INDIAN STATISTICAL INSTITUTE

Class Test III

M Tech (CS), 2023 (Semester – I) *Probability and Stochastic Processes*

Date: 10.11.2023 Maximum Marks : 30 Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course:(MTech/JRF/PLP) ____________ **Name:** ______________________________ **Roll Number:** ____________

(Q1) Let $X_1, \ldots, X_n$ be independent uniform random variables over $[0, 1]$. Let $Y = \min(X_1, \ldots, X_n)$. Show that $\mathrm{E}[Y] = \frac{1}{n+1}$. [10]

(Ans 1:) We know for a continuous random variable X, that takes only nonnegative values, and has a density function $f(x)$,

$$\begin{aligned}
\int_{x=0}^{\infty} \Pr(X \geq x)\ dx &= \int_{x=0}^{\infty} \int_{y=x}^{\infty} f(y)\ dy\ dx \\
&= \int_{y=0}^{\infty} \int_{x=0}^{y} f(y)\ dx\ dy \\
&= \int_{y=0}^{\infty} y f(y)\ dy \\
&= E[X].
\end{aligned}$$

Now, as per the question, $Y_1 = \min(X_1, \ldots, X_n)$ and Y_1 takes value in the range $[0, 1]$. We want to find $E[Y_1]$ which is $\int_{y=0}^{1} \Pr(Y_1 \geq y)$. So, we need to figure out the probability of $\Pr(Y_1 \geq y)$.

$$\begin{aligned}
\Pr(Y_1 \geq y) &= \Pr(\min(X_1, \ldots, X_n) \geq y) \\
&= \Pr\left((X_1 \geq y) \cap (X_2 \geq y) \cap \cdots \cap (X_n \geq y)\right) \\
&= \prod_{i=1}^{n} \Pr(X_i \geq y) \\
&= (1-y)^n
\end{aligned}$$

Therefore, $E[Y_1] = \int_{y=0}^{1} (1-y)^n\ dy = \frac{1}{n+1}$. ◀

(Q2) (i) Let f and g be two PDFs corresponding to two random variables X and Y, respectively. Show that the function $h = \alpha f + (1-\alpha)g$ is also a PDF, where $0 \leq \alpha \leq 1$.

(ii) Compute the moment generating function of a normal random variable $X \sim N(\mu, \sigma^2)$. [5+5=10]

(Ans 2(i)) As $0 \leq \alpha \leq 1$, and $f, g \geq 0$, so $h \geq 0$. For h to be a pdf, we need to show $\int_{\mathbb{R}} h dx = 1$. Indeed it is, because $\int_{\mathbb{R}} h\, dx = \int_{\mathbb{R}} \alpha\, f\, dx + \int_{\mathbb{R}} (1-\alpha)\, g\, dx = \alpha \int_{\mathbb{R}} f\, dx + (1-\alpha) \int_{\mathbb{R}} g\, dx = \alpha \cdot 1 + (1-\alpha) \cdot 1 = 1.$ ◀

(Ans 2(ii)) Done in the class.

(Q3) Let $N, X_1, X_2, \ldots$ be independent random variables, where X_i's are Bernoulli with parameter p, and N takes nonnegative integral values. Let $Y = X_1 + \ldots + X_N$ and $Y = 0$ when $N = 0$. Show that if N is binomial with parameters m and q, then Y is binomial with parameters m and pq. [10]

(Ans 3:) The moment generating function of Y, i.e., $M_Y(t) = E\left[e^{tY}\right]$.

$$\begin{aligned}
E\left[e^{tY}\right] &= E\left[E[e^{tY} \mid N]\right] \\
&= E\left[M_{X_1}(t)^N\right] \\
&= E\left[(pe^t + (1-p))^N\right] \\
&= \sum_{i=0}^{m} \Pr(N = i) \cdot \left(pe^t + (1-p)\right)^i \\
&= \sum_{i=0}^{m} \left(\binom{m}{r} q^i (1-q)^{m-i} \cdot \left(pe^t + (1-p)\right)^i\right) \\
&= \sum_{i=0}^{m} \left(\binom{m}{r} \left(pqe^t + (1-p)q\right)^i (1-q)^{m-i}\right) \\
&= \left(pqe^t + (1-pq)\right)^m
\end{aligned}$$

This implies Y follows the binomial distribution with parameters m and pq. ◀

(Q4) (i) A sequence X_n of random variables is said to converge to a number c in the mean square, if $\lim_{n\to\infty} E[(X_n - c)^2] = 0$. Show that convergence in the mean square implies convergence in probability.

(ii) Jobs are processed, one at a time. The processing times of jobs are independent random variables, uniformly distributed in $[1, 10]$. Find or approximate the probability that the number of jobs processed within 200 time units is at least 75. You can use standard normal distribution table. [5+5=10]

(Ans 4(i)) Let X_n converges to c in the mean square. Using the Markov inequality, we get

$$\Pr(|X_n - c| \geq \epsilon) \leq \frac{E\left[(X_n - c)^2\right]}{\epsilon^2}.$$

This implies $\lim_{n\to\infty} \Pr(|X_n - \epsilon|) = 0$, which is the desired result. ◀

(Ans 4(ii)) Let X_i be the random variable that denotes the processing time of the i^{th} job, $i \in \mathbb{N}$. Note that $X_i \in [1, 10]$ and, is uniformly distributed. $Z = \frac{X_1 + \ldots + X_{75} - 75\cdot\mu}{\sqrt{75}\sigma}$, where μ and σ are the mean and standard deviation of each X_i. So, $\mu = 11/2$ and $\sigma \approx 2.59$.
Observe that we have to compute the following.

$$\begin{aligned} & \Pr(X_1 + \ldots + X_{75} \leq 200) \\ \approx\ & \Pr\left(Z \leq -9.47\right) \\ \approx\ & \Phi(-9.47) \text{ (by CLT)} \end{aligned}$$

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