

INDIAN STATISTICAL INSTITUTE

Class Test I

M Tech (CS), 2024 (Semester – I)

Probability and Stochastic Processes

Date: 02.09.2024

Maximum Marks : 30

Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course:(MTech/JRF/PLP) _____ **Name:** _____ **Roll Number:** _____

- (Q1) (a) For n events, A_1, \dots, A_n , show that $\Pr\left(\bigcap_{i=1}^n A_i\right) \geq 1 - n + \sum_{i=1}^n \Pr(A_i)$
- (b) Let A and B be events with probabilities $\Pr(A) = 0.8$ and $\Pr(B) = 0.4$. Find out an upper and lower bound on $\Pr(A \cap B)$.

[5+5]

(Ans:) (a)

$$\begin{aligned} 1 - \Pr\left(\bigcap_{i=1}^n A_i\right) &= \Pr\left(\overline{\bigcap_{i=1}^n A_i}\right) \\ &= \Pr\left(\bigcup_{i=1}^n \overline{A_i}\right) \\ &\leq \sum_{i=1}^n \Pr(\overline{A_i}) \quad [\text{by union bound}] \\ &= \sum_{i=1}^n (1 - \Pr(A_i)) \\ &= n - \sum_{i=1}^n \Pr(A_i) \end{aligned}$$

(b)

$$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) \geq \Pr(A) + \Pr(B) - 1 = (0.8 + 0.4) - 1 = 0.2$$

$$\text{Also, since } A \cap B \subseteq A \text{ and } A \cap B \subseteq B, \Pr(A \cap B) \leq \min\{\Pr(A), \Pr(B)\} = 0.4$$

$$\text{So, } 0.2 \leq \Pr(A \cap B) \leq 0.4$$

- (Q2) (a) Find out the probability that exactly one of the events A and B occur in terms of probability of events A , B and their intersection.
- (b) Consider n independent tosses of a fair coin. Which sequence of outcome is more probable to happen – $\underbrace{HH \cdots H}_{n \text{ H's}}$ or $\underbrace{HTHT \cdots HT}_{n/2 \text{ pairs of H's and T's}}$? Argue with a short justification.
- (c) If an event A is independent of itself, what is the value of $\Pr(A)$?

[5+2+3=10]

(Ans:)

(a)

$$\Pr(A\Delta B) = \Pr((A \cup B) \setminus (A \cap B)) = \Pr(A \cup B) - \Pr(A \cap B) = \Pr(A) + \Pr(B) - 2\Pr(A \cap B).$$

(b) As tosses are independent and the coin is fair, so both possibilities are equiprobable with a probability of $(\frac{1}{2})^n$.

(c) Let $\Pr(A) = p$. If A is independent of itself, then

$$p = \Pr(A) = \Pr(A \cap A) = \Pr(A) \cdot \Pr(A) = p^2.$$

So, $p = 0$ or $p = 1$.

(Q3) We toss n coins and each turns up heads with probability p independently of other tosses. There are q rounds of tosses. In the first round, all coins are tossed. In the second round, each coin that shows up heads in the first round, is tossed again. In any subsequent round i , $1 < i \leq q$, each coin that shows up heads in round $i - 1$, is tossed again. Deduce the PMF of the number of heads resulting from the q -th round of toss. [10]

(Ans:)

Let X be the number of heads in the q -th round. We view it in a different light.

X is the same as if we toss all the coins q times and count the number that show heads on all the q occasions. Each coin shows heads q times with probability p^q . So,

$$\Pr(X = k) = \binom{n}{k} p^{qk} (1 - p^q)^{n-k}$$

(Q4) Let X_1, \dots, X_n be independent and identically distributed (i.i.d.) geometric random variables, each with success probability p .

(a) Show that $\Pr(X_i > t) = (1 - p)^t$.

(b) Using the above result, prove that $\Pr(\max_i X_i > t) \leq n(1 - p)^t$.

[3+7=10]

(a) X_i 's are i.i.d. geometric random variables. So, we have

$$\begin{aligned}\Pr(X_i > t) &= 1 - \Pr(X_i \leq t) \\ &= 1 - (1 - (1 - p)^t) \\ &= (1 - p)^t\end{aligned}$$

(b) Now, we have

$$\begin{aligned}\Pr(\max_i X_i > t) &= \Pr(\text{at least one } X_i > t) \\ &= \Pr\left(\bigcup_{i=1}^n (X_i > t)\right) \\ &\leq \sum_{i=1}^n \Pr(X_i > t) \text{ (using union bound)} \\ &= n(1 - p)^t\end{aligned}$$