Probability and Stochastic Processes (2023-24)
Problem Sheet 2

1. Let X be a random variable taking values in the range $[a, b]$. Then, show that $\mathbf{V}[X] \leq \frac{(b-a)^2}{4}$.
2. For a geometric random variable X with parameter p and for $n > 0$, show that

$$\Pr(X = n + k \mid X > k) = \Pr(X = n)$$

3. If f is a convex function, then show that $\mathbf{E}[f(X)] \geq f(\mathbf{E}[X])$.
4. Does $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$ implies X, Y are independent? Prove or give a counterexample.
5. Are there discrete random variables X and Y such that $\mathbf{E}[X] > 100\mathbf{E}[Y]$ but Y is greater than X with probability at least $\frac{1}{100}$?
6. There are 100 shoelaces in a box. At each stage, you pick two random ends and tie them together. Either this results in a longer shoelace (if the two ends came from different pieces), or it results in a loop (if the two ends came from the same piece). What are the expected number of steps until everything is in loops, and the expected number of loops after everything is in loops?
7. Randomly, k distinguishable balls are placed into n distinguishable boxes, with all possibilities equally likely. Find the expected number of empty boxes, the expected number of boxes with exactly one ball, and the expected number of boxes with more than one ball.
8. For $n \geq 2$, what is the average number of local maxima of a random permutation of $1, 2, \ldots, n$ with all $n!$ permutations equally likely?
9. Suppose you want to find a minimum of an array A of length n using the following algorithm:

Algorithm 1: FindMin(A)

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x ← ∞;
for i = 1 to n do
    if x ≥ A[i] then
        x ← A[i];
    end
end
return x
```

What is the expected number of times the algorithm executes the Line: "$x \leftarrow A[i]$", that is, what is the expected number of times x is being replaced?

10. Suppose that we roll twice a fair k-sided die with the numbers 1 through k on the die's faces, obtaining values X_1 and X_2 . What is $\mathbf{E}[\max(X_1, X_2)]$? What is $\mathbf{E}[\min(X_1, X_2)]$? (b) Also show that $\mathbf{E}[\max(X_1, X_2)] + \mathbf{E}[\min(X_1, X_2)] = \mathbf{E}[X_1] + \mathbf{E}[X_2]$.

11. You need a new staff assistant, and you have n people to interview. You want to hire the best candidate for the position. When you interview a candidate, you can give them a score, with the highest score being the best and no ties being possible. You interview the candidates one by one. Because of your company's hiring practices, after you interview the kth candidate, you either offer the candidate the job before the next interview or you forever lose the chance to hire that candidate. We suppose the candidates are interviewed in a random order, chosen uniformly at random from all $n!$ possible orderings. We consider the following strategy. First, interview m candidates but reject them all: these candidates give you an idea of how strong the field is. After the mth candidate. hire the first candidate you interview who is better than all of the previous candidates you have interviewed. Let E be the event that we hire the best assistant, and let E_i be the event that i-th candidate is the best and we hire him. Determine $\Pr(E_i)$, and show that

$$\Pr(E) = \frac{m}{n} \sum_{j=m+1}^{n} \frac{1}{j-1}$$

12. Suppose that the number of people entering a department store on a given day is a random variable with mean 50. Suppose further that the amounts of money spent by these customers are independent random variables having a common mean of Rs. 8. Finally, suppose also that the amount of money spent by a customer is also independent of the total number of customers who enter the store. What is the expected amount of money spent in the store on a given day?