INDIAN STATISTICAL INSTITUTE

Tutorial Sheet I

B. Stat. - III Year, 2014-2015 (Semester - VI)

Design and Analysis of Algorithms

Note: This is a collection of problems.

(Q1) Show that $\log n! = O(n \log n)$ and $\log n! = \Omega(n \log n)$.

(Q2) The asymptotic notation O satisfies the transitive property, i.e. if $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$.

Now, $2^{n+1} = O(2^n)$. Extending this further, we can write $2^n = O(2^{n-1})$, $\ldots$, $2^i = O(2^{i-1})$, $\ldots$. Using the transitive property, we can write $2^{n+1} = O(2^{i-1})$. We can go on extending this, so that finally $2^{n+1} = O(2^k)$, where k is a constant. So, we can write $2^{n+1} = O(1)$.

Question: Do you agree to what has been proved? If not, where is the fallacy?

(Q3) $f(n) \prec g(n)$ denotes that $f(n) = o(g(n))$. Using this notation, find the hierarchy of the following functions: $\log^2 n$, 2^{n^2}, $\log\log n$, $n!$, 2^n, $n^{4/5}$, $\sqrt(n)$; and fill up the following.

	$\prec$		$\prec$		$\prec$		$\prec$		$\prec$		$\prec$	

(Q4) We define a palindrome as a string over the alphabet set $\Sigma = \{a, b, c, \ldots, z\}$ (where $|\Sigma| = 26$) that reads the same both forward and backward. tenet is an example of a palindrome. Using recurrence relation, find out a closed form expression of the number $P(n)$ of palindromes of length n over Σ.

(Q5) $X = \{x_1, x_2, \ldots, x_n\}$ is a sorted array of n integers and x is an integer. Design an efficient algorithm to determine whether there are two elements in X whose sum is exactly x. Also, find out the complexity of the algorithm you designed.

(Q6) Let T be the tree representation of a binary heap. Let x and y $(x, y > 0)$ be two elements in the heap and let the depth of x and y be d_x and d_y respectively. Now, consider the following statement. $\mathbf{x > y}$ **implies** $\mathbf{d_x > d_y}$. Prove or disprove the statement made.

(Q7) Can a heap also have the property of a binary search tree? If yes, try to give an example of such a tree of depth 4. Can you generalize such a construction for trees of arbitrary depth?

(Q8) Given an array $X = [x_1, x_2, \ldots, x_n]$ such that $0 \le x_i \le k$ $(i = 1, 2, \ldots, n)$. You are allowed to preprocess X such that *counting queries* can be reported efficiently. A *counting query* is a query such that given $[a, b]$ $(0 \le a, b \le k)$, you are to report the number of elements (and not the elements themselves!) that lie within $[a, b]$. Try designing an efficient preprocessing and query algorithm to answer such queries.

(Q9) Given two strings $X = [x_1, x_2, \ldots, x_n]$ and $Y = [y_1, y_2, \ldots, y_m]$, the *shortest common supersequence* (SCS) is a minimum length string Z such that both X and Y are subsequences of Z. For example, if $X = [abcbdab]$ and $Y = [bdcaba]$, a SCS is $Z = [abdcabdab]$. Give a dynamic programming solution that given X and Y finds out the length of the SCS Z only. Explain the correctness.

(Q10) The following problem is known as the *edit distance* problem. Given two strings S_1 of n_1 characters and S_2 of n_2 characters, the problem is to modify S_1 and convert it into S_2 through a sequence of character edits. The character edits can be insertion, deletion and overwrite. Each character edit takes unit amount of time. Design and analyze an efficient algorithm that minimizes the number of edits (and hence the time) required. The characters belong to the same alphabet set.

(Q11) We are given n matrices $M_1, M_2, \ldots, M_n$. In how many ways can we multiply them?

(Q12) We studied a dynamic programming algorithm to find the length of the LCS. Modify the algorithm suitably to find the LCS.

(Q13) Given n nodes, how many distinct binary trees can be formed?

(Q14) Given a convex polygon $P = \{p_1, p_2, \ldots, p_n\}$ of n vertices, given in a clockwise order, and a point p, devise an $O(\log n)$ time algorithm to find the tangents from p to P.

(Q15) Find the number of triangulations possible for a convex polygon of n vertices.

(Q16) Is the convex hull the smallest perimeter set that encloses the set of points?

(Q17) Show that the number of edges in a tree with n vertices is $n-1$.

(Q18) Let $G = (V, E)$ be a planar connected graph with F faces. Show that $|V| - |E| + |F| = 2$.

(Q19) Show that the points corresponding to the unbounded cells of the Voronoi diagram of the point set P are the vertices of the convex hull of P.

(Q20) Show that each point of $\mathbb{R}^2$ belongs to the Voronoi diagram of a point set P.

(Q21) Let P be a set of n points in general position. Prove that the Voronoi diagram of P is connected.

(Q22) What is the average number of vertices per Voronoi cell? Prove your result.

(Q23) Let P be a set of n points in general position. Consider the dual graph of the Voronoi diagram, known as Delaunay triangulation. Find the number of triangles and the number of edges in the Delaunay triangulation.

(Q24) Let P be a set of n points. Find the closest point for each point $p \in P$.

(Q25) Let S be a set of n line segments in the plane. Prove that the convex hull of S is exactly the same as the convex hull of the endpoints of the segments in S.

(Q26) (a) Let P_1 and P_2 be two disjoint convex polygons with n vertices in total. Give an $O(n)$ time algorithm that computes the convex hull of $P_1 \cup P_2$.

(b) Use the algorithm from (a) to develop an $O(n \log n)$ time divide-and-conquer algorithm to compute the convex hull of a set of n points in the plane.

(Q27) Let S be a set of n (possibly intersecting) unit circles in the plane. Compute the convex hull of S.