INDIAN STATISTICAL INSTITUTE

<u>Class Test I</u>

M Tech (CS), 2023 (Semester – I) *Probability and Stochastic Processes*

Date: 08.09.2023 Maximum Marks : 30 Duration : 1.0 hours

Note: The question paper is of 40 marks. Answer as much as you can, but the maximum you can score is 30. Answer a question within its allotted box.

Course:(MTech/JRF/PLP) ________ **Name:** __________________ **Roll Number:** ________

(Q1) Let A and B be events with probabilities $\Pr(A) = \frac{3}{4}$ and $\Pr(B) = \frac{1}{3}$.

(a) Show that $\frac{1}{12} \leq \Pr(A \cap B) \leq \frac{1}{3}$.

(b) Find the corresponding bounds for $\Pr(A \cup B)$. [5+5]

(a)
$\Pr(A \cap B) = \Pr(A) + \Pr(B) - \Pr(A \cup B) \geq \Pr(A) + \Pr(B) - 1 = \frac{1}{12}$.
Also, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, $\Pr(A \cap B) \leq \min\{\Pr(A), \Pr(B)\} = \frac{1}{3}$.

(b)
For $\Pr(A \cup B) \leq \min\{\Pr(A) + \Pr(B), 1\} = 1$, and $\Pr(A \cup B) \geq \max\{P(A), P(B)\} = \frac{3}{4}$.

(Q2) Let there be n sticks each of which is broken into one long and one short part. The $2n$ parts are arranged into n pairs from which new sticks are formed. Find the probability that

(a) the parts will be joined in the original order.

(b) that all long parts are paired with short parts. [6+4=10]

(a)
The possible number of arrangements is $(2n)!$. Of them, fix a couple (a long and a short part of the same stick). There are n of them. They can be permuted in $n!$ ways. Now, in an arrangement, each couple can be arranged in 2 ways, giving 2^n for n couples. So, the said probability is $\frac{2^n n!}{(2n)!}$.

(b)
Now permute n long sticks in $n!$ ways and n short sticks in $n!$ ways and pair them up in $(n!)^2$ ways. For each such arrangement, one can again order the long and short in 2 ways, leading to 2^n for n couples. Thus, the said probability is $\frac{2^n (n!)^2}{(2n)!}$ which for a better form is $\frac{2^n}{\binom{2n}{n}}$.

(Q3) (a) Consider the following gambling game. A player holds a bet on any one of the numbers $\{1, 2, 3, 4, 5, 6\}$. Three dice are then rolled, and if the number bet by the player appears i times, $i = 1, 2, 3$, then the player wins i units. On the other hand, if the number bet by the player does not appear on any of the dice, then the player loses 1 unit. Find out if the game is fair to the player.

(b) Let X be a random variable defined over a sample space Ω such that $\mathrm{E}[X] = \mu$. Show that $\Pr(X \geq \mu) > 0$.

[7+3=10]

(a)

Let X be a random variable that denotes the amount won by the player. X takes the value -1, 1, 2 and 3. The dice are independent. Let Y be a random variable that denotes the number of times the number bet by the player matches the number in the three dice rolled. Y can take the values 0, 1, 2 and 3. Also, Y follows binomial with parameters $(3, \frac{1}{6})$.

So, $\Pr(X = -1) = \Pr(Y = 0) = \binom{3}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^3 = \frac{125}{216}$. Similarly, $\Pr(X = 1) = \Pr(Y = 1) = \binom{3}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2 = \frac{75}{216}$; and $\Pr(X = 2) = \Pr(Y = 2) = \binom{3}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1 = \frac{15}{216}$; and $\Pr(X = 3) = \Pr(Y = 3) = \binom{3}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^0 = \frac{1}{216}$.
$E[X]$, the expectation of X, is basically the amount won by the player. $E[X] = \frac{-125+75+30+3}{216} = \frac{-17}{216}$.
As $E[X] < 0$, the player would lose in the expected case and as such the game is not fair to the player.

(b)

Assume, for a contradiction, $\Pr(X \geq \mu) = 0$. Then,

$$\mu = \mathrm{E}[X] = \sum_{x} x \Pr(X = x) = \sum_{x<\mu} x \Pr(X = x) < \sum_{x<\mu} \mu \Pr(X = x) = \mu$$

which can not be. So, $\Pr(X \geq \mu) \neq 0$ and as $\Pr(\cdot) \geq 0$, we have the result.

(Q4) Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) geometric random variables, each with success probability p.

(a) Show that $\Pr(X_i > t) = (1-p)^t$.

(b) Using the above result, prove that $\Pr(\max_i X_i > t) \leq n(1-p)^t$. [3+7=10]

(a)

X_i's are i.i.d. geometric random variables. So, we have

$$\begin{aligned}\Pr(X_i > t) &= 1 - \Pr(X_i \leq t) \\ &= 1 - \left(1 - (1-p)^t\right) \\ &= (1-p)^t\end{aligned}$$

(b)

Now, we have

$$\begin{aligned}\Pr(\max_i X_i > t) &= \Pr(\text{ at least one } X_i > t) \\ &= \Pr\left(\bigcup_{i=1}^{n} (X_i > t)\right) \\ &\leq \sum_{i=1}^{n} \Pr(X_i > t) \text{ (using union bound)} \\ &= n(1-p)^t\end{aligned}$$