i.e., $A + A^T = 0$, and $\mathbf{b} = -\mathbf{c}$.

(c) Prove that $\mathcal{P}$ has an optimal solution if and only if the following set of constraints has a feasible solution.

$$\begin{aligned} A\mathbf{x} &\leq \mathbf{b} \\ A^T\mathbf{y} &\geq \mathbf{c} \\ \mathbf{c}^T\mathbf{x} &\geq \mathbf{b}^T\mathbf{y} \\ \mathbf{x}, \mathbf{y} &\geq \mathbf{0} \end{aligned}$$

(Ans:) Follows from weak and strong duality.

[7+3+5=15]

(Q5) (a) Describe the steps of a randomized incremental algorithm for solving LPs where the dimension $d = 2$. Clearly mention any assumptions that you make.

(b) Analyze the expected time taken by your algorithm when the dimension $d = 2$.

(c) Generalize the above analysis of the expected time taken by the algorithm when d is a variable parameter and not a constant by forming a recurrence involving n and d. There is no need of solving the recurrence.

(Ans:) Let $T(n, d)$ denote the expected running time of the algorithm for n halfspaces in dimension d. For $d + 1 in$, let p_i denote the probability that the insertion of the i-th hyperplane in the random order, results in changing the optimum vertex. To handle the case, when there is no change, takes $O(d)$ time and this happens with probability $(1 - p_i)$. For the case, when there is a change in the optimum vertex, we need to intersect the previous $i - 1$ halfspaces with the current halfspace; this takes $O(d(i - 1))$ time. We then invoke a $d - 1$-dimensional LP with $i - 1$ halfspaces in dimension $d - 1$. Thus, we have with $p_i = d/i$,

$$\begin{aligned} T(n, d) &\leq \textstyle\sum_{i=d+1}^{n} \left((1 - p_i)d + p_i(di + T(i, d - 1))\right) \\ &\leq \textstyle\sum_{i=d+1}^{n} \left(d + p_i(di + T(i, d - 1))\right) \end{aligned}$$

[7+3+5=15]

(Q6) In the *set cover* problem, we have an universe $\mathcal{U} = \{u_1, \ldots, u_n\}$ of n elements. Let $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a set of m sets, where each set $S_i \subseteq \mathcal{U}$. Each set S_i has a weight $w_i \geq 0$. The problem in *set cover* is to find a minimum weight collection of subsets of $\mathcal{S}$ that covers all elements of $\mathcal{U}$.

(a) Write an integer linear program (ILP) for the *set cover problem* using decision variables x_i to indicate whether the set S_i is included in the solution or not.

(b) Relax the above ILP and round the optimal solution of the linear program as follows: given the optimal solution $\mathbf{x}^*$ of the linear program, we include the subset S_i in our solution if and only if $x_i^* \geq \frac{1}{f}$, where f is the maximum number of sets in which any element appears and x_i^* is the i-th component of $\mathbf{x}$.

For this rounding scheme, show that the set generated is a set cover and is an f-factor approximation algorithm.

[5+10=15]

(Ans:) (a) In order to ensure that every element u_i is covered, it must be the case that at least one of the subsets S_j containing u_i is selected, i.e., $\sum_{S_j:u_i\in S_j} x_j \geq 1$ for each u_i. Let $x_j = 1$ if S_j is included in the solution; otherwise, $x_j = 0$. The resulting ILP is:

$$\begin{array}{lccc} \text{Minimize} & \sum_{j=1}^{m} w_j x_j & & \\ \text{subject to} & \sum_{S_j:u_i\in S_j} x_j \quad \geq & 1 & \forall u_i \in \mathcal{U} \\ & x_j \quad\quad \in & \{0,1\} & \end{array}$$

(b) The ILP is relaxed by replacing $x_j \in \{0,1\}$ by $x_j \geq \{0,1\}$. Let the optimal values of the ILP and the relaxed LP are Z^{ILP} and Z^{LP}. Clearly, $Z^{LP} \leq Z^{ILP}$. Let x^* denote the optimal solution of the LP. The rounding scheme is as follows: We include the set S_j in optimal solution if $x_j^* \geq \frac{1}{f}$ where f is the maximum number of sets in which any element appears. Let I denote the indices j of the subsets in this solution. We round the fractional solution x_j^* to an integer solution $x^{'}$ by setting $x_j^{'} = 1$ if $x_j^* \geq \frac{1}{f}$, and $x_j^{'} = 0$ otherwise.

We call an element u_i is covered if this solution contains some subset containing u_i. Because the optimal solution x^* is a feasible solution to the linear program, we know that $\sum_{S_j:u_i\in S_j} x_j^* \geq 1$ for element u_i. So atleast one term must be atleast $\frac{1}{f}$. Therefore $j \in I$ and the element u_i is covered. Hence, the collection of subsets $S_j, j \in I$ is a set cover.

It is clear that the algorithm runs in polynomial time. By our construction, $1 \leq f.x_j^*$ for each $j \in I$. From this and the fact that each term $f.w_j.x_j^*$ is nonnegative for $j = 1, 2, \ldots, m$, we see that

$$\sum_{j\in I} w_j \leq \sum_{j=1}^{m} w_j.(f.x_j^*) = f\sum_{j=1}^{m} = f.Z^{LP} \leq f.Z^{ILP}.$$