INDIAN STATISTICAL INSTITUTE

<u>Class Test II</u>

M Tech (CS), 2023 (Semester – I) *Probability and Stochastic Processes*

Date: 18.10.2023 Maximum Marks : 30 Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course:(MTech/JRF/PLP) ____________ **Name:** ____________________________________ **Roll Number:** ____________

(Q1) Let X and Y be independent Bernoulli random variables with parameter $\frac{1}{2}$. Show that $X + Y$ and $|X - Y|$ are dependent though uncorrelated. [5]

(Ans:) Let us look at the covariance of the random variables $X + Y$, and $|X - Y|$.

$$\text{cov}\,(X + Y, |X - Y|) = E[(X + Y)(|X - Y|)] - E[X + Y] \cdot E[|X - Y|]$$

The values of the random variables and their corresponding probabilities are as follows:

$(X + Y)\,\lvert X - Y\rvert$	0	1	2
Probability	$\frac{5}{8}$	$\frac{1}{4}$	$\frac{1}{8}$

$(X + Y)$	0	1	2
Probability	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$(X - Y)$	0	1
Probability	$\frac{1}{2}$	$\frac{1}{2}$

$$\begin{aligned}\text{cov}\,(X + Y, |X - Y|) &= E[(X + Y)(|X - Y|)] - E[X + Y] \cdot E[|X - Y|] \\ &= \frac{1}{4} + \frac{1}{4} - 1 \cdot \frac{1}{2} \\ &= 0\end{aligned}$$

So, $X + Y$ and $|X - Y|$ are uncorrelated.

Looking at the probabilities $\Pr(X + Y = 0, |X - Y| = 0)$, $\Pr(X + Y = 0)$ and $\Pr(|X - Y| = 0)$, we see that $\Pr(X + Y = 0, |X - Y| = 0) \neq \Pr(X + Y = 0) \cdot \Pr(|X - Y| = 0)$. So, they are dependent. ◀

(Q2) A *tournament* on a set V of n players is an orientation $T = (V, E)$ of the edges of the complete graph on the set of vertices V. Thus for every two distinct elements x and y of V, either (x, y) or (y, x) belongs to E, but not both. A simple interpretation of *tournament* is in terms of games where each distinct pair x, y of players, $x, y \in V$, play a single match; the outcome of the games are either win or loss. (x, y) is in the *tournament* if and only if x beats y.

T has the property S_k if for every set of k players there is one who beats them all.

Show that if $\binom{n}{k}(1 - 2^{-k})^{n-k} < 1$, then there is a tournament on n vertices that has the property S_k. [10]

[(Ans:)] Consider a fixed subset K (($K \subset V$) of size k from a random tournament $V = \{1, 2, \ldots, n\}$.

Notice that for each vertex $v \in V \setminus K$, the probability that v does not beat all the members of K is $1 - 2^{-k}$ and all of these $n - k$ events for a choice of v are independent.

Let $\mathcal{E}_K$ be the event that there is no player (vertex) who beats all the members of K. So, $\Pr(\mathcal{E}_K) = (1 - 2^{-k})^{n-k}$. We next use the union bound and the condition given in the question to deduce the result, as follows:

$$\Pr\left(\bigcup_{K \subset V, |K|=k} \mathcal{E}_K\right) = \sum_{K \subset V, |K|=k} \Pr\left(\mathcal{E}_K\right) = \binom{n}{k}\left(1 - 2^{-k}\right)^{n-k} < 1.$$

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(Q3) We toss a biased coin n times. The probability of heads for the biased coin is given by q, the value of a random variable Q having mean μ and positive variance σ^2. Let X_i's be Bernoulli random variables modelling the outcome of the i-th toss; $X_i = 1$ if and only if the i-th toss is a head. Let the X_i's ($i = 1, \ldots, n$) be conditionally independent, given $Q = q$. Let X be the number of heads obtained in the n tosses. Find (i) $E[X_i]$ and $E[X]$; (ii) Find cov(X_i, X_j). Are $X_1, \ldots, X_n$ independent?; (iii) Find var(X). [(2+2)+(4+1)+6=15]

[(Ans (i):] As each X_i is a 0-1 random variable with the probability of head following a random variable Q, we have $E[X_i \mid Q] = \Pr(X_i = 1) = Q$. So, $E[X_i] = E[E[X_i \mid Q]] = E[Q] = \mu$. Since $X = \sum_{i=1}^{n} X_i$, $E[X] = n\mu$.

[(Ans (ii):)] Using the assumption on conditional independence, we have for $i \neq j$,

$$E[X_i X_j \mid Q] = E[X_i \mid Q] \cdot E[X_j \mid Q] = Q^2.$$

Also, $E[X_i X_j] = E[E[X_i X_j \mid Q]] = E[Q^2]$. So,

$$\text{cov}(X_i, X_j) = E[X_i X_j] - E[X_i] \cdot E[X_j] = E[Q^2] - \mu^2 = \sigma^2 > 0.$$

Since, cov$(X_i, X_j) > 0$, $X_1, \ldots, X_n$ are not independent.

[(Ans (iii):)] As X_i is a 0-1 random variable, we know that $X_i^2 = X_i$. Also, for $i = j$,

$$\text{var}(X_i) = E[X_i^2] - (E[X_i])^2 = E[X_i] - (E[X_i])^2 = \mu - \mu^2.$$

Now, use the law of total variance, to get

$$\begin{aligned}
\text{var}(X) &= E[\text{var}(X \mid Q)] + \text{var}(E[X \mid Q]) \\
&= E[\text{var}(X_1 + \ldots + X_n \mid Q)] + \text{var}(E[X_1 + \ldots + X_n \mid Q]) \\
&= E[nQ(1-Q)] + \text{var}(nQ) \\
&= nE[Q - Q^2] + n^2 \text{var}(Q) \\
&= n(\mu - \mu^2 - \sigma^2) + n^2\sigma^2 \\
&= n(\mu - \mu^2) + n(n-1)\sigma^2
\end{aligned}$$

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(Q4) (i) Let X be a random variable that assumes only nonnegative values. Show that for all $a > 0$, $\Pr(X \geq a) \leq \frac{E[X^m]}{a^m}$, where m is a positive integer ≥ 2.

(ii) Let $X_1, \ldots X_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. Let the sample mean $M_n = \frac{X_1 + \cdots + X_n}{n}$. Show that $\Pr(|M_n - \mu| \geq \varepsilon) \to 0, \text{ as } n \to \infty$. [5+5=10]

[Ans (i):] For $a > 0$, let

$$I = \begin{cases} 1 & \text{if } X^m \geq a^m; \\ 0 & \text{otherwise.} \end{cases}$$

Since $X \geq 0$, we have $I \leq 1 \leq \frac{X^m}{a^m}$. As I is a 0-1 r.v., $E[I] = \Pr(I = 1) = \Pr(X^m \geq a^m)$. Take expectations on both sides to get

$$\Pr(X^m \geq a^m) = E[I] \leq E\left[\frac{X^m}{a^m}\right] = \frac{E\left[X^m\right]}{a^m}.$$

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[Ans (ii):] Let $X_1, \ldots X_n$ be a sequence of i.i.d. random variables with mean μ and variance σ^2. The sample mean $M_n = \frac{X_1 + \cdots + X_n}{n}$. The mean and variance of M_n are μ and $\frac{\sigma^2}{n}$. Now, applying Chebyshev inequality with $\varepsilon > 0$, we get

$$\Pr(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2}.$$

Now for every $\varepsilon > 0$ as $n \to \infty$, we have

$$\Pr(|M_n - \mu| \geq \varepsilon) = \Pr\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| \geq \varepsilon\right) \leq \frac{\sigma^2}{n\varepsilon^2} \to 0.$$

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